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A VARIABLE FRACTIONAL ORDER NETWORK MODEL OF ZIKA VIRUS

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Abstract. A variable fractional-order network model of Zika is presented in this paper. We also carry out a detailed analysis on the equilibrium points and its stability. Numerical solutions are obtained using a predictor-corrector method to handle the fractional derivatives. The fractional derivatives are described in the Caputo sense. Numerical simulations are presented to illustrate the results. Also, the numerical simulations show that, modeling the Zika variable fractional order model has more advantages than classical integer-order modeling.

1. Introduction

Human societies have been shocked because of the global rapid spread of Zika virus infection in the last few years. World health organization (WHO) sounds the alarms to make the world pay attention to the severity of this serious epidemic. Zika virus is mosquito-borne disease. Zika virus can be transmitted to people primarily through the bite of infected Aedes species mosquitoes. Also, it can be transmitted by many ways such as during pregnancy, blood transfusion, breastfeeding and a person with Zika virus can pass it to his or her sex partners [47]. Historically, it was detected first in forests of Uganda in the middle of the previous century. Recently, the virus hit South American continent and infected thousands of babies with microcephaly. Mathematical models of infectious have become the backbones of epidemiology. Decision making centers are interested in using mathematical models of infectious diseases to take the necessary actions to eradicate infectious diseases [13, 19, 20, 46]. The first mathematical model of Zika virus transmission has been investigated in [23]. Also in [12], mathematical models of dengue and

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Zika outbreaks in Tropical Island have been illustrated. The impact of mosquito-borne and sexual transmission on the spread of Zika virus has been studied in [14]. Different preliminary population models [18, 21] like SIR, SEIR and SEI were used to describe the transmission of dengue, Chikungunya or Zika virus through populations and to study the dynamics of co-infection between two diseases in the same population [16, 22, 43]. All these mentioned examples of Zika models are integer order models in which the memory effect is ignored. In this paper, a variable fractional order SIR model of Zika virus is presented to study the impact of the memory on the spread of Zika virus through a population.

The fractional calculus has been investigated from more than 300 years [24, 27, 32]. It is considered as a mathematical tool for characterizing memory biological and physical systems. The variable-order fractional derivative, which is an extension of constant-order fractional derivative, has been introduced in several scientific fields [5, 6, 25, 33, 35]. Therefore, the main contribution of this study is to investigate the system behavior when the differential order of the system is fractional, continuous function or discontinuous function.

The rest of this paper is organized as follows. Section 2 introduces some rules about fractional calculus. Section 3 presents a variable fractional order model on homogenous network of Zika. Section 4 illustrates the equilibrium points and stability. Section 5 presents illustrative results of numerical simulations. Section 6 concludes this paper.

2. Preliminaries

To distinguish memory properties of dynamical systems in many scientific fields is a crucial issue in modeling of complex systems. Fractional derivative has the significant feature of capturing the history of the variable which cannot be easily done in case of the integer order derivatives [31, 40]. The variable-order fractional derivative, which is an extension of constant- fractional order derivative, is a powerful tool to characterize memory that may vary from point to point. So it is clear that the integer order derivative can be used to characterize the short memory of systems, while the constant-order fractional derivative has advantages in characterizing the long memory of systems. Furthermore, the variable-order fractional derivative can be employed to depict the variable memory of systems [40]. In other words, variable-order fractional derivative is good at depicting the memory property which changes with time or spatial location [44]. It is clear that the integer order derivative can be used to characterize the short memory of systems, while the constant-order fractional derivative has advantages in characterizing the long memory of systems. Furthermore, the variable-order fractional derivative can be employed to depict the variable memory of systems [40].

Several physical phenomena are often better described by fractional order models because fractional order operators are global not only local characteristics [42]. Furthermore, integer order models sometimes contradict the experimental results. Recently, it has been found that, the order of derivative can be generalized to be a bounded function. In other words, fractional order can vary with the variables of fractional order differential equations (FDEs) [45]. The variable fractional order behavior arises in numerous applications. Several theoretical studies coupled with experimental studies related to the fractional variable order have been presented in the last few years. In [40], the variable fractional order is applied in some applications of control theory. A generalization of the van der Pol equation using the VODE formulation is analyzed in this paper. A new generalization of the Schrodinger equation via the concept of space fractional variable-order derivative has been presented in [7]. The stability and the convergence of the space fractional variable-order Schrodinger equation were studied as well. Some experimental results have shown that variable-order fractional models are better to describe many physical phenomena. Smit and de Vries [39] demonstrated that the stress-stain behavior of viscoelastic materials with changing strain level can be characterized by variable-order fractional differential equations. In [15] the authors showed that the relaxation processes and reaction kinetics of proteins under different temperatures show variable-order fractional operator properties. The experimental results presented by H. Sheng et al [36] show that the order of the fractional operator is the function of the temperature variable. D. Sierociuk et al [38] have presented an experimental study of two types of electrical circuits. They verified that the two types of circuits should be presented by variable-order systems. The fractional differential order is a key factor that determines the final quality of the enhanced image, whereas an extremely high fractional order will lead to bad quality of the enhanced image. Hence, variable-order fractional differential operators may solve this problem [44].

In this part, we give some definitions of variable-order fractional derivative which is an extension of constant-order fractional derivative. There exist different approaches for defining the fractional derivatives.

Definition 2.1. (Riemann-Liouville fractional derivatives of order $\alpha(t)$)

Let $\alpha(t)$ be a continuous and bounded function, then Riemann-Liouville variable-order fractional derivative of $f(t) : [a, b] \to \mathbb{R}$ is defined as [45]:

(i) Left Riemann–Liouville derivative of order $\alpha(t)$ is defined by

$$ {}_{a}^{RL}D_{t}^{\alpha(t)} f(t) = \frac{1}{\Gamma(1-\alpha(t))} \frac{d}{dt} \int_{a}^{t} (t-\tau)^{-\alpha(t)} f(\tau)\, d\tau, \quad 0 < \alpha(t) \leq 1 $$

(ii) Right Riemann–Liouville derivative of order $\alpha(t)$ is defined by

$$ {}_{t}^{RL}D_{b}^{\alpha(t)}f(t) = \frac{-1}{\Gamma(1-\alpha(t))}\frac{d}{dt}\int_{t}^{b}(\tau-t)^{-\alpha(t)}f(\tau)\,d\tau, \quad 0<\alpha(t)\le 1 $$

Definition 2.2. (Caputo fractional derivatives of order $\alpha(t)$)

Let $\alpha(t)$ be a continuous and bounded function, then the Caputo variable-order fractional derivative of $f(t):[a,b]\to\mathbb{R}$ is defined as [45]:

(i) Left Caputo derivative of order $\alpha(t)$ is defined by

$$ {}_{a}^{c}D_{t}^{\alpha(t)}f(t) = \frac{1}{\Gamma(1-\alpha(t))}\int_{a}^{t}(t-\tau)^{-\alpha(t)}f'(\tau)\,d\tau, 0<\alpha(t)\le 1 $$

(ii) Right Caputo fractional order derivative of order $\alpha(t)$ is defined by

$$ {}_{t}^{c}D_{b}^{\alpha(t)}f(t) = \frac{-1}{\Gamma(1-\alpha(t))}\int_{t}^{b}(\tau-t)^{-\alpha(t)}f'(\tau)\,d\tau, 0<\alpha(t)\le 1 $$

Definition 2.3. (Grünwald-Letnikov fractional derivatives of order $\alpha(t)$)

Let $\alpha(t)$ be a continuous and bounded function, then the Grünwald-Letnikov variable-order fractional derivative of $f(t):[a,b]\to\mathbb{R}$ is defined as [41]:

$$ {}_{0}^{GL}D_{t}^{\alpha(t)}f(t) = \lim_{h\to 0} h^{-\alpha}\sum_{j=0}^{[n]}(-1)^{j}\binom{\alpha(t)}{j}f(t-jh) $$

Where h is the step size, $n=\frac{t}{h}$, $[n]$ is the integer part of n and $0<\alpha(t)\le 1$.

