COEFFICIENTS ESTIMATE FOR CERTAIN SUBCLASSES OF BI-UNIVALENT FUNCTIONS ASSOCIATED WITH QUASI-SUBORDINATION

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Abstract. In this paper we introduce and investigate certain new subclasses of the function class Σ of bi-univalent function defined in the open unit disk, which are associated with the quasi-subordination. We find estimates on the Taylor-Maclaurin coefficient $a_2$ and $a_3$ for functions in these subclasses. Several known and new consequences of these results are also pointed out.

1. Introduction and definitions

Let $A$ denote the class of analytic functions in the open unit disk $U = \{ z : |z| < 1 \}$ that have the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in U),$$

(1.1)

and let $S$ be the class of all functions from $A$ which are univalent in $U$. The Koebe one quarter theorem [5] states that the image of $U$ under every function $f$ from $S$ contains a disk of radius $\frac{1}{4}$. Thus such univalent function has an inverse $f^{-1}$ which satisfies $f^{-1}(f(z)) = z$, $(z \in U)$ and $f(f^{-1}(w)) = w$, $(|w| < r_0(f), r_0(f) \geq \frac{1}{4})$. In fact the inverse function $f^{-1}$ is given by

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^2 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots.$$ (1.2)

A function $f \in A$ is said to be bi-univalent in $U$ if both $f$ and $f^{-1}$ are univalent in $U$. Let $\Sigma$ denotes the class of bi-univalent functions defined in the unit disc $U$.

Ma - Minda [9] introduce the following classes by means of subordination:

$$\mathcal{S}^*(h) = \{ f \in A : \frac{zf'(z)}{f(z)} < h(z) \},$$

where $h$ is an analytic function with positive real part on $U$ with $h(0) = 1, h(0)' > 0$ which maps the unit disc $U$ onto a region starlike with respect to 1 and which is...
symmetric with respect to real axis. A function \( f \in S^*(h) \) is called Ma- Minda starlike. \( C(h) \) is the class of convex function \( f \in A \) for which

\[
1 + \frac{zf''(z)}{f'(z)} < h(z).
\]

The classes \( S^*(h) \) and \( C(h) \) include several well-known subclasses of starlike and convex function as special case. The concept of subordination is generalized in 1970 by Robertson [18] through introducing a new concept of quasi-subordination.

For two analytic functions \( f \) and \( h \), the function \( f \) is quasi subordination to \( h \) written as

\[
f(z) \prec_q h(z) \quad (z \in \mathbb{U}) \tag{1.3}
\]

if there exist analytic functions \( \phi \) and \( \omega \), with \( |\phi(z)| \leq 1, \omega(0) = 0 \) and \( |\omega(z)| < 1 \) such that

\[
\frac{f(z)}{\phi(z)} \prec h(z),
\]

which is equivalent to

\[
f(z) = \phi(z)h(\omega(z)) \quad (z \in \mathbb{U}).
\]

Observe that if \( \phi(z) = 1 \), then \( f(z) = h(\omega(z)) \), so that \( f(z) \prec h(z) \) in \( \mathbb{U} \), also if \( \omega(z) = z \), then \( f(z) = \phi(z)h(z) \) and it is said that \( f(z) \) is majorized by \( h(z) \) and written as \( f(z) \preceq h(z) \) in \( \mathbb{U} \). Hence it is obvious that the quasi-subordination is a generalization of the usual subordination as well as majorization. The work on quasi - subordination is quite extensive which includes some recent investigations [2,7,8,10,12,17,18].

