INEQUALITIES FOR A CLASS OF FUNCTIONS STARLIKE WITH RESPECT TO SYMMETRIC POINTS

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Abstract. The purpose of the present paper is to investigate a subordination theorem, boundedness properties associated with partial sums and an integral mean inequality for a class of functions starlike with respect to symmetric points.

1. Introduction

Let \( S \) denote the class of functions \( f(z) \) normalized by \( f(0) = f'(0) - 1 = 0 \), analytic and univalent in the open unit disk \( U = \{ z; z \in \mathbb{C}; |z| < 1 \} \), then \( f(z) \) can be expressed as:

\[
   f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}
\]

Consider the subclass \( T \) of the class \( S \) consisting of functions of the form

\[
   f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n. \tag{1.2}
\]

If the functions \( g(z) \) and \( h(z) \) belonging to the class \( S \) are, respectively, given by \( g(z) = z + \sum_{n=2}^{\infty} b_n z^n \) and \( h(z) = z + \sum_{n=2}^{\infty} c_n z^n \) then the Hadamard product (or convolution) denoted by \((g \ast h)(z)\) of the two functions \( g(z) \) and \( h(z) \) is defined by

\[
   (g \ast h)(z) = z + \sum_{n=2}^{\infty} b_n c_n z^n = (h \ast g)(z). \tag{1.3}
\]

A domain \( D \subset \mathbb{C} \) is convex if the line segment joining any two points in \( D \) lies entirely in \( D \), while a domain is starlike with respect to a point \( w_0 \in D \) if the line segment joining any point of \( D \) to \( w_0 \) lies inside \( D \). A function \( f \in S \) is starlike if \( f(U) \) is a starlike domain with respect to origin, and convex if \( f(U) \) is convex. Analytically, \( f \in S \) if and only if \( \text{Re} \left( \frac{zf'(z)}{f(z)} \right) > 0 \), whereas \( f \in S \) is convex if and

\[\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > 0.\]
only if \( \text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \). The classes consisting of starlike and convex functions are denoted by \( S^* \) and \( K \) respectively. The classes \( S^*(\alpha) \) and \( K(\alpha) \) of starlike and convex functions of order \( \alpha \), \( 0 \leq \alpha < 1 \), are respectively characterized by \( \text{Re}\left(\frac{zf''(z)}{f'(z)}\right) > \alpha \) and \( \text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \).

Let \( S^*_s \) be the subclass of \( S \) consisting of functions given by (1.1), satisfying

\[
\text{Re}\left\{\frac{zf'(z)}{f(z) - f(-z)}\right\} > 0 \quad (z \in U).
\]  

(1.4)

Function \( f(z) \in S^*_s \) are called starlike with respect to symmetric points and were introduced by Sakaguchi [2]. A subclass \( S^*_s(\alpha, \beta) \) of \( S^*_s \) of functions \( f(z) \), regular and univalent in \( U \) given by (1.1) and satisfying the condition

\[
\left|\frac{zf'(z)}{f(z) - f(-z)} - 1\right| < \beta \left|\frac{\alpha zf'(z)}{f(z) - f(-z)} + 1\right| \quad (z \in U, 0 \leq \alpha \leq 1, 1/2 < \beta \leq 1)
\]  

(1.5)

was introduced in [4]. Further, we let

\[
T S^*_s(\alpha, \beta) = S^*_s(\alpha, \beta) \cap T
\]  

(1.6)

The objective of the present paper is to investigate the integral means inequality, a subordination theorem and partial sums for the class \( S^*_s(\alpha, \beta) \). For this we need the following results:

**Lemma 1.1.** A function of the form (1.1) is in

\[
\sum_{n=2}^{\infty} \psi(n; \alpha, \beta)|a_n| \leq 1,
\]  

(1.7)

where

\[
\psi(n; \alpha, \beta) = \frac{n(1 + \alpha \beta) + (\beta - 1)[1 - (-1)^n]}{\beta(2 + \alpha) - 1} \quad (0 \leq \alpha \leq 1, 1/2 < \beta \leq 1),
\]  

(1.8)

then \( f(z) \in S^*_s(\alpha, \beta) \).

**Lemma 1.2.** A function of the form (1.2) is in \( T S^*_s(\alpha, \beta) \) \((0 \leq \alpha \leq 1, 1/2 < \beta \leq 1)\) if and only if

\[
\sum_{n=2}^{\infty} \psi(n; \alpha, \beta)|a_n| \leq 1,
\]  

(1.9)

where \( \psi(n; \alpha, \beta) \) is given by (1.8).