Some problems appeared when discrete time fractional order derivative is used as follows [29]:

- Fractionalizing the discrete-time systems using classic tools have resulted in finite dimension integer-order systems that are difficult to manipulate.
- Integer order discrete-time systems that is used to approximate continuous-time fractional systems which have long memory, are known for their short memory.

The above definitions show that the memory effect of considered system changes with time and is determined by the current state. Therefore, variable order fractional derivative can be used to characterize variable memory effect of the system. Caputo derivative is attractive when physical models are presented due to the clarity of the physical interpretation of the prescribed data [8]. Also Caputo derivative is essential because the initial conditions for the fractional-order models with the Caputo derivatives are the same as for the integer-order models [38].

Unfortunately, most variable-order fractional differential equations do not have an exact analytical solution, so it is needed to use numerical methods to solve such equations [1, 2].

3. Model derivation

In most of all studies of epidemic spreading on complex networks, the population dynamics factors are not considered. Authors in [17] have introduced a modified epidemic model with birth and death on homogeneous and heterogeneous networks. Through mean field analysis, they found that on homogeneous network, there is an epidemic threshold while for a heterogeneous network, the epidemic threshold is absent in the thermodynamic limit. Because as mentioned in [10, 11] that fractional order models are useful in epidemic models to predict the spread of disease and how to prevent epidemics and so much more, therefore, A fractional order network model for Zika is presented in [9] and the authors expected that in the future the second rote will be more difficult to control so they concluded that their model will be useful a conceptual tool for modeling the impact of interventions aiming to control the disease.

For the SIRS model, each individual can be in three states. $S(t)$,$I(t)$ and $R(t)$ introduce the susceptible, infected, and recovered or vaccinated individuals at time t, respectively. First, a susceptible individual can acquire infection from an infected neighbor at rate β. Then, an infected individual is cured at rate ε and becomes susceptible again at rate γ. At the same time, the susceptible individual is vaccinated at rate δ. The constant λ is the recruitment rate of susceptible corresponding to births and immigration. μ is the natural death rate of population and α is the disease related death rate [17]. The authors in [17] show that in order to construct SIRS model based on homogenous network, we have to consider two hypotheses

1. Homogeneity: to make the network simpler, we consider each node's degree is $< \mathrm{k} >$ where $< \mathrm{k} >$ is the average connectivity of the network, so the network is homogenous.
2. Homogenous mixing: the strength of infection is proportional to the population density.

The variable fractional order SIRS epidemic model on homogenous networks is given by

$$\begin{aligned} D^{\alpha_1(t)}S(t) &= \lambda - \frac{\beta\,\langle k\rangle\, SI}{S+I+R} + \gamma\, R - (\delta+\mu)S, \\ D^{\alpha_2(t)}I(t) &= \frac{\beta\,\langle k\rangle\, SI}{S+I+R} - (\varepsilon+\mu+\alpha)I, \\ D^{\alpha_3(t)}R(t) &= \varepsilon\, I - (\mu+\gamma)R + \delta\, S, \end{aligned} \tag{1}$$

where $\lambda, \beta, \gamma, \delta, \mu, \alpha$ and ε are positive constants, and $\langle k\rangle$ is the average connectivity in network neglecting the heterogeneity of the node degrees [17]. Beside the variable order $\alpha_i(t), i = 1, 2, 3$ is a function of time

4. Equilibrium Points and Stability

Consider the system (1) as follows:

$$D^{\alpha_1(t)}(S) = g_1(S, I, R),$$
$$D^{\alpha_2(t)}(I) = g_2(S, I, R),$$
$$D^{\alpha_3(t)}(R) = g_3(S, I, R).$$

With $\alpha_i(t) \in (0, 1], i = 1, 2, 3$ and the initial conditions $(S(0), I(0), R(0))$.

To evaluate the equilibrium points of the fractional-order system (1), let

$$D^{\alpha_1(t)}(S) = D^{\alpha_2(t)}(I) = D^{\alpha_3(t)}(R) = 0$$
$$\Rightarrow g_i(S_{eq}, I_{eq}, R_{eq}) = 0, i = 1, 2, 3.$$

From which we can get the equilibrium points (S_{eq}, I_{eq}, R_{eq}).

To evaluate the asymptotic stability let

$$S(t) = S_{eq} + \xi_1(t),$$
$$I(t) = I_{eq} + \xi_2(t),$$
$$R(t) = R_{eq} + \xi_3(t).$$

So the equilibrium point (S_{eq}, I_{eq}, R_{eq}) is locally asymptotically stable if all eigenvalues of Jacobian matrix evaluated at the equilibrium point satisfy [26]

$$|\arg(\sigma_i)| > \frac{\alpha(t)\pi}{2}, \alpha(t) \in (0, 1], t \geq 0, i = 1, 2, 3.$$

Now we evaluate the equilibrium points for system (1)

It has two steady states: $E_0 = (\frac{\lambda(\gamma+\mu)}{\mu(\gamma+\mu+\delta)}, 0, \frac{\lambda\delta}{\mu(\gamma+\mu+\delta)})$ and $E_1 = (S^*, I^*, R^*)$ where:

$$S^* = \frac{\lambda(\varepsilon+\mu+\alpha)(\mu+\gamma+\varepsilon)}{(\varepsilon+\mu+\alpha)(\mu+\gamma+\delta)[\mu R_0+\alpha(R_0-1)]+\beta\langle k\rangle\mu\varepsilon},$$
$$I^* = \frac{\lambda(\varepsilon+\mu+\alpha)(\mu+\gamma+\delta)(R_0-1)}{(\varepsilon+\mu+\alpha)(\mu+\gamma+\delta)[\mu R_0+\alpha(R_0-1)]+\beta\langle k\rangle\mu\varepsilon},$$
$$R^* = \frac{\varepsilon I^*+\delta S^*}{\mu+\gamma}.$$

Where the basic reproductive number $R_0 = \frac{\beta\langle k\rangle(\mu+\gamma)}{(\varepsilon+\mu+\alpha)(\mu+\gamma+\delta)}$.