In 1967, Lewin [8] investigated the class \( \Sigma \) of bi-univalent functions and obtained the bound for the second coefficient \( a_2 \). Brannan and Taha [3] considered certain subclasses of bi-univalent functions similar to the familiar subclasses of univalent functions consisting of starlike, strongly starlike and convex functions. They introduced the bi-starlike function, bi-convex function classes and obtained non sharp estimates on the first two Taylor-Maclaurin coefficients \( |a_2| \) and \( |a_3| \). Recently Ali et al. [1], Deniz [4], Tang et al. [19], Peng et al. [14] Ramachandran et al. [16], Mugurugusundaramoorthy et al. [11] etc. have introduced and investigated Ma-Minda type subclasses of bi-univalent functions class \( \Sigma \). Further generalization of Ma-Minda type subclasses of class \( \Sigma \) have been made several authors including ( [6], [13], [10], [20] ) by means of quasi - subordination. Motivated by work in [7, 12] on quasi-subordination, we introduce and study here certain new subclasses of class \( \Sigma \).

Throughout this paper it is assumed that \( h(z) \) is analytic in \( \mathbb{U} \) with \( h(0) = 1 \) and let

\[
\phi(z) = A_0 + A_1z + A_2z^2 + \cdots \quad (|\phi(z)| \leq 1, z \in \mathbb{U}) \tag{1.3}
\]

and

\[
h(z) = 1 + B_1z + B_2z^2 + \cdots \quad (B_1 \in \mathbb{R}^+). \tag{1.4}
\]

**Definition 1.1.** For \( 0 \leq \lambda \leq 1 \) and \( \gamma \in \mathbb{C}\setminus\{0\} \), a function \( f \in \Sigma \) is said to be in the class \( S^*_\Sigma(\lambda, \gamma, h) \), if the following two conditions are satisfied:

\[
\frac{1}{\gamma} \left( \frac{zf'(z)}{(1-\lambda)z + Af(z)} - 1 \right) \prec_q (h(z) - 1) \tag{1.5}
\]
and
\[
\frac{1}{\gamma} \left( \frac{w g'(w)}{(1-\lambda)w + \lambda g(w)} - 1 \right) \prec_q (h(w) - 1),
\]
where \( g = f^{-1} \) and \( h \) is given by (1.5) and \( z, w \in \mathbb{U} \).

It follows that a function \( f \) is in the class \( S_2^h(\lambda, \gamma, h) \) if and only if there exists an analytic function \( \phi \) with \( |\phi(z)| \leq 1, (z \in \mathbb{U}) \) such that
\[
\frac{1}{\gamma} \left( \frac{z f'(z) + z^2 f''(z)}{(1-\lambda)z + \lambda z f'(z)} - 1 \right) \prec_q (h(z) - 1),
\]
and
\[
\frac{1}{\gamma} \left( \frac{w g'(w) + w^2 g''(w)}{(1-\lambda)w + \lambda w g'(w)} - 1 \right) \prec_q (h(w) - 1),
\]
where \( g = f^{-1} \) and \( h \) is given by (1.5) and \( z, w \in \mathbb{U} \).

**Definition 1.2.** For \( 0 \leq \lambda \leq 1 \) and \( \gamma \in \mathbb{C}\setminus\{0\} \), a function \( f \in \Sigma \) is said to be in the class \( K_2^h(\lambda, \gamma, h) \), if the following two conditions are satisfied:
\[
\frac{1}{\gamma} \left( z f'(z) + z^2 f''(z) \right) \prec_q (h(z) - 1),
\]
and
\[
\frac{1}{\gamma} \left( w g'(w) + w^2 g''(w) \right) \prec_q (h(w) - 1),
\]
where \( g = f^{-1} \) and \( h \) is given by (1.5) and \( z, w \in \mathbb{U} \).

In the present paper, we find estimates on the Taylor-MacLaurin coefficients \( |a_n| \) and \( |a_3| \) for function \( f \) belonging in the classes \( S_2^h(\lambda, \gamma, h) \) and \( K_2^h(\lambda, \gamma, h) \). Several known and new consequences of these results are also pointed out.

In order to derive our main results, we have to recall here the following well-known Lemma:

**Lemma 1.3.**[15] Let \( p \in \mathcal{P} \) be family of all functions \( p \) analytic in \( \mathbb{U} \) for which \( \Re\{p(z)\} > 0 \) and have the form \( p(z) = 1 + p_1 z + p_2 z^2 + \ldots \) for \( z \in \mathbb{U} \), then \( |p_n| \leq 2 \) for each \( n \).

