SOLUTION OF THE FRACTIONAL EPIDEMIC MODEL BY L-ADM

S. Z. RIDA, A. A. M. ARAFA, Y. A. GABER

Abstract. In this paper the fractional order epidemic model of a non-fatal disease in a population which is assumed to have a constant size over the period of the epidemic is considered. Laplace-adomian decomposition method (for short L-ADM) is used to compute an analytical solution of the system of nonlinear fractional differential equations governing the problem. The results are compared with the results obtained by the classical Runge-Kutta method in the case of integer-order derivatives.

1. Introduction

A simple deterministic model predicting the behavior of epidemic outbreaks was formulated by A. G. McKendrick and W. O. Kermack in 1927 (see [1]). In their mathematical epidemic model, called the Susceptible-Infected-Recovered (which this is called an SIR model, or the $xyz$ model), to describe the spread of diseases, McKendrick and Kermack proposed the following nonlinear system of ordinary differential equations [1]

\[
\begin{align*}
\frac{dx(t)}{dt} & = -\beta x(t)y(t), \\
\frac{dy(t)}{dt} & = \beta x(t)y(t) - \gamma y(t), \\
\frac{dz(t)}{dt} & = \gamma y(t).
\end{align*}
\] (1.1)

In this model the fixed population consists of three types where, at time $t$, $x(t)$ is the number of susceptible individuals, $y(t)$ is the number of infected individuals, able to spread the disease by contact with susceptible, $z(t)$ is the number of isolated individuals, who cannot get or transmit the disease for various reasons. Moreover, the constant $\beta$ and $\gamma$ give the transition rates between compartments. The transition rate between $x$ (Susceptible) and $y$ (infected) is $\beta y$, where $\beta$ is the contact rate, which takes into account the probability of getting the disease in a contact between a susceptible and an infectious subject [2]. The transition rate

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between $y$ (Infected) and $z$ (recovered), is $\gamma$, which has the meaning of the rate of recovery or death. Since $\beta$ and $\gamma$ are interpreted as transition rates (probabilities), their range is $0 \leq \beta \leq 1$ and $0 \leq \gamma \leq 1$. The problem (1.1) was solved by Biazar [3] using Adomian decomposition method (ADM), Rafei et al. [4] using homotopy perturbation method (HPM), Rafei et al. [5] by variation iteration method (VIM), Fadi et al. [6] using homotopy analysis method (HAM) and Abdul-Monim et al. [7] by the differential transformation method (DTM).

The fractional order extension of this model have been first studied in [9, 11]. The reason of using fractional differential quations (FDEs) is that FDEs are naturally related to systems with memory which exists in most biological system. Also they show the realistic biphasic decline behavior of infection of diseases but at a slower rate. The new system is described by the following system of fractional differential quations (FDEs).

$$
\begin{align*}
D^{\alpha_1} x(t) &= -\beta x(t)y(t), \\
D^{\alpha_2} y(t) &= \beta x(t)y(t) - \gamma y(t), \quad \text{where } \alpha_1, \alpha_2, \alpha_3 > 0 \\
D^{\alpha_3} z(t) &= \gamma y(t),
\end{align*}
$$

(1.2)

subject to the initial conditions

$$
x(0) = N_1, \quad y(0) = N_2, \quad z(0) = N_3.
$$

(1.3)

For this model the initial conditions are not independent, since they must satisfy the condition $N_1 + N_2 + N_3 = N$, where $N$ is the total fixed number of the individuals in the given population.

The motivation of this paper is to find analytical solution for general class of fraction order model of non-fatal epidemic by using the L-ADM.

2. Preliminary

Here, we present some necessary definitions and notations related to fractional calculus (see e.g. [10]). The most commonly used definitions are Riemann-Liouville and Caputo.

**Definition 2.1.** The Riemann-Liouville fractional integration of order $\alpha$ is defined as:

$$
(J_0^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(s)ds, \quad \alpha > 0, \quad t > t_0,
$$

and

$$
(J_0^0 f)(t) = f(t).
$$

The Riemann-Liouville derivative has certain disadvantages such that the fractional derivative of a constant is not zero. Therefore, we will make use of Caputo’s definition owing to its convenience for initial conditions of the fractional differential equations.