Lemma 1.1 and Lemma 1.2 were earlier proved by Rosy et al. [4]. From (1.8) it is easy to check that

\[
\psi(n+1; \alpha, \beta) - \psi(n; \alpha, \beta) = \begin{cases} \frac{\alpha \beta + 2 \beta - 1}{\beta(2 + \alpha) - 1}, & n \text{ even} \\ \frac{1 + \alpha \beta + 2 (1 - \beta)}{\beta(2 + \alpha) - 1}, & n \text{ odd} \end{cases}
\]  

(1.10)

which is positive for \( 0 \leq \alpha \leq 1, 1/2 < \beta \leq 1 \). Hence sequence (1.8) is non-decreasing sequence. Again \( \psi(2; \alpha, \beta) = \frac{2(1 + \alpha \beta)}{\beta(2 + \alpha) - 1} \) which is positive for \( 0 \leq \alpha \leq 1, 1/2 < \beta \leq 1 \), hence all the terms of sequence \( \psi(n; \alpha, \beta) \) are positive. Similarly

\[
\psi(n; \alpha, \beta) - n = \begin{cases} \frac{2n(1 - \beta)}{\beta(2 + \alpha) - 1}, & n \text{ even} \\ \frac{2(n - 1)(1 - \beta)}{\beta(2 + \alpha) - 1}, & n \text{ odd} \end{cases}
\]  

(1.11)
which is positive for $0 \leq \alpha \leq 1, 1/2 < \beta \leq 1$. Hence all the terms of the sequence $\langle \psi(n; \alpha, \beta) - n \rangle_{n=2}^{\infty}$ are positive.

2. Integral Means Inequalities

The following subordination result due to Littlewood [1] will be required in our investigation.

Lemma 2.1. If $f(z)$ and $g(z)$ are analytic in $\mathbb{U}$ with $f(z) \prec g(z)$, then

$$\int_0^{2\pi} |f(re^{i\theta})|^\mu \, d\theta \leq \int_0^{2\pi} |g(re^{i\theta})|^\mu \, d\theta,$$

(2.1)

where $\mu > 0$, $z = re^{i\theta}$ ($0 < r < 1$).

Theorem 2.1. Let $\mu > 0$. If $f(z) \in TS^*_\alpha(\alpha, \beta)$ ($0 \leq \alpha \leq 1, 1/2 < \beta \leq 1$) is given by (1.2) then for $z = re^{i\theta}$ ($0 < r < 1$):

$$\int_0^{2\pi} |f(re^{i\theta})|^\mu \, d\theta \leq \int_0^{2\pi} |f_1(re^{i\theta})|^\mu \, d\theta,$$

(2.2)

where

$$f_1(z) = z - \frac{\beta(2 + \alpha) - 1}{2(1 + \alpha \beta)} z^2.$$

(2.3)

The proof of the above theorem is simple so we leave it here.

3. Subordination Theorem

Before stating and proving our subordination theorem, we need the following definition and a lemma due to Wilf [6].

Definition 3.1. If $f, g \in \mathcal{H}$ where $\mathcal{H}$ denote the class of all holomorphic functions, then the function $f$ is said to be subordinate to $g$, written as $f(z) \prec g(z)$ ($z \in \mathbb{U}$), if there exists a Schwarz function $w \in \mathcal{H}$ with $w(0) = 0$ and $|w(z)| < 1$ ($z \in \mathbb{U}$) such that $f(z) = g(w(z))$. In particular, if $g$ is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

Definition 3.2. An infinite sequence $\{b_n\}_{1}^{\infty}$ of complex numbers will be called a subordinating factor sequence if whenever

$$f(z) = \sum_{n=1}^{\infty} a_n z^n$$

(3.1)

is analytic, univalent and convex in $\mathbb{U}$, then

$$\sum_{n=1}^{\infty} a_n b_n z^n \subseteq f(z) \ (z \in \mathbb{U}, a_1 = 0).$$

(3.2)

Lemma 3.1. The sequence $\{b_n\}_{1}^{\infty}$ is a subordinating factor sequence if and only if

$$\Re \left\{ 1 + 2 \sum_{k=1}^{\infty} b_k z^k \right\} > 0 \ (z \in \mathbb{U}).$$

(3.3)
Theorem 3.1. Let \( f(z) \) of the form (1.1) satisfy the coefficient inequality (1.7), then
\[
1 + \frac{\alpha \beta}{1 + 3\alpha \beta + 2\beta} (f * g)(z) \prec g(z),
\]
for every function \( g(z) \in K \) (Class of convex functions). In particular:
\[
\Re \{ f(z) \} > -\frac{1 + 3\alpha \beta + 2\beta}{2(1 + \alpha \beta)} (z \in \mathbb{U}).
\]
The constant factor \( \frac{1 + \alpha \beta}{1 + 3\alpha \beta + 2\beta} \) in the subordination result (3.4) cannot be replaced by any larger one.