2. Coefficient bounds for the function class \( S_2^h(\lambda, \gamma, h) \)

**Theorem 2.1.** Let \( 0 \leq \lambda \leq 1 \) and \( \gamma \in \mathbb{C}\setminus\{0\} \). If \( f \in \mathcal{A} \) of the form (1.1) belonging to the class \( S_2^h(\lambda, \gamma, h) \), then
\[
|a_2| \leq \min \left\{ \frac{B_1|\gamma||A_0|}{(2-\lambda)}, \sqrt{\frac{(B_1 + |B_2 - B_1||\gamma||A_0|)}{\lambda^2 - 3\lambda + 3}} \right\}
\]
and
\[
|a_3| \leq \min \left\{ \frac{|\gamma|}{\lambda^2 - 3\lambda + 3} (B_1 + |B_2 - B_1||A_0| + \frac{|\gamma|}{(3-\lambda)} |A_1| B_1), \frac{|\gamma|\lambda B_1^2}{2 - \lambda} |A_0|^2 + (B_1 + |B_2 - B_1||A_0| + B_1 |A_1|) \right\}.
\]
Proof. Let \( f \in S_1^1(\lambda, \gamma, h) \). In view of Definition 1.1, there exist then Schwarz functions \( r(z) \), \( s(z) \) and an analytic function \( \phi(z) \) such that

\[
\frac{1}{\gamma} \left( \frac{zf'(z)}{(1 - \lambda)z + \lambda f(z)} - 1 \right) = \phi(z)(h(r(z)) - 1) \tag{2.3}
\]

and

\[
\frac{1}{\gamma} \left( \frac{wg'(w)}{(1 - \lambda)w + \lambda g(w)} - 1 \right) = \phi(w)(h(s(w)) - 1). \tag{2.4}
\]

Define the functions \( p(z) \) and \( q(z) \) by

\[
p(z) = \frac{1 + r(z)}{1 - r(z)} = 1 + c_1z + c_2z^2 + \cdots \tag{2.5}
\]

and

\[
q(z) = \frac{1 + s(z)}{1 - s(z)} = 1 + d_1z + d_2z^2 + \cdots, \tag{2.6}
\]

which are equivalently

\[
r(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{1}{2} \left[ c_1z + \left( c_2 - \frac{c_1^2}{2} \right) z^2 + \cdots \right] \tag{2.7}
\]

and

\[
s(z) = \frac{q(z) - 1}{q(z) + 1} = \frac{1}{2} \left[ d_1z + \left( d_2 - \frac{d_1^2}{2} \right) z^2 + \cdots \right]. \tag{2.8}
\]

It is clear that \( p(z), q(z) \) are analytic and have positive real parts in \( \mathbb{U} \). In view of (2.3), (2.4), (2.7) and (2.8), clearly

\[
\frac{1}{\gamma} \left( \frac{zf'(z)}{(1 - \lambda)z + \lambda f(z)} - 1 \right) = \phi(z) \left[ h \left( \frac{p(z) - 1}{p(z) + 1} \right) - 1 \right] \tag{2.9}
\]

and

\[
\frac{1}{\gamma} \left( \frac{wg'(w)}{(1 - \lambda)w + \lambda g(w)} - 1 \right) = \phi(w) \left[ h \left( \frac{q(w) - 1}{q(w) + 1} \right) - 1 \right]. \tag{2.10}
\]

The series expansions for \( f(z) \) and \( g(w) \) as given in (1.1) and (1.2) respectively, provide us

\[
\frac{1}{\gamma} \left( \frac{zf'(z)}{(1 - \lambda)z + \lambda f(z)} - 1 \right) = \frac{1}{\gamma} \left[ (2 - \lambda)a_2z + [(3 - \lambda)a_3 - \lambda(2 - \lambda)a_2^2]z^2 + \cdots \right] \tag{2.11}
\]

and

\[
\frac{1}{\gamma} \left( \frac{wg'(w)}{(1 - \lambda)w + \lambda g(w)} - 1 \right) = \frac{1}{\gamma} \left[ (\lambda - 2)a_2w + [(3 - \lambda)(2a_2^2 - a_3) - \lambda(2 - \lambda)a_2^2]w^2 + \cdots \right]. \tag{2.12}
\]