To study the stability we will get the eigenvalues of Jacobian matrix for the system (1)

Firstly, at a disease-free equilibrium point $E_0 = (\frac{\lambda(\gamma+\mu)}{\mu(\gamma+\mu+\delta)}, 0, \frac{\lambda\delta}{\mu(\gamma+\mu+\delta)})$

$$J(E_0) = \begin{bmatrix} -\delta-\mu & \frac{\beta\langle k\rangle(\mu+\gamma)}{(\mu+\gamma+\delta)} & \gamma \\ 0 & \frac{\beta\langle k\rangle(\mu+\gamma)}{(\mu+\gamma+\delta)} - (\varepsilon+\mu+\alpha) & 0 \\ \delta & \varepsilon & -(\mu+\gamma) \end{bmatrix}$$

And the eigenvalues are

$$\sigma_1 = -\mu < 0,$$
$$\sigma_2 = -(\mu+\gamma+\delta) < 0,$$
$$\sigma_3 = -(\varepsilon+\mu+\alpha)(1-R_0) < 0 \quad if\ R_0 < 1.$$

Therefore, if $R_0 < 1$, a disease-free equilibrium point E_0 is locally asymptotically stable since $|\arg(\sigma_i)| = |-\pi| > \frac{\alpha(t)\pi}{2}, \alpha(t) \in (0,1], t \geq 0, i = 1,2,3$ and it is unstable if $R_0 > 1$.

Secondly, at an endemic equilibrium point $E_1 = (S^*, I^*, R^*)$

The characteristic polynomial is given by:

$$\sigma^3 + A\sigma^2 + B\sigma + C = 0,$$

To facilitate analysis, let$p = \frac{I^*}{S^*}, \quad \tau = \frac{I^*}{S^*+I^*+R^*}, \quad w = (\alpha + \mu + \varepsilon)$.

Then

$$\begin{aligned} &A = \gamma + 2\mu + \delta + pw > 0, \\ &B = (p - \tau)w^2 + (\gamma p + \varepsilon\tau + p\mu + \tau\mu)w + \gamma\mu + \mu\delta + \mu^2 > 0, \\ &C = wp[\mu(\gamma + \mu) + \varepsilon\mu] = w\lambda(R_0 - 1)[\mu(\gamma + \mu) + \varepsilon\mu] > 0 \; if \; R_0 > 1. \end{aligned}$$

By using Theorem (Routh-Hurwitz criteria) [1] which states that stability conditions are $C > 0$ and $AB - C > 0$ to get eigenvalues which are negative or have negative real parts. Hence, if $R_0 > 1$, the endemic equilibrium point E_1 is locally asymptotically stable since

$$|\arg(\sigma_i)| = |-\pi| > \frac{\alpha(t)\pi}{2}, \alpha(t) \in (0,1], t \geq 0, i = 1,2,3$$

5. Numerical Simulation and Discussion

We will introduce an algorithm of predictor-corrector method for solving the following system of variable fractional order differential equations

$$\begin{aligned} &D^{\alpha_1(t)}x(t) = f_1(x(t), y(t), z(t)), \\ &D^{\alpha_2(t)}y(t) = f_2(x(t), y(t), z(t)), \qquad 0 \leq t \leq T \\ &D^{\alpha_3(t)}z(t) = f_3(x(t), y(t), z(t)), \end{aligned}$$

With $0 < \alpha_i(t) \leq 1 \, (i = 1,2,3)$ and initial condition (x_0, y_0, z_0).

(1) Evaluate the predicted values as follows:

$$\begin{aligned} x^p_{n+1} &= x_0 + \textstyle\sum_{j=0}^{n} \frac{\beta_{1,j,n+1}}{\Gamma(\alpha_1(t_{n+1}))} f_1(x_j, y_j, z_j), \\ y^p_{n+1} &= y_0 + \textstyle\sum_{j=0}^{n} \frac{\beta_{2,j,n+1}}{\Gamma(\alpha_2(t_{n+1}))} f_1(x_j, y_j, z_j), \\ z^p_{n+1} &= x_0 + \textstyle\sum_{j=0}^{n} \frac{\beta_{3,j,n+1}}{\Gamma(\alpha_3(t_{n+1}))} f_1(x_j, y_j, z_j), \end{aligned}$$

Where

$$\beta_{i,j,n+1} = \frac{h^{\alpha_i(t_{n+1})}}{\alpha_i(t_{n+1})}[(n-j+1)^{\alpha_i(t_{n+1})} - (n-j)^{\alpha_i(t_{n+1})}]. \; h = T/N, \; T_n = nh.$$

(2) Evaluate the corrected values as follows

$$x_{n+1} = x_0 + \frac{h^{\alpha_1(t_{n+1})}}{\Gamma(\alpha_1(t_{n+1})+2)} + f_1(x^p_{n+1}, y^p_{n+1}, z^p_{n+1}) + \sum_{j=0}^{n} \frac{h^{\alpha_1(t_{n+1})}\gamma_{1,j,n+1}}{\Gamma(\alpha_1(t_{n+1}))+2)} f_1(x_j, y_j, z_j),$$
$$y_{n+1} = y_0 + \frac{h^{\alpha_2(t_{n+1})}}{\Gamma(\alpha_2(t_{n+1})+2)} + f_2(x^p_{n+1}, y^p_{n+1}, z^p_{n+1}) + \sum_{j=0}^{n} \frac{h^{\alpha_2(t_{n+1})}\gamma_{2,j,n+1}}{\Gamma(\alpha_2(t_{n+1}))+2)} f_2(x_j, y_j, z_j),$$
$$z_{n+1} = z_0 + \frac{h^{\alpha_3(t_{n+1})}}{\Gamma(\alpha_3(t_{n+1})+2)} + f_3(x^p_{n+1}, y^p_{n+1}, z^p_{n+1}) + \sum_{j=0}^{n} \frac{h^{\alpha_3(t_{n+1})}\gamma_{3,j,n+1}}{\Gamma(\alpha_3(t_{n+1}))+2)} f_3(x_j, y_j, z_j),$$

where

$$\gamma_{i,j,n+1} = \begin{cases} n^{\alpha_i(t_{n+1})+1} - (n-\alpha_i(t_{n+1}))(n+1)^{\alpha_i(t_{n+1})} & , \ j=0, \\ (n-j-2)^{\alpha_i(t_{n+1})+1} + (n-j)^{\alpha_i(t_{n+1})+1} - 2(n-j+1)^{\alpha_i(t_{n+1})} & , \ 1 \le j \le n, \\ 1 & , \ j = n+1 \end{cases}$$

We applied the above predictor-corrector method to get numerical solution of the system (1). We investigate the system behavior in three cases. First case when the variable order is $\alpha_i(t) = 1 - 0.004t$. Second case when the variable order is a periodic function $\alpha_i(t) = 0.7 - 0.01\sin(\pi t)$. Finally we look into the case of the function of variable order is discontinuous as follows:

$$\alpha_i(t) = \begin{cases} 1, & t \in [0, 50] \\ 0.8, & t \in (50, 100]. \end{cases} \quad , i = 1, 2, 3 \tag{2}$$

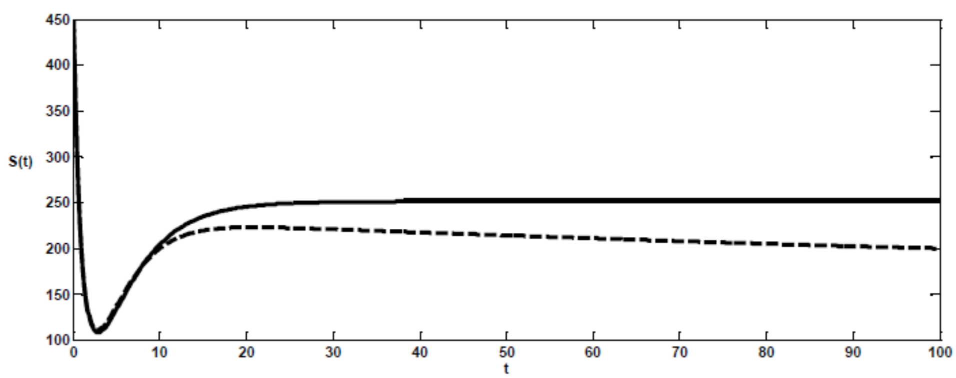

FIGURE 1. The dynamic trajectory $S(t)$ at $\alpha(t) = 1$ (the solid line) and at $\alpha(t) = 1 - 0.004t$ (the dashed line)