Proof. Let \( f(z) \) defined by (1.1) satisfy the coefficient inequality (1.7). In view of Definition 3.2, the subordination (3.4) will hold true if the sequence
\[
\left\{ \frac{1 + \alpha \beta}{1 + 3\alpha \beta + 2\beta} a_n \right\}_{n=1}^{\infty} (a_1 = 1)
\]
is a subordinating factor sequence which by virtue of Lemma 3.1 is equivalent to the inequality
\[
\Re \left\{ 1 + 2 \sum_{n=1}^{\infty} \frac{(1 + \alpha \beta)}{1 + 3\alpha \beta + 2\beta} a_n z^n \right\} > 0 \; (z \in \mathbb{U}).
\]
Now for \( |z| = r (0 < r < 1) \), we obtain
\[
\Re \left\{ 1 + 2 \sum_{n=1}^{\infty} \frac{2(1 + \alpha \beta)}{1 + 3\alpha \beta + 2\beta} a_n z^n \right\} = \Re \left\{ 1 + \frac{2(1 + \alpha \beta)}{1 + 3\alpha \beta + 2\beta} \sum_{n=2}^{\infty} \frac{2(1 + \alpha \beta)}{1 + 3\alpha \beta + 2\beta} a_n z^n \right\}
\geq 1 - \frac{2(1 + \alpha \beta)}{1 + 3\alpha \beta + 2\beta} r - \sum_{n=2}^{\infty} \frac{n(1 + \alpha \beta) + (\beta - 1)(1 - (-1)^n)}{1 + 3\alpha \beta + 2\beta} |a_n| r^n
\geq 1 - \frac{2(1 + \alpha \beta)}{1 + 3\alpha \beta + 2\beta} r - \frac{\beta(\alpha + 2) - 1}{1 + 3\alpha \beta + 2\beta} r.
\]
This evidently establishes the inequality (3.6) and consequently the subordination result (3.4) of Theorem 3.1 is proved. The assertion (3.5) follows readily from (3.4) when the function \( g(z) \) is selected as
\[
g(z) = z - \frac{\beta(\alpha + 2) - 1}{1 + 3\alpha \beta + 2\beta} z^2.
\]
The sharpness of the multiplying factor in (3.4) can be established by considering a functions \( h(z) \) defined by
\[
h(z) = z - \frac{\beta(\alpha + 2) - 1}{1 + 3\alpha \beta + 2\beta} z^2,
\]
which belongs to the class \( TS^\ast_s(\alpha, \beta) \). Using (3.4), we infer that
\[
1 + \frac{\alpha \beta}{1 + 3\alpha \beta + 2\beta} h(z) \prec \frac{z}{1 - z},
\]
and it follows that
\[
\min_{|z| \leq 1} \left\{ \Re \left( \frac{1 + \alpha \beta}{1 + 3\alpha \beta + 2\beta} h(z) \right) \right\} = -\frac{1}{2}.
\]
This completes the proof. \( \square \)
4. Partial Sums

In this section we investigate the ratio of real parts of functions involving (1.1) and its sequence of partial sums defined by

\[ f_1(z) = z \quad \text{and} \quad f_N(z) = z - \sum_{n=2}^{N} a_n z^n \quad (\text{for all } n \in \mathbb{N} \setminus \{1\}), \]

and determine sharp lower bounds for \( \Re \{ f(z)/f_N(z) \}, \Re \{ f_N(z)/f(z) \}, \Re \{ f'(z)/f'_N(z) \} \) and \( \Re \{ f'_N(z)/f'(z) \} \).

