ON CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS ASSOCIATED WITH GEGENBAUER POLYNOMIALS

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Abstract. In this work, the authors considered a new subclass $T_{G_t}(\alpha, \beta)$ consisting of analytic univalent functions with negative coefficients defined by Gegenbauer polynomials. Coefficient inequalities, extreme points and integral means inequalities for the class $T_{G_t}(\alpha, \beta)$ were determined.

1. Introduction

Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in U)$$

which are analytic in the unit disk $U = \{ z : |z| < 1 \}$ and normalized by $f(0) = f'(0) - 1 = 0$ in $U$. Recall that, $S$ denote the subclass of $A$ consisting of functions that are univalent. Also, denote by $T$ a subclass of $A$ consisting functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0 \quad (z \in U)$$

introduced and studied by Silverman [5].

The class $T(\lambda), \lambda \geq 0$ were introduced and investigated by Szynal [8] as the subclass of $A$ consisting of functions of the form

$$f(z) = \int_{-1}^{1} k(z, t) d\mu(t),$$

where

$$k(z, t) = \frac{z}{(1 - 2tz + z^2)^{\lambda}} \quad (z \in U), \quad t \in [-1, 1]$$

and $\mu$ is a probability measure on the interval $[-1, 1]$. The collection of such measures on $[a, b]$ is denoted by $P_{[a, b]}$.

The Taylor series expansion of the function in (4) gives

$$k(z, t) = z + c_1^{(\lambda)}(t) z^2 + c_2^{(\lambda)}(t) z^3 + ...$$

2010 Mathematics Subject Classification. 30C45, 30C50.
Key words and phrases. analytic function, univalent function, integral means inequalities, convolution, subordination, Gegenbauer polynomials.
and the coefficients for (5) were given below:

\[ c_0^{(\lambda)}(t) = 1, c_1^{(\lambda)}(t) = 2\lambda, c_2^{(\lambda)}(t) = 2\lambda(\lambda+1)t^2 - \lambda, c_3^{(\lambda)}(t) = \frac{4}{3}\lambda(\lambda+1)(\lambda+2)t^3 - 2\lambda(\lambda+1)t, \ldots \]  
(6)

where \( c_n^{(\lambda)}(t) \) denotes the Gegenbauer polynomial of degree \( n \). Varying the parameter \( \lambda \) in (5), we obtain the class of typically real functions studied by [1], [3], [4] and [6].

For \( g(z) = z + \sum_{n=2}^{\infty} b_n z^n \), the Hadamard product of \( f \) and \( g \) is defined by

\[ (f \ast g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n \quad (z \in U). \]

Also, for two analytic functions \( g \) and \( h \) with \( g(0) = h(0) \), \( g \) is said to be subordinate to \( h \), denoted by \( g \prec h \), if there exists an analytic function \( \psi \) such that \( \psi(0) = 0 \); \( |\psi'(z)| < 1 \) and \( g(z) = h(\psi(z)) \), for all \( (z \in U) \).

Let \( G_{\lambda,t} : A \to A \) defined in terms of the convolution by

\[ G_{\lambda,t}f(z) = k(z,t) * f(z), \]

we have

\[ G_{\lambda,t}f(z) = z + \sum_{n=2}^{\infty} c_n^{(\lambda)}(t) a_n z^n. \]  
(7)

A class \( UCD(\alpha) \), \( \alpha \leq 0 \) consisting of functions \( f \in A \) satisfying

\[ \text{Re}\left\{ f'(z) \right\} \geq \alpha\text{Re}\left\{ f''(z) \right\}, (z \in U) \]

was introduced and investigated in [10].

A related class \( SD(\alpha) \) have been introduced and studied in [7] and [9]. A function \( f \) of the form (1) is said to be in the class \( SD(\alpha) \) if

\[ \text{Re}\left\{ \frac{f(z)}{z} \right\} \geq \alpha \left| \frac{f'(z)}{z} - \frac{f(z)}{z} \right|, \text{ for } \alpha \geq 0. \]

Recently, [11] extended the class of functions studied by [7] and [9] by making use of Hurwitz-Lerch Zeta Function, the coefficient inequalities, extreme points, integral means inequalities and subordination results for the class \( T_{j_{\mu,b}}(\alpha,\beta) \) were obtained in which

\[ \text{Re}\left\{ \frac{j_{\mu,b}f(z)}{z} \right\} \geq \alpha \left| \frac{j_{\mu,b}f(z)}{z} \right| + \beta, \text{ for } \alpha \geq 0. \]

For \( \alpha \geq 0, \beta \in [0,1), \lambda > 0, t \in [-1,1] \), we let \( G_{\lambda,t}(\alpha,\beta) \) be the subclass of \( A \) consisting of functions of the form (1) and its geometrical condition satisfy

\[ \text{Re}\left\{ \frac{G_{\lambda,t}f(z)}{z} \right\} \geq \alpha \left| \frac{G_{\lambda,t}f(z)}{z} \right| + \beta, \]  
(8)

where \( G_{\lambda,t}f(z) \) is given by (7).