Using (2.5) and (2.6) together with (1.4) and (1.5)

\[
\phi(z) \left[ h \left( \frac{p(z) - 1}{p(z) + 1} \right) - 1 \right] = \frac{1}{2} A_0B_1c_1z + \left[ \frac{1}{2} A_1B_1c_1 + \frac{1}{2} A_0B_1 \left( c_2 - \frac{c_1^2}{2} \right) \right] + \frac{A_0B_2c_1^2}{4} \left[ z^2 + \cdots \right] \tag{2.13}
\]

and

\[
\phi(w) \left[ h \left( \frac{q(w) - 1}{q(w) + 1} \right) - 1 \right] = \frac{1}{2} A_0B_1d_1z + \left[ \frac{1}{2} A_1B_1d_1 + \frac{1}{2} A_0B_1 \left( d_2 - \frac{d_1^2}{2} \right) \right] + \frac{A_0B_2d_1^2}{4} \left[ z^2 + \cdots \right] \tag{2.14}
\]
Further, from (2.15) and (2.18), we deduce that
\[ \frac{2 - \lambda}{\gamma}a_2 = \frac{1}{2}A_0B_1c_1 \] (2.15)
and
\[ \frac{(3 - \lambda)a_3 - \lambda(2 - \lambda)a_3}{\gamma} = \frac{1}{2}A_1B_1c_1 + \frac{1}{2}A_0B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{A_0B_2c_1^2}{4}. \] (2.16)

Similarly (2.10) gives us
\[ \frac{2 - \lambda}{\gamma}a_2 = \frac{1}{2}A_0B_1d_1 \] (2.17)
and
\[ \frac{(3 - \lambda)(2a_2^2 - a_3) - \lambda(2 - \lambda)a_3^2}{\gamma} = \frac{1}{2}A_1B_1d_1 + \frac{1}{2}A_0B_1\left(d_2 - \frac{d_1^2}{2}\right) + \frac{A_0B_2d_1^2}{4}. \] (2.18)

From (2.15) and (2.17), we find that
\[ a_2 = \frac{A_0B_1c_1\gamma}{2(2 - \lambda)} = -\frac{A_0B_1d_1\gamma}{2(2 - \lambda)} \] (2.19)
which implies
\[ |a_2| \leq \frac{|A_0\gamma|B_1}{2 - \lambda}. \] (2.20)

Adding (2.16) and (2.18), we obtain
\[ \frac{2(\lambda^2 - 3\lambda + 3)}{\gamma}a_2^2 = \frac{A_0B_1}{2}(c_2 + d_2) + \frac{A_0(B_2 - B_1)}{4}\left(c_1^2 + d_1^2\right), \] (2.21)
which implies
\[ |a_2|^2 \leq \frac{|A_0\gamma|(B_1 + |B_2 - B_1|)}{\lambda^2 - 3\lambda + 3}, \] (2.22)
hence, using (2.20) and (2.22) we get the bounds on $|a_2|$ as asserted in (2.1).

Next, in order to find the upper bound for $|a_3|$, by subtracting (2.18) from (2.16), we get
\[ \frac{2(3 - \lambda)}{\gamma}a_3 = \frac{2(3 - \lambda)}{\gamma}a_2^2 + \frac{A_1B_1}{2}(c_1 - d_1) + \frac{A_0B_1}{2}(c_2 - d_2), \] (2.23)
by using Lemma 1.2 and (2.21) in (2.23), we obtain
\[ |a_3| \leq \left[ \frac{|A_0B_1|}{\lambda^2 - 3\lambda + 3} + \frac{|A_0(B_2 - B_1)|}{\lambda^2 - 3\lambda + 3} + \frac{|A_1B_1|}{3 - \lambda} \right]|\gamma|. \] (2.24)