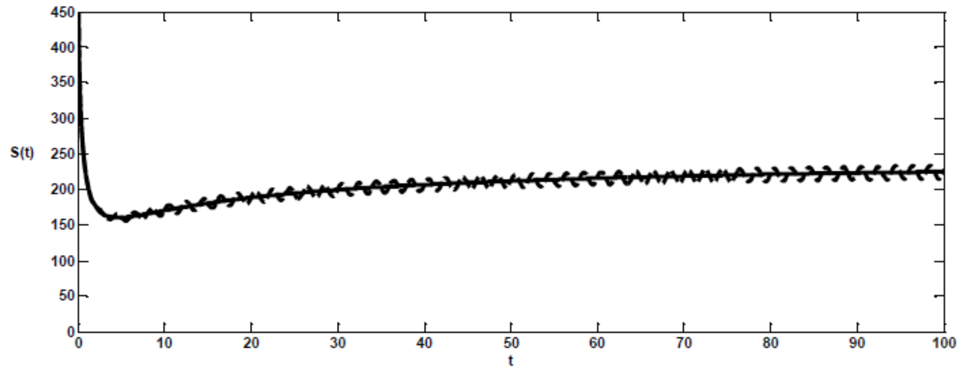

FIGURE 2. The dynamic trajectory $S(t)$ at $\alpha(t) = 0.7$ (the solid line) and at $\alpha(t) = 0.7 - 0.01sin(\pi t)$ (the dashed line)

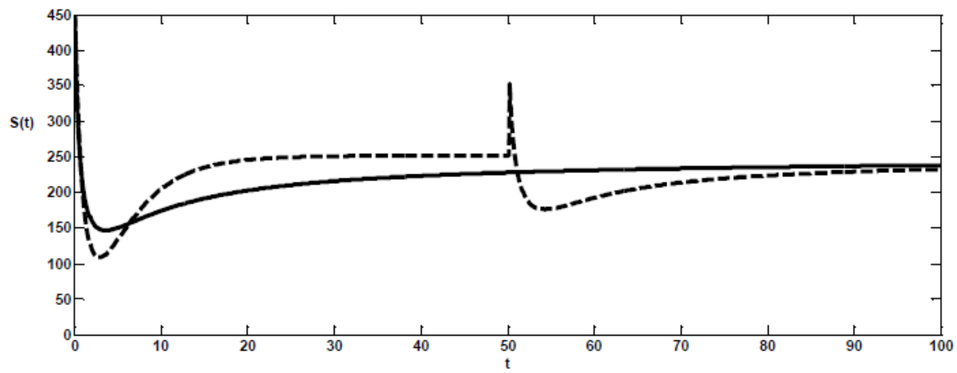

FIGURE 3. The dynamic trajectory $S(t)$ at $\alpha(t) = 0.8$ (the solid line) and at $\alpha(t) = \begin{cases} 1\,, & t = [0, 50] \\ 0.8, & t = (50, 100]. \end{cases}$ (the dashed line).

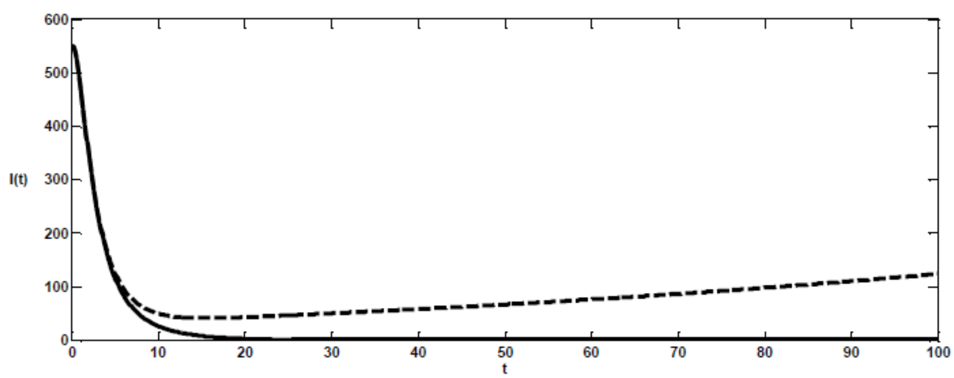

FIGURE 4. The dynamic trajectory $I(t)$ at $\alpha(t) = 1$ (the solid line) and at $\alpha(t) = 1 - 0.004t$ (the dashed line).

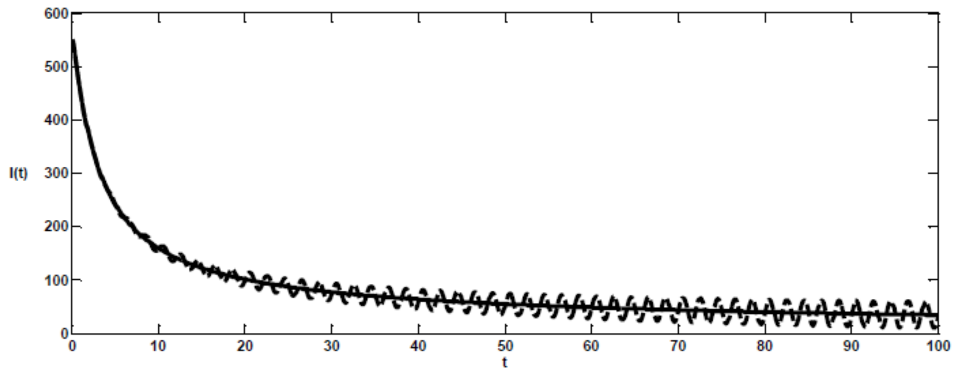

FIGURE 5. The dynamic trajectory $I(t)$ at $\alpha(t) = 0.7$ (the solid line) and at $\alpha(t) = 0.7 - 0.01sin(\pi t)$ (the dashed line).

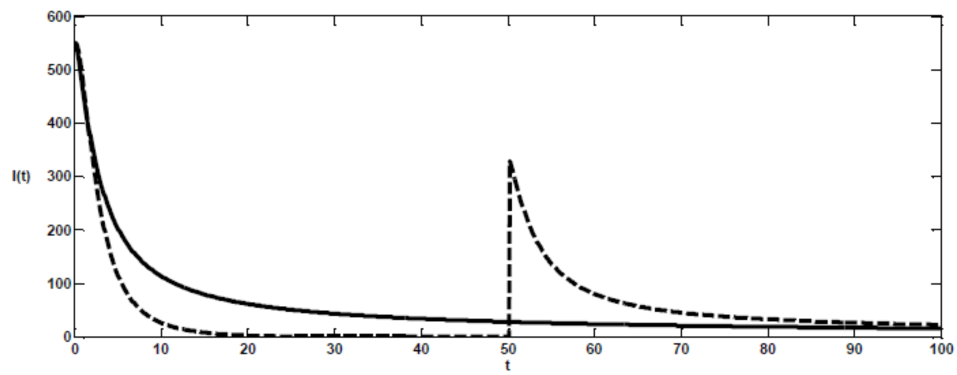

FIGURE 6. The dynamic trajectory $I(t)$ at $\alpha(t) = 0.8$ (the solid line) and at $\alpha(t) = \begin{cases} 1\,, & t = [0, 50] \\ 0.8, & t = (50, 100]. \end{cases}$ (the dashed line).