Motivated by earlier works of [11] and [12], in this paper, we investigate the coefficient inequalities, extreme points and the integral means inequalities for the class \( T_{G_{\lambda,t}}(\alpha,\beta) \).
2. Main Results

Theorem 2.1 A function \( f(z) \) be the form (1) is in \( G_{\lambda,t}(\alpha, \beta) \) if
\[
\sum_{n=2}^{\infty} (1 + \alpha(n-1)) c_{n-1}^\lambda(t) |a_n| \leq 1 - \beta
\]
where \( \alpha \geq 0, \beta \in [0,1), \lambda > 0, t \in [-1,1] \).

Proof It suffices to show that
\[
\alpha \left| (G_{\lambda,t} f(z))' - \frac{G_{\mu,b} f(z)}{z} \right| - \text{Re} \left\{ \frac{G_{\lambda,t} f(z)}{z} - 1 \right\} \leq 1 - \beta.
\]
We have
\[
\alpha \left| (G_{\lambda,t} f(z))' - \frac{G_{\mu,b} f(z)}{z} \right| - \text{Re} \left\{ \frac{G_{\lambda,t} f(z)}{z} - 1 \right\} \\
\leq \alpha \left| (G_{\lambda,t} f(z))' - \frac{G_{\mu,b} f(z)}{z} \right| - \text{Re} \left\{ \frac{G_{\lambda,t} f(z)}{z} - 1 \right\} \\
\leq \alpha \left| \sum_{n=2}^{\infty} (n-1) c_{n-1}^\lambda(t) a_n z^n \right| + \left| \sum_{n=2}^{\infty} c_{n-1}^\lambda(t) a_n z^n \right|
\]
\[
= \sum_{n=2}^{\infty} \left( 1 + \alpha(n-1) \right) c_{n-1}^\lambda(t) |a_n|.
\]
The last expression is bounded above by \((1 - \beta)\) if
\[
\sum_{n=2}^{\infty} (1 + \alpha(n-1)) c_{n-1}^\lambda(t) |a_n| \leq 1 - \beta
\]
and this completes the proof.

For the next theorem, the necessary and sufficient conditions for the functions of the class \( T G_{\lambda,t}(\alpha, \beta) \)

Theorem 2.1 A function \( f(z) \) be the form (2) is in \( T G_{\lambda,t}(\alpha, \beta) \) if
\[
\sum_{n=2}^{\infty} (1 + \alpha(n-1)) c_{n-1}^\lambda(t) |a_n| \leq 1 - \beta
\]
where \( \alpha \geq 0, \beta \in [0,1), \lambda > 0, t \in [-1,1] \).

Proof Suppose \( f(z) \) of the form (2) is in the class \( T G_{\lambda,t}(\alpha, \beta) \). Then
\[
\text{Re} \left\{ \frac{G_{\lambda,t} f(z)}{z} \right\} - \alpha \left| (G_{\lambda,t} f(z))' - \frac{G_{\mu,b} f(z)}{z} \right| \geq \beta
\]
Equivalently
\[
\text{Re} \left[ 1 - \sum_{n=2}^{\infty} c_{n-1}^\lambda(t) |a_n| z^{n-1} \right] - \alpha \left| \sum_{n=2}^{\infty} (n-1) c_{n-1}^\lambda(t) a_n z^{n-1} \right| \geq \beta
\]
Letting \( z \) to take real values and as \( |z| \to 1 \), we have
\[
1 - \sum_{n=2}^{\infty} c_{n-1}^\lambda(t) |a_n| - \alpha \sum_{n=2}^{\infty} (n-1) c_{n-1}^\lambda(t) |a_n| \geq \beta
\]
which implies Theorem 2.2.

**Corollary 2.3:** A function \( f(z) \) be the form (2) is in \( T\mathcal{G}_{\lambda,t}(\alpha, \beta) \) if

\[
|a_n| \leq \frac{1 - \beta}{(1 + \alpha(n - 1)) c_{n-1}^\lambda(t)}
\]

where \( \alpha \geq 0, \beta \in [0, 1), \lambda > 0, t \in [-1, 1] \).

**Theorem 2.4:** Let \( f_1(z) = z \) and \( f_n(z) = z - \frac{1 - \beta}{(1 + \alpha(n - 1)) c_{n-1}^\lambda(t)} z^n, n \geq 2 \) for where \( \alpha \geq 0, \beta \in [0, 1), \lambda > 0 \) and \( t \in [-1, 1] \). Then \( f(z) \) is in the class \( T\mathcal{G}_{\lambda,t}(\alpha, \beta) \) if and only if it can be expressed in the form

\[
f(z) = \sum_{n=1}^{\infty} \psi_n f_n(z)
\]

where \( \psi \geq 0 \) and \( \sum_{n=1}^{\infty} \psi_n = 1 \).