Next, from (2.15) and (2.16), we have
\[ \frac{(3 - \lambda)a_3}{\gamma} = \frac{\lambda\gamma A_0^2B_1^2c_1^2}{4(2 - \lambda)} + \frac{1}{2}A_1B_1c_1 + \frac{1}{2}A_0B_1c_2 + \frac{1}{4}A_0(B_2 - B_1)c_1^2, \]
which implies
\[ |a_3| \leq \frac{|\gamma|}{3 - \lambda}\left[ B_1\left(\frac{\lambda}{2 - \lambda}\right)|A_0|^3|B_1| + |A_1| + |A_0| + |A_0(B_2 - B_1)| \right]. \] (2.25)

Further, from (2.15) and (2.18), we deduce that
\[ |a_3| \leq \frac{|\gamma|}{3 - \lambda}\left[ B_1\left(\frac{\lambda^2 - 4\lambda + 6}{(2 - \lambda)^2}\right)|A_0|^2|B_1| + |A_1| + |A_0| + |A_0(B_2 - B_1)| \right] \] (2.26)
and thus we obtain the conclusion (2.2) of our theorem.

**Remarks 2.2.**
(i) For $\lambda = 1$, Theorem 2.1 provides improvement over the estimates obtained in [10], Corollary 9, p. 5.
(ii) For $\lambda = \gamma = 1$, Theorem 2.1 reduces to a result in [13], Theorem 3.2, p. 8.
(iii) For $\lambda = 0, \gamma = 1$, Theorem 2.1 reduces to a result in [13], Corollary 2.4, p. 8.

For $\phi(z) \equiv 1$, the above theorem reduces to following corollary:

**Corollary 2.3.** For $0 \leq \lambda \leq 1$ and $\gamma \in \mathbb{C} \setminus \{0\}$, if $f \in A$ of the form (1.1) satisfy the following subordination:

$$1 + \frac{1}{\gamma} \left( \frac{zf'(z)}{(1 - \lambda)z + \lambda f(z)} - 1 \right) < h(z) \quad (2.27)$$

and

$$1 + \frac{1}{\gamma} \left( \frac{wg'(w)}{(1 - \lambda)w + \lambda g(w)} - 1 \right) < h(w), \quad (2.28)$$

where $g = f^{-1}$ and $h$ is given by (1.5) and $z, w \in \mathbb{U}$, then

$$|a_2| \leq \min \left\{ \frac{B_1|\gamma|}{|\lambda|^{\frac{1}{2}} \sqrt{|B_1 + |B_2 - B_1||\gamma|}} \right\} \quad (2.29)$$

and

$$|a_3| \leq \min \left\{ \frac{|\gamma|}{\lambda^2 - 3\lambda + 3} (B_1 + |B_2 - B_1|), \quad \frac{|\gamma|}{|3 - \lambda|} (\frac{|\gamma|}{2 - \lambda} B_1^2 + B_1 + |B_2 - B_1|) \right\}. \quad (2.30)$$

For $\lambda = \gamma = 1$, Corollary 2.4 gives the coefficient estimates for Ma-Minda bi-starlike functions. **Remark 2.4.** For $\lambda = 0$ and $\gamma = 1$ Corollary 2.4 reduces to a result in [1, Theorem 2.1, p. 345].

### 3. Coefficient bounds for the function class $K^q_\lambda(\lambda, \gamma, h)$

**Theorem 3.1.** Let $0 \leq \lambda \leq 1$ and $\gamma \in \mathbb{C} \setminus \{0\}$. If $f \in A$ of the form (1.1) belonging to the class $K^q_\lambda(\lambda, \gamma, h)$, then

$$|a_2| \leq \min \left\{ \frac{B_1|\gamma||A_0|}{2|2 - \lambda| \sqrt{|B_1 + |B_2 - B_1||\gamma||A_0|}}} \right\} \quad (3.31)$$

and

$$|a_3| \leq \min \left\{ \frac{|\gamma|}{4\lambda^2 - 11\lambda + 9} (B_1 + |B_2 - B_1||A_0| + \frac{|\gamma|}{3(3 - \lambda)} A_1|B_1|, \quad \frac{|\gamma|}{3(3 - \lambda)} \left| \frac{\gamma|\lambda B_1^2}{2 - \lambda} |A_0|^2 + (B_1 + |B_2 - B_1||A_0| + B_1|A_1|) \right| \right\}. \quad (3.32)$$