**Theorem 4.1.** Let \( f(z) \) of the form (1.1) satisfy the coefficient inequality (1.7), then

\[ \Re \left( \frac{f(z)}{f_N(z)} \right) \geq 1 - \frac{1}{\psi(N+1; \alpha, \beta)}, \]  

and

\[ \Re \left( \frac{f_N(z)}{f(z)} \right) \geq \frac{\psi(N+1; \alpha, \beta)}{\psi(N+1; \alpha, \beta + 1)} \]

where \( \psi(N+1; \alpha, \beta) \) is given by (1.8). The results are sharp for every \( N \), with the extremal functions given by

\[ f(z) = z + \frac{1}{\psi(N+1; \alpha, \beta)} z^{N+1} \quad (N \in \mathbb{N} \setminus \{1\}) \]

**Proof.** We prove (4.2) by setting

\[ g(z) = \psi(N+1; \alpha, \beta) \left\{ \frac{f(z)}{f_N(z)} - \left( 1 - \frac{1}{\psi(N+1; \alpha, \beta)} \right) \right\} \]

\[ = 1 + \frac{\psi(N+1; \alpha, \beta) \sum_{n=N+1}^{\infty} a_n z^{n-1}}{1 + \sum_{n=2}^{N} a_n z^{n-1}}, \]

\[ \frac{|g(z) - 1|}{|g(z) + 1|} \leq \frac{\psi(N+1; \alpha, \beta) \sum_{n=N+1}^{\infty} |a_n|}{2 - 2 \sum_{n=2}^{N} |a_n| - \psi(N+1; \alpha, \beta) \sum_{n=N+1}^{\infty} |a_n|} \]

Now \( \frac{|g(z)-1|}{|g(z)+1|} \leq 1 \), if

\[ \sum_{n=2}^{N} |a_n| + \psi(N+1; \alpha, \beta) \sum_{n=N+1}^{\infty} |a_n| \leq 1 \]

In view of (1.7), this is equivalent to showing that

\[ \sum_{n=2}^{N} (\psi(n; \alpha, \beta) - 1) |a_n| + \sum_{n=N+1}^{\infty} (\psi(n; \alpha, \beta) - \psi(N+1; \alpha, \beta)) |a_n| \geq 0 \]

Which is true in view of (1.10) and (1.11). Finally it can be verified that equality in (4.2) is attained for the function given by (4.4), when \( z = re^{i\pi/N} \) and \( r \to 1^- \). The proof of (4.3) is similar hence omitted here. \( \square \)
Theorem 4.2. Let $f(z)$ of the form (1.1) satisfy the coefficient inequality (1.7), then
\[
\left( \frac{f'(z)}{f_N'(z)} \right) \geq 1 - \frac{N + 1}{\psi(N + 1; \alpha, \beta)},
\]
and
\[
\Re \left( \frac{f_N'(z)}{f'(z)} \right) \geq \frac{\psi(N + 1; \alpha, \beta)}{N + 1 + \psi(N + 1; \alpha, \beta)}
\]
where $\psi(N + 1; \alpha, \beta)$ is given by (1.8). The results are sharp for every $N$, with the extremal functions given by (4.4).

Proof. We prove (4.5) by setting
\[
g(z) = \frac{\psi(N + 1; \alpha, \beta)}{N + 1} \left\{ \frac{f'(z)}{f_N'(z)} - \left( 1 - \frac{N + 1}{\psi(N + 1; \alpha, \beta)} \right) \right\}
\]
\[
= 1 + \sum_{n=2}^{\infty} \frac{\psi(N+1;\alpha,\beta)}{N+1} n a_n z^{n-1} + \sum_{n=2}^{N} n a_n z^{n-1},
\]
\[
\left| \frac{g(z) - 1}{g(z) + 1} \right| \leq \frac{\psi(N+1;\alpha,\beta)}{2 - 2 \sum_{n=2}^{N} n |a_n| - \frac{\psi(N+1;\alpha,\beta)}{N+1} \sum_{n=N+1}^{\infty} n |a_n|}.
\]

Now $\left| \frac{g(z) - 1}{g(z) + 1} \right| \leq 1$, if
\[
\sum_{n=2}^{N} n |a_n| + \frac{\psi(N + 1; \alpha, \beta)}{N + 1} \sum_{n=N+1}^{\infty} n |a_n| \leq 1
\]
In view of (1.7), this is equivalent to showing that
\[
\sum_{n=2}^{N} (\psi(n; \alpha, \beta) - n) |a_n| + \sum_{n=N+1}^{\infty} (\psi(n; \alpha, \beta) - \frac{\psi(N + 1; \alpha, \beta)}{N + 1} n) |a_n| \geq 0
\]
Which is true in view of (1.10) and (1.11). This completes the proof of (4.5). The proof of (4.6) is similar, hence omitted.

References