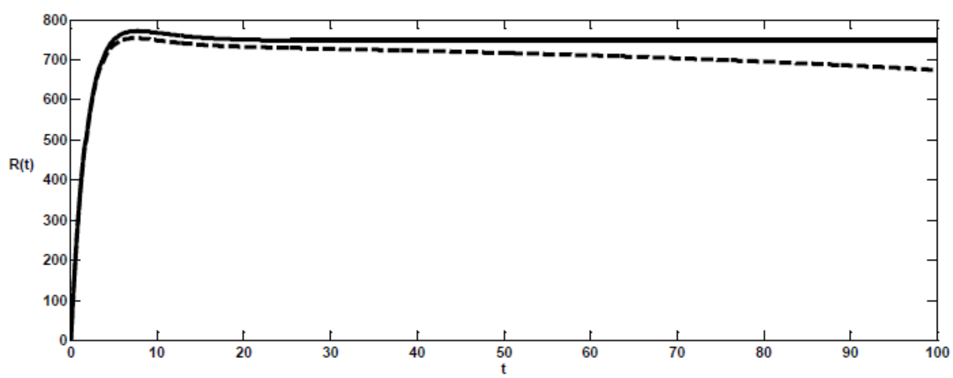

FIGURE 7. The dynamic trajectory $R(t)$ at $\alpha(t) = 1$ (the solid line) and at $\alpha(t) = 1 - 0.004t$ (the dashed line).

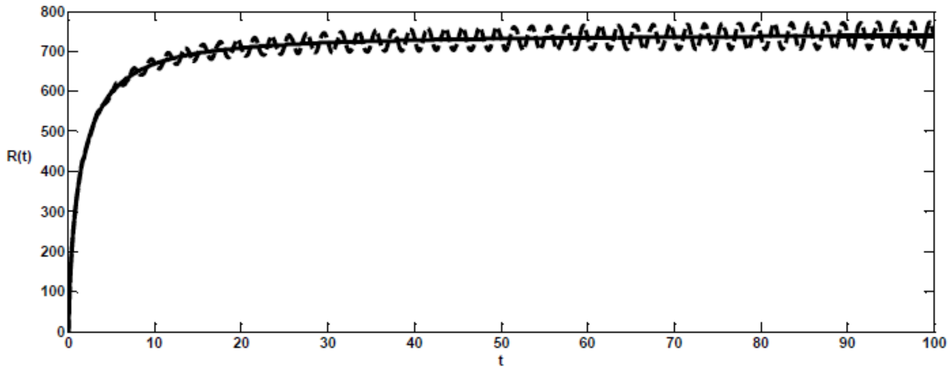

FIGURE 8. The dynamic trajectory $R(t)$ at $\alpha(t) = 0.7$ (the solid line) and at $\alpha(t) = 0.7 - 0.01sin(\pi t)$ (the dashed line).

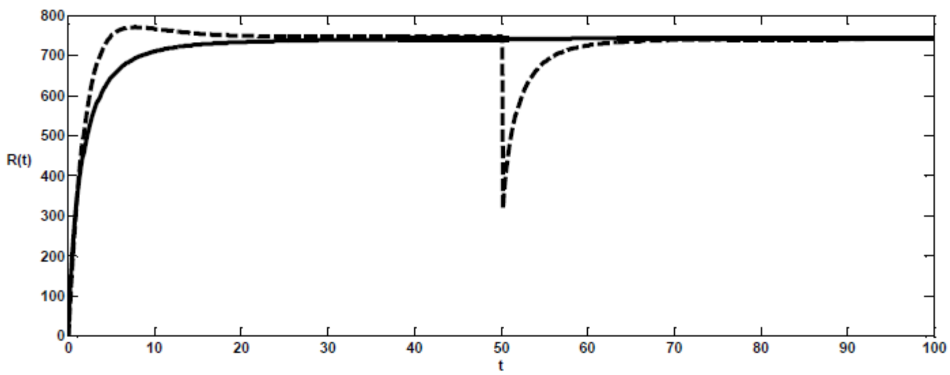

FIGURE 9. The dynamic trajectory $R(t)$ at $\alpha(t) = 0.8$ (the solid line) and at $\alpha(t) = \begin{cases} 1, & t = [0, 50] \\ 0.8, & t = (50, 100]. \end{cases}$ (the dashed line).

The numerical results displayed in Figs. 1-9 for $\langle k \rangle = 6, \lambda = 1, \beta = 0.2, \mu = 0.001, \gamma = 0.1, \alpha = 0.00087, \varepsilon = 0.5, \delta = 0.3$ and initial conditions are $S(0) = 450, I(0) = 550, R(0) = 0$ we found that a disease equilibrium point $E_0 = (251.87, 0, 748.13)$ is locally asymptotically stable , the disease will eventually disappear where $R_0 = 0.6022 < 1$.

In Figs. 10-18 we take $\langle k \rangle = 6, \lambda = 1, \beta = 0.2, \mu = 0.001, \gamma = 0.1, \alpha = 0.00087, \varepsilon = 0.5, \delta = 0.005$ and initial conditions are $S(0) = 800, I(0) = 200, R(0) = 0$ we found that a disease free equilibrium point $E_0 = (952.83, 0, 47.1698)$ is unstable where $R_0 = 2.2783 > 1$ and a unique endemic equilibrium point $E_1 = (386.518,\ 87.1414, 450.528)$ is locally asymptotically stable.

In Figs. 1, 4, 7, 10, 13, 16 we take the variable order $\alpha_i(t) = 1 - 0.004t$ means the memory in the system is a decreasing function so the system behavior is slower with time. In Figs. 2, 5, 8,11,14,17 we take the variable order $\alpha_i(t) = 0.7 - 0.01\sin(\pi t)$ means the memory in the system is a periodic function so the resulted system behavior is a periodic. In Figs. 3, 6, 9, 12, 15, 18 we present the variable order as stated in (2) means the memory has a jump so the behavior of the system also has a jump.

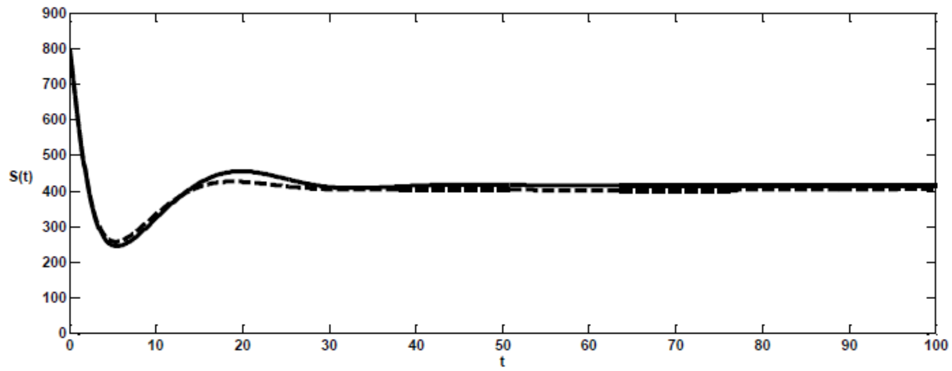

FIGURE 10. The dynamic trajectory $S(t)$ at $\alpha(t) = 1$ (the solid line) and at $\alpha(t) = 1 - 0.004t$ (the dashed line).