**Proof.** Let $f \in K^q_\lambda(\lambda, \gamma, h)$. In view of Definition 1.2, there exist then Schwarz functions $r(z), s(z)$ and an analytic function $\phi(z)$ such that

$$\frac{1}{\gamma} \left( \frac{zf'(z)}{(1 - \lambda)z + \lambda f(z)} - 1 \right) = \phi(z)(h(z) - 1) \quad (3.33)$$

and

$$\frac{1}{\gamma} \left( \frac{wg'(w)}{(1 - \lambda)w + \lambda g(w)} - 1 \right) = \phi(z)(h(w) - 1), \quad (3.34)$$
where \( r(z) \) and \( s(z) \) are defined by (2.7) and (2.8) respectively. Under the same restrictions for \( p(z), q(z), c_i \) and \( d_i \) as mentioned in Theorem 2.1, obviously we have

\[
\frac{1}{\gamma} \left( z f'(z) + z^2 f''(z) - 1 \right) = \phi(z) \left[ h\left( \frac{p(z) - 1}{p(z) + 1} \right) - 1 \right]
\]

(3.35)

and

\[
\frac{1}{\gamma} \left( w g'(w) + w^2 g''(w) - 1 \right) = \phi(w) \left[ h\left( \frac{q(w) - 1}{q(w) + 1} \right) - 1 \right].
\]

(3.36)

The series expansions for \( f(z) \) and \( g(w) \) as given in (1.1) and (1.2) respectively, provides us

\[
\frac{1}{\gamma} \left( z f'(z) + z^2 f''(z) - 1 \right) = \frac{1}{\gamma} \left[ 2(2 - \lambda) a_2 z + ((3 - \lambda) a_3 - 4\lambda(2 - \lambda)a_2^2) z^2 + ... \right]
\]

(3.37)

and

\[
\frac{1}{\gamma} \left( w g'(w) + w^2 g''(w) - 1 \right) = \frac{1}{\gamma} \left[ -2(2 - \lambda) a_2 w + (3(3 - \lambda)(2a_2^2 - a_3) - 4\lambda(2 - \lambda)a_2^2) w^2 + ... \right].
\]

(3.38)

Now using (2.13) and (3.7) in (3.5) and comparing the coefficients of \( z \) and \( z^2 \), we get

\[
\frac{2(2 - \lambda)}{\gamma} a_2 = \frac{1}{2} A_0 B_1 c_1
\]

(3.39)

and

\[
\frac{1}{\gamma} \left( 3(3 - \lambda)a_3 - 4\lambda(2 - \lambda)a_2^2 \right) = \frac{1}{2} A_1 B_1 c_1 + \frac{1}{2} A_0 B_1 \left( c_2 - \frac{c_1^2}{2} \right) + \frac{A_0 B_2 c_1^2}{4}.
\]

(3.40)

Similarly (2.14), (3.6) and (3.8) yields

\[
\frac{2(2 - \lambda)}{\gamma} a_2 = \frac{1}{2} A_0 B_1 d_1
\]

(3.41)

and

\[
\frac{1}{\gamma} \left( 3(3 - \lambda)(2a_2^2 - a_3) - 4\lambda(2 - \lambda)a_2^2 \right) = \frac{1}{2} A_1 B_1 d_1 + \frac{1}{2} A_0 B_1 \left( d_2 - \frac{d_1^2}{2} \right) + \frac{A_0 B_2 d_1^2}{4}.
\]

(3.42)