**Definition 2.2.** Riemann-Liouville and Caputo fractional derivatives of order $\alpha$ can be defined respectively as:

$$
D_0^\alpha f(t) = D^n (J^{n-\alpha} f(t)), \\
D_0^\alpha f(t) = J^{n-\alpha} (D^n f(t)),
$$
where \( n - 1 < \alpha \leq n, \ n \in \mathbb{N} \), \( f \) is a given function, and \( \Gamma(\cdot) \) denotes the gamma function. It is known that \( (D_0^\alpha f)(t) \rightarrow f'(t) \) as \( \alpha \rightarrow 1 \).

Now, we recall the definitions of Laplace transform of Caputo’s derivative and Mittag-Leffler function in two arguments.

**Definition 2.3.**

\[
\mathcal{L}\{D_0^\alpha f(t), s\} = s^\alpha F(s) - \sum_{i=0}^{n-1} s^{\alpha-i-1} f^{(i)}(0), \ (n - 1 < \alpha \leq n); \ n \in \mathbb{N}.
\]

3. The Laplace-Adomian Decomposition Method (L-ADM)

Consider the fractional-order epidemic model (1.2) subject to the initial condition (1.3). The nonlinear term in this model Eqs. (1.2) is \( xy \) and \( \beta, \gamma \) are known constants. For \( \alpha_1 = \alpha_2 = \alpha_3 = 1 \) the fractional order model converts to the classical epidemic model (see e.g. [8]). Applying the Laplace transform on both sides of Eqs. (1.2)

\[
\begin{align*}
\mathcal{L}\{D_0^{\alpha_1}(x)\} &= -\beta \mathcal{L}\{x(t)y(t)\}, \\
\mathcal{L}\{D_0^{\alpha_2}(y)\} &= \beta \mathcal{L}\{x(t)y(t)\} - \gamma \mathcal{L}\{y(t)\}, \\
\mathcal{L}\{D_0^{\alpha_3}(z)\} &= \gamma \mathcal{L}\{y(t)\},
\end{align*}
\]

using property of the Laplace transform, we get

\[
\begin{align*}
S^{\alpha_1} \mathcal{L}\{x\} - S^{\alpha_2-1} x(0) &= -\beta S \mathcal{L}\{x(t)y(t)\}, \\
S^{\alpha_2} \mathcal{L}\{y\} - S^{\alpha_3-1} y(0) &= \beta S \mathcal{L}\{x(t)y(t)\} - \gamma S^{\alpha_2} \mathcal{L}\{y(t)\}, \\
S^{\alpha_3} \mathcal{L}\{z\} - S^{\alpha_3-1} z(0) &= \gamma S \mathcal{L}\{y(t)\},
\end{align*}
\]

using initial condition from (1.3)

\[
\begin{align*}
\mathcal{L}\{x\} &= \frac{N_1}{S} - \frac{\beta}{S^2} \mathcal{L}\{x(t)y(t)\}, \\
\mathcal{L}\{y\} &= \frac{N_2}{S} + \frac{\beta}{S^2} \mathcal{L}\{x(t)y(t)\} - \frac{\gamma}{S^2} \mathcal{L}\{y(t)\}, \\
\mathcal{L}\{z\} &= \frac{N_3}{S} + \frac{\gamma}{S^2} \mathcal{L}\{y(t)\}.
\end{align*}
\]

The method assumes the solution as an infinite series:

\[
x = \sum_{k=0}^{\infty} x_k, \quad y = \sum_{k=0}^{\infty} y_k, \quad z = \sum_{k=0}^{\infty} z_k.
\]

The nonlinearity \( xy \) is decomposed as

\[xy = \sum_{k=0}^{\infty} A_k,\]

where \( A_k \) so-called Adomian Polynomials given as

\[
A_k = \frac{1}{k!} \frac{d^k}{d\lambda^k} \left[ \sum_{j=0}^{k} \lambda^j x_j \sum_{j=0}^{k} \lambda^j y_j \right]_{\lambda=0}.
\]