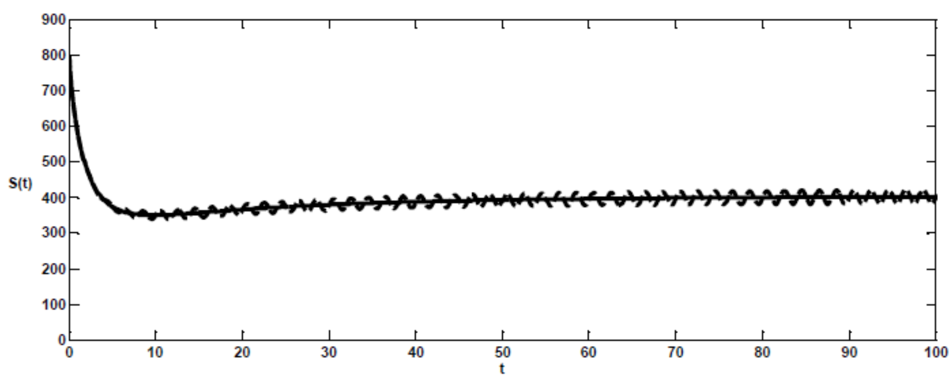

FIGURE 11. The dynamic trajectory $S(t)$ at $\alpha(t) = 0.7$ (the solid line) and at $\alpha(t) = 0.7 - 0.01sin(\pi t)$ (the dashed line).

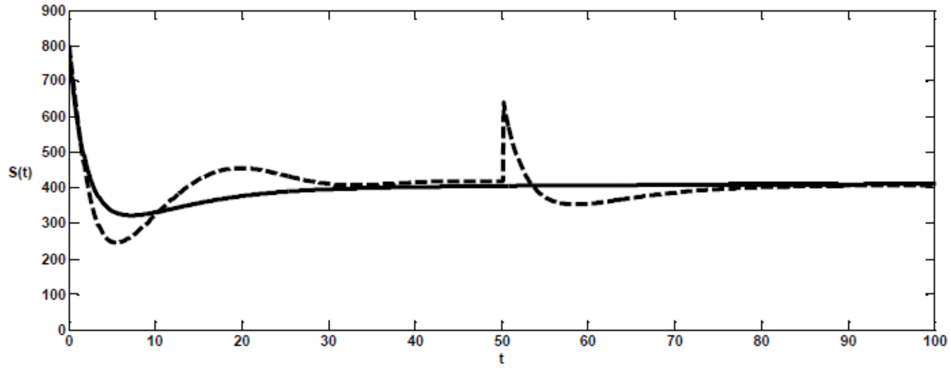

FIGURE 12. The dynamic trajectory $S(t)$ at $\alpha(t) = 0.8$ (the solid line) and at $\alpha(t) = \begin{cases} 1, & t = [0, 50] \\ 0.8, & t = (50, 100]. \end{cases}$ (the dashed line).

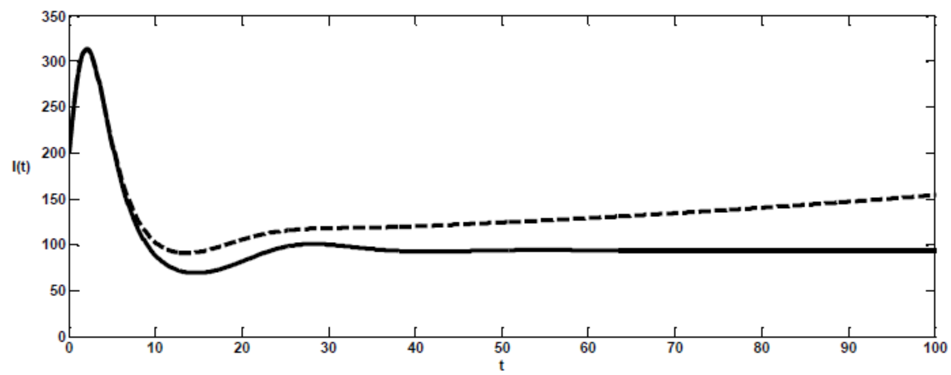

FIGURE 13. The dynamic trajectory $I(t)$ at $\alpha(t) = 1$ (the solid line) and at $\alpha(t) = 1 - 0.004t$ (the dashed line).

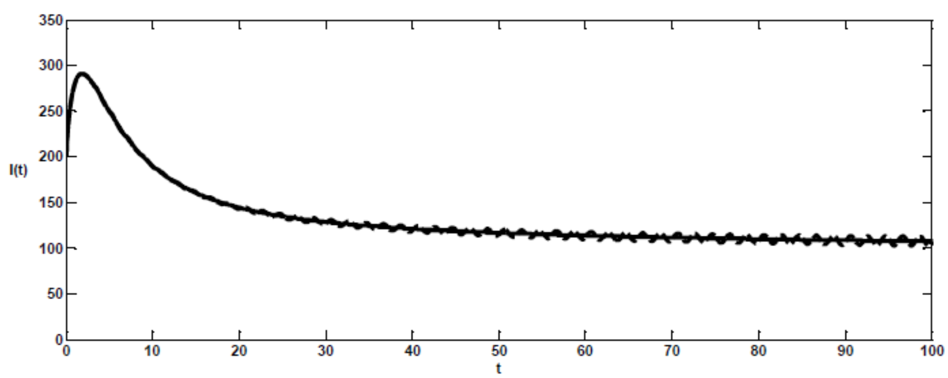

FIGURE 14. The dynamic trajectory $I(t)$ at $\alpha(t) = 0.7$ (the solid line) and at $\alpha(t) = 0.7 - 0.01sin(\pi t)$ (the dashed line).

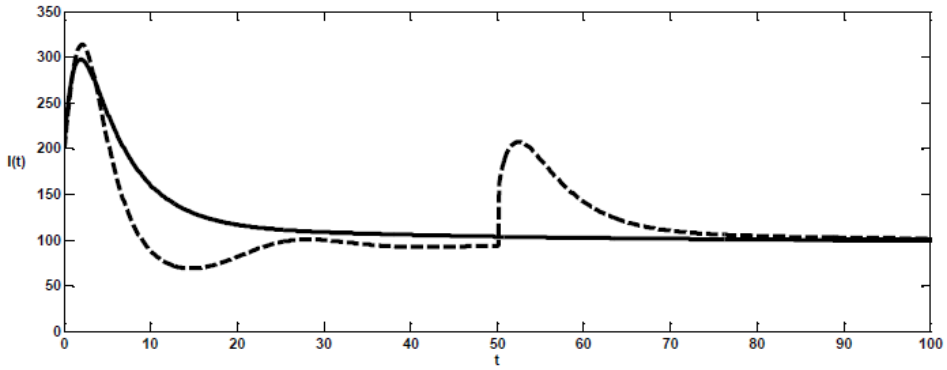

FIGURE 15. The dynamic trajectory $I(t)$ at $\alpha(t) = 0.8$ (the solid line) and at $\alpha(t) = \begin{cases} 1, & t = [0, 50] \\ 0.8, & t = (50, 100]. \end{cases}$ (the dashed line).