From (3.9) and (3.11), we have

\[
a_2 = \frac{\gamma A_0 B_1 c_1}{4(2 - \lambda)} = -\frac{\gamma A_0 B_1 d_1}{4(2 - \lambda)},
\]

(3.43)

further by adding (3.10) and (3.12), we obtain

\[
\frac{2(4\lambda^2 - 11\lambda + 9)}{\gamma} a_2^2 = \frac{A_0 B_1}{2} (c_2 + d_2) + \frac{A_0 (B_2 - B_1)}{4} (c_1^2 + d_1^2).
\]

(3.44)

On using the Lemma 1.3 in (3.13) and (3.14), we can get the desired bounds on \( |a_2| \) as given in (3.1). Next, in order to find the upper bound for \( |a_3| \), by subtracting (3.12) from (3.10) and using (3.14), we get

\[
|a_3| \leq \frac{|\gamma|}{4(2 - \lambda)} \|A_0|B_1 + |A_0(B_2 - B_1)|\| + \frac{|\gamma|}{3(3 - \lambda)} |A_1|B_1.
\]

(3.45)
For another bound on \(|a_3|\), we substitute the value of \(a_2^2\) from (3.9) into (3.10) and use the Lemma 1.3, which gives us

\[
|a_3| \leq \frac{\gamma}{3(3 - \lambda)} |\frac{\gamma}{2 - \lambda} B^2_1 + (B_1 + |B_2 - B_1|)A_0 + B_1A_1|.
\] (3.46)

With the help of (3.9) and (3.12) we obtain one more bound on \(|a_3|\) that is

\[
|a_3| \leq \frac{\gamma}{3(3 - \lambda)} \left|\frac{\gamma}{2(2 - \lambda)^2} B^2_2(2\lambda^2 - 7\lambda + 9) + (B_1 + |B_2 - B_1|)A_0 + B_1A_1\right|.
\] (3.47)

Obviously the RHS of (3.17) is greater than the RHS of (3.16), so the desired bound on \(|a_3|\) is obtained from (3.15) and (3.16). For \(\phi(z) \equiv 1\), the above theorem reduces to following corollary: **Corollary 3.2.** For \(0 \leq \lambda \leq 1\) and \(\gamma \in \mathbb{C}\setminus \{0\}\), if \(f \in \mathcal{A}\) of the form (1.1) satisfy the following subordinations:

\[1 + \frac{1}{\gamma} \left( \frac{zf'(z) + z^2f''(z)}{(1 - \lambda)z + \lambda zf'(z) - 1} \right) \prec (h(z)) \] (3.48)

and

\[1 + \frac{1}{\gamma} \left( \frac{wg'(w) + w^2g''(w)}{(1 - \lambda)w + \lambda wg'(w) - 1} \right) \prec (h(w)), \] (3.49)

where \(g = f^{-1}\) and \(h\) is given by (1.5) and \(z, w \in \mathbb{U}\), then

\[|a_2| \leq \min \left\{ \frac{B_1|\gamma|}{2(2 - \lambda)} \sqrt{\frac{(B_1 + |B_2 - B_1|)|\gamma|}{4\lambda^2 - 11\lambda + 9}} \right\} \] (3.50)

and

\[|a_3| \leq \min \left\{ \frac{|\gamma|}{4\lambda^2 - 11\lambda + 9} (B_1 + |B_2 - B_1|), \right. \]
\[\left. \frac{|\gamma|}{2 - \lambda} \left|\frac{\gamma}{2 - \lambda} B^2_1 + (B_1 + |B_2 - B_1|)\right| \right\}. \] (3.51)

**Remarks 3.3.** (i) For \(\lambda = 1\), Theorem 3.1 provides improvement over the estimates obtained in [10], Corollary 11, p 5.

(ii) For \(\lambda = \gamma = 1\), Theorem 3.1 provides improvement over the estimates obtained in [13], Theorem 3.3, p. 9.

Other interesting corollaries and consequences of Theorem 3.1 could be derived by specializing the parameters.

**References**


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