**Proof:** Let \( f(z) \) be expressible in the form \( f(z) = \sum_{n=1}^{\infty} \psi_n f_n(z) \). Then

\[
f(z) = \psi_1 f_1(z) + \sum_{n=2}^{\infty} \psi_n f_n(z) = \psi_1 z + \sum_{n=2}^{\infty} \psi_n \left[ z - \frac{1 - \beta}{(1 + \alpha(n - 1)) c_{n-1}^\lambda(t)} z^n \right]
\]

\[
= z - \frac{1 - \beta}{(1 + \alpha(n - 1)) c_{n-1}^\lambda(t)} z^n.
\]

Now

\[
\sum_{n=2}^{\infty} \frac{(1 + \alpha(n - 1)) c_{n-1}^\lambda(t)}{1 - \beta} \frac{1 - \beta}{(1 + \alpha(n - 1)) c_{n-1}^\lambda(t)} \psi_n = \sum_{n=1}^{\infty} \psi_n = 1 - \psi_1 \leq 1.
\]

Thus \( f \in T\mathcal{G}_{\lambda,t}(\alpha, \beta) \).

Conversely, suppose \( f \in T\mathcal{G}_{\lambda,t}(\alpha, \beta) \). Then corollary 2.3 gives

\[
a_n \leq \frac{1 - \beta}{(1 + \alpha(n - 1)) c_{n-1}^\lambda(t)}, n \geq 2
\]

Set \( \psi_n = \frac{(1 + \alpha(n - 1)) c_{n-1}^\lambda(t)}{1 - \beta} a_n, n \geq 2 \), where \( \psi_1 = 1 - \sum_{n=2}^{\infty} \psi_n \). Then \( f(z) = z - \sum_{n=2}^{\infty} a_n z^n \)

\[
= \sum_{n=2}^{\infty} \psi_n \frac{1 - \beta}{(1 + \alpha(n - 1)) c_{n-1}^\lambda(t)} z^n
\]

\[
= z - \sum_{n=2}^{\infty} \psi_n z + \sum_{n=2}^{\infty} \psi_n f_n(z)
\]

\[
= z \left[ 1 - \sum_{n=2}^{\infty} \psi_n \right] + \sum_{n=2}^{\infty} \psi_n f_n(z)
\]

\[
= \psi_1 f_1 z + \sum_{n=2}^{\infty} \psi_n f_n(z)
\]

\[
= \sum_{n=1}^{\infty} \psi_n f_n(z)
\]

Hence the proof.

For the purpose of the last theorem, the lemma below shall be necessary.

**Lemma:**[12]: If the functions \( f(z) \) and \( g(z) \) are analytic in \( (z \in U) \) with \( g(z) \prec f(z) \), then

\[
\int_0^{2\pi} |g(re^{i\theta})|^p d\theta \leq \int_0^{2\pi} |f(re^{i\theta})|^p d\theta, (0 \leq r < 1, p > 0).
\]
Theorem 2.5 Suppose \( f \in T_{G}(\alpha, \beta), p > 0, \alpha \geq 0, \lambda > 0, \beta \in [0, 1), t \in [-1, 1] \) and \( f_{2}(z) \) is defined by \( f_{2}(z) = z - \frac{1 - \beta}{2\lambda(1 + \alpha)}z^{2} \). Then for \( z = re^{i\theta}, 0 \leq r < 1 \), we have

\[
\int_{0}^{2\pi} |f(z)|^{p}d\theta \leq \int_{0}^{2\pi} |f_{2}(z)|^{p}d\theta. \tag{11}
\]

Proof For \( f(z) = z - \sum_{n=2}^{\infty} |a_n|z^n \), (11) is equivalent to proving that

\[
\int_{0}^{2\pi} |z - \sum_{n=2}^{\infty} |a_n|z^n|^{p}d\theta \leq \int_{0}^{2\pi} |z - \frac{1 - \beta}{2\lambda(1 + \alpha)}z^{2}|^{p}d\theta \quad (p > 0). \tag{12}
\]

By applying Littlewood's subordination theorem, it will be sufficient to show that

\[
1 - \sum_{n=2}^{\infty} |a_n|z^{n-1} \prec 1 - \frac{1 - \beta}{2\lambda(1 + \alpha)} \omega(z). \tag{13}
\]

Setting

\[
1 - \sum_{n=2}^{\infty} |a_n|z^{n-1} = 1 - \frac{1 - \beta}{2\lambda(1 + \alpha)} \omega(z), \tag{14}
\]

we obtain \( \omega(z) = \frac{2\lambda(1 + \alpha)}{1 - \beta} \sum_{n=2}^{\infty} a_{n}z^{n-1} \) and \( \omega(z) \) is analytic in \( (z \in \mathbb{U}) \) with \( \omega(0) = 0 \).

Moreover, it suffices to prove that \( \omega(z) \) satisfies \( |\omega(z)| < 1, (z \in \mathbb{U}) \). Now

\[
|\omega(z)| = \left| \sum_{n=2}^{\infty} \frac{2\lambda(1 + \alpha)}{1 - \beta} a_{n}z^{n-1} \right| \leq |z| \sum_{n=2}^{\infty} \frac{2\lambda(1 + \alpha)}{1 - \beta} |a_{n}| \leq |z| < 1. \tag{15}
\]

In view of the inequality (15) the subordination (13) follows, which proves the theorem.

References

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