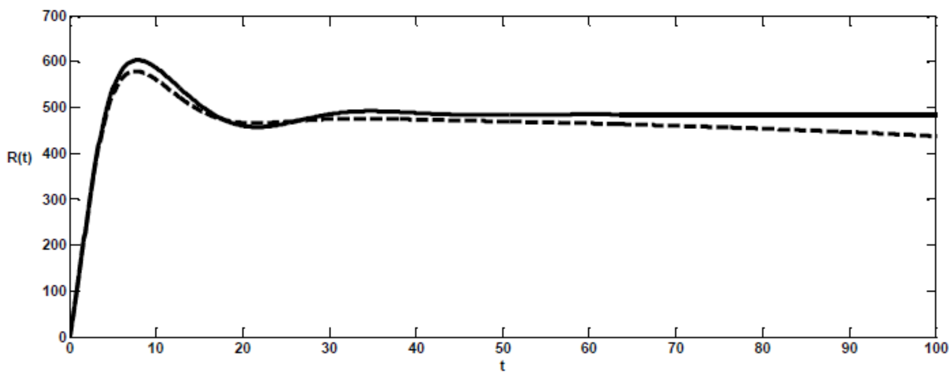

FIGURE 16. The dynamic trajectory $R(t)$ at $\alpha(t) = 1$ (the solid line) and at $\alpha(t) = 1 - 0.004t$ (the dashed line).

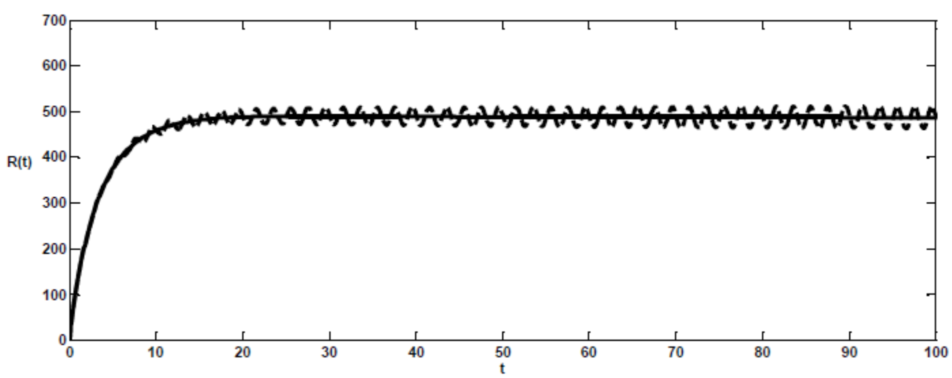

FIGURE 17. The dynamic trajectory $R(t)$ at $\alpha(t) = 0.7$ (the solid line) and at $\alpha(t) = 0.7 - 0.01sin(\pi t)$ (the dashed line).

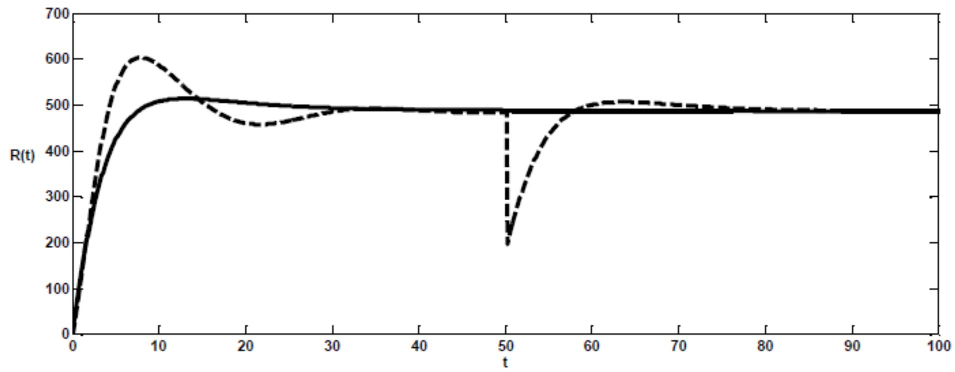

FIGURE 18. The dynamic trajectory $R(t)$ at $\alpha(t) = 0.8$ (the solid line) and at $\alpha(t) = \begin{cases} 1, & t = [0, 50] \\ 0.8, & t = (50, 100]. \end{cases}$ (the dashed line).

6. Conclusion

This paper introduced a variable fractional order network model of Zika. It is presented the equilibrium points and stability of the system. We used the numerical results to show that according to the formula of the memory of the system such as a decreasing function, periodic or discontinuous the behavior of the system has the same properties of the memory function i.e. since the memory function is a decreasing function so the system behavior is slower in time. And if the memory function is a periodic function hence the behavior of the system is also periodic. In addition, the system behavior has a jump when the memory function is discontinues function.

References

[1] R. L. J. S. Allen, “An Introduction to Mathematical Biology”: New Jersey, Pearson Prentice Hall, Upper Saddle River, 2006.

[2] A. A. M. Arafa, S.Z. Rida, and M. Khalil,” The effect of anti-viral drug treatment of human immunodeficiency”, Appl. Math. Model., vol.37, no.4, pp. 2189–2196, 2013.

[3] A. Atangana, and A.H. Cloot, “Stability and convergence of the space fractional variable-order Schrodinger equation”, Adv. Differ. Equ., vol.2013, no.80, 2013.

[4] A. H. Bhrawy, and M. A. Zaky, “Numerical simulation for two- dimensional variable-order fractional nonlinear cable equation”, Nonlinear Dyn., vol.80, no. (1-2), pp.101-116, 2015.

[5] A. V. Chechkin, R. Gorenflo, and I.M. Sokolov, ”Fractional diffusion in inhomogeneous media”, J. Phys. A: Math. Gen., vol.38, no. 42, 2005.

[6] A. V. Chechkin, V. Yu. Gonchar, R. Gorenflo, N. Korabel, and I.M. Sokolov, “Generalized fractional diffusion equations for accelerating sub diffusion and truncated Levy flights”, Phys. Rev., vol. 78, no.2, 2008.

[7] G. Diaz, and C.F.M Coimbra, “Nonlinear dynamics and control of a variable order oscillator with application to the van del Pol equation”, Nonlinear Dyn., vol. 56,no.(1-2),pp.145-157,2009.

[8] K. Diethelm, “Analysis of Fractional Differential Equation: An Application-Oriented Exposition Using Differential Operators of Caputo Type”, Springer-Verlag, Berlin, Heidelberg, 2010.

[9] H. El saka, E. Ahmed,” a fractional order network model of Zika” ,BioRxiv,2016.

[10] H. A. El-Saka and A. El-Sayed, “Fractional Order Equations and Dynamical Systems”, Lambert Academic Publishing, Germany, 2013.

[11] H. A. El-Saka, “The fractional-order SIR and SIRS epidemic models with variable population size”, Math. Sci. Lett., vol.2, No. 3, pp. 1-6, 2013.

[12] S. Funk, A. J. Kucharski, A. Camacho, R. M. Eggo, L. Yakob, L. M. Murray, and W. John Edmunds,” Comparative analysis of dengue and Zika outbreaks reveals differences by setting and virus”, PLOS Negl.Trop. Dis., 2016.

[13] K. Gao, and D.Y. Hua,” Effects of immunity on global oscillations in epidemic spreading in small-world networks”, Physics Procedia, vol.3, no.5, pp.1801–1809, 2010.

[14] D. Gao, Y. Lou, D. He, T. C. Porco, Y. Kuang, G. Chowell, and S. Ruan,” Prevention and control of Zika as a mosquito-borne and sexually transmitted disease: A mathematical modeling analysis”, Scientific Reports, vol.6, 2016.

[15] W. G. Glockle, T. F. Nonnenmacher,”A fractional calculus approach of self-similar protein dynamics”, Biophys. J., vol.68, no.1, pp. 46–53, 1995.

[16] H. W. Hethcote,” The mathematics of infectious diseases”. SIAM Rev., vol. 42, no.4, pp.599–653, 2000.

[17] Y. Hua, Lequan Minab and Y. Kuang, “Modeling the dynamics of epidemic spreading on homogenous and heterogeneous networks”, Applicable Analysis: An International Journal, vol. 94, no.11, pp.1308-2330, 2015.

[18] R. Isea, and E. Karl Lonngren, “A Preliminary Mathematical Model for the Dynamic Transmission of Dengue, Chikungunya and Zika”, American Journal of Modern Physics and Application, Vol. 3, No. 2, pp.11-15, 2016.

[19] V. Isham, S. Harden, and M. Nekovee, “Stochastic epidemics and rumors on finite random networks,” Physica A, vol. 389, no. 3, pp. 561–576, 2010.

[20] M. J. Keeling, and P. Rohani,” Modeling Infectious Diseases in Humans and Animals”, NJ, Princeton University Press, 2008.

[21] M. Khalid, and Fareeha Sami Khan, “Stability Analysis of Deterministic Mathematical Model for Zika Virus”, British Journal of Mathematics & Computer Science, vol.19, no.4, pp.1-10, 2016.

[22] G. C. Korobeinikov Wake, ”Lyapunov functions and global stability for SIR, SIRS, and SIS epidemiological models”, Appl. Math. Lett., vol.15, no.8, pp. 955–960, 2002.

[23] A. j. Kucharski, S. Funk, R.M. Eggo, H.P. Mallet, W. J. Edmunds, E. J. Nilles, “Transmission dynamics of Zika virus in island populations: a modeling analysis of the 2013-14 French Polynesia outbreak”, PLOS Negl. Trop. Dis., vol. 10, no. 5, pp. e0004726, 2016.

[24] C. Li, and W. Deng, ”Remarks on fractional derivatives”, Appl. Math. Comput., vol.187, no.2, pp.777-784, 2007.

[25] C. F. Lorenzo, and T.T. Hartley, ”Variable order and distributed order fractional operators”, Nonlinear Dyn., vol.29, no.1, pp.57-98, 2002.

[26] D. Matignon, “Stability results for fractional differential equations with applications to control processing”, In: IEEE-SMC Proceedings, Computational Engineering in Systems Applications Multi-Conference ,IMACS , vol. 2, pp. 963–968,1996.

[27] C. A. Monje, Y. Q. Chen , B. M. Vinagre, D. Xue, and V. Feliu , "Fractional-order Systems and Controls": Fundamentals and Applications. Springer-Verlag, London, 2010.

[28] Z. Odibat, and N. Shawagfeh, "Generalized Taylor's formula", Appl. Math. Comput., vol.186, no.1, pp.286-293, 2007.

[29] M. D. Ortigueira, F.J. V. Coito and J.J. Trujillo," Discrete-time differential systems. Signal Processing", vol. 107, no.22, pp.198–217, 2015.

[30] L. M. Petrovic, D.T. Spasic, and T. M. Atanackovic, "On a mathematical model of a human root dentin", Dental Materials, vol. 21, no.2, pp. 125-128, 2005.

[31] C. M. A. Pinto and J. A. T. Machado,"Fractional model for malaria transmission under control strategies", Computers and Mathematics with Applications, vol. 66, no.5,pp. 908–916, 2013.

[32] I. Podlubny, "Fractional Differential Equations", Academic Press, San Diego. California, USA, 1999.

[33] S. G. Samko, "Fractional integration and differentiation of variable order", Analysis Mathematics, vol. 21, no.3, p.213-236, 1995.

[34] S. Samko, "Fractional integration and differentiation of variable order: an overview", Nonlinear Dyn., vol.71, no.4, pp.653-662, 2013.

[35] S. G. Samko, and B. Ross,"Integration and differentiation to a variable fractional order", Integral Transform. Spec. Funct., vol.1, no.4, pp. 277-300, 1993.

[36] H. Sheng, H. G. Sun, C. Coopmans, Y. Q. Chen, and G. W. Bohannan ,"A Physical experimental study of variable-order fractional integrator and differentiator", Eur. Phys. J. Spec. Top.,vol.193,no.1,pp.93-104 ,2011.

[37] Li. Shujuan , D.H. Gouge, K. Walker, Al Fournier, Shaku Nair, M. Wierda, and J. Hurley, "The Zika Virus", University of Arizona Cooperative Extension, Arizona pest management center,2016.

[38] D. Sierociuk, I. Podlubny, and I. Petras," Experimental evidence of variable-order behavior of ladders and nested ladders", IEEE Trans. Control Syst. Technol., vol. 21, no.2, pp.459-466, 2013.

[39] W. Smit, and H. de Vries, "Rheological models containing fractional derivatives", Rheol Acta,vol.9,no.4,pp.525-534,1970.

[40] H. G. Sun, W. Chen, H. Wei, and Y. Q. Chen, "A comparative study of constant-order and variable-order fractional models in characterizing memory property of systems", Eur. Phys. J. Special Topics,vol.193,no.1,pp.185-192,2011.

[41] N. H. Sweilam, S.M.AL-Mekhla?, "Numerical study for multi-strain tuberculosis (TB) model of variable-order fractional derivatives", J.Adv. Res., vol.7, no.2, pp.271-283, 2015.

[42] D. Tavares, R. Almeida, D. F. M. Torres, "Caputo derivatives of fractional variable order: numerical approximations". Commun. Nonlinear. Sci. Numer. Simul., vol.35, pp.69-87, 2016.

[43] Yi. Xiao, Yicang Zhou, Sanyi Tang," Modeling disease spread in dispersal networks at two levels". Math Med Biol, vol.28, no.3,pp.22-244,2011.

[44] M. Xu, J. Yang, D. Zhao and H. Zhao, "An image-enhancement method based on variable-order fractional differential operators", Bio-Medical Materials and Engineering, vol. 26, no.s1, pp.S1325-S1333, 2015.

[45] Xu. Yufeng and He. Zhimin,"Existence and uniqueness results for Cauchy problem of variable-order fractional differential equations", J. Appl. Math. Comput vol.43, no.(1-2), pp.295-306, 2013.

[46] G. Zhu, X. Fu, and G. Chen," Spreading dynamics and global stability of a generalized epidemic model on complex heterogeneous networks", Appl. Math. Model., vol.36, no.12, pp.5808-5817, 2012.

[47] Center of disease control and prevention: https://www.cdc.gov/zika/transmission/index.html

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