Substituting from Eqs. (3.4), (3.5) into (3.3) the result is

\[
\begin{align*}
\mathcal{L}\{x_0\} &= \frac{N_1}{S}, \\
\mathcal{L}\{y_0\} &= \frac{N_2}{S}, \\
\mathcal{L}\{z_0\} &= \frac{N_3}{S},
\end{align*}
\]
\[
\mathcal{L}\{x_1\} = -\frac{\beta}{S^{\alpha_1}} \mathcal{L}\{A_0\}, \quad \mathcal{L}\{y_1\} = \frac{\beta}{S^{\alpha_2}} \mathcal{L}\{A_0\} - \frac{\gamma}{S^{\alpha_2}} \mathcal{L}\{y_0\}, \quad \mathcal{L}\{z_1\} = \frac{\gamma}{S^{\alpha_3}} \mathcal{L}\{y_0\}, \quad (3.7)
\]

The aim is to study the mathematical behavior of the solution \(x(t), y(t), z(t)\) for the different values of \(\alpha\). By applying the inverse Laplace transform to both sides of Eqs.\((3.6)\) the values of \(x_0, y_0, z_0\) are obtained. Substituting these values of \(A_0, y_0\) into Eqs.\((3.7)\), the first component \(x_1, y_1, z_1\) are obtained. The other terms of \(x_2, x_3, ..., y_2, y_3, ...\) and \(z_2, z_3, ...\) can be calculated recursively in a similar way and we can write the solution

\[
x(t) = x_0 + x_1 + x_2 + ... , \quad y(t) = y_0 + y_1 + y_2 + ... , \quad z(t) = z_0 + z_1 + z_2 + ... . \quad (3.8)
\]

4. NUMERICAL RESULTS AND DISCUSSION

The L-ADM provides an analytical approximate solution in terms of an infinite power series. For numerical results, the following values, for parameters, are considered \([3]\). The first few components of L-ADM solution \(x(t), y(t)\) and \(z(t)\) are

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(N_1 = 20)</td>
<td>initial population of (x(t)), who are susceptible</td>
</tr>
<tr>
<td>(N_2 = 15)</td>
<td>initial population of (y(t)), who are infective</td>
</tr>
<tr>
<td>(N_3 = 10)</td>
<td>initial population of (z(t)), who are immune</td>
</tr>
<tr>
<td>(\beta = 0.01)</td>
<td>rate of change of susceptibles to infective population</td>
</tr>
<tr>
<td>(\gamma = 0.02)</td>
<td>rate of change of infectives to immune population</td>
</tr>
</tbody>
</table>

Table 1. Parameters values.

calculated. We computed the first three terms of the L-ADM series solution for the system \([1.2]\). We present two of them as follows:

\[
x_1 = \frac{-3 t^{\alpha_1}}{\Gamma(\alpha_1 + 1)} , \quad y_1 = \frac{2.7 t^{\alpha_2}}{\Gamma(\alpha_2 + 1)} , \quad \text{and} \quad z_1 = \frac{0.3 t^{\alpha_3}}{\Gamma(\alpha_3 + 1)},
\]

\[
x_2 = \frac{0.45 t^{2\alpha_1}}{\Gamma(2\alpha_1 + 1)} - \frac{0.54 t^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1 + \alpha_2 + 1)}, \quad y_2 = \frac{0.486 t^{2\alpha_2}}{\Gamma(2\alpha_2 + 1)} - \frac{0.45 t^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1 + \alpha_2 + 1)}, \quad \text{and} \quad z_2 = \frac{0.054 t^{2\alpha_2+\alpha_3}}{\Gamma(\alpha_2 + \alpha_3 + 1)},
\]

thus, the L-ADM series solution of the system \([1.2]\) can be given by Eqs.\((3.8)\) at \(\alpha_1 = \alpha_2 = \alpha_3 = 1\), we obtain three terms approximations which are the same solution obtained in \([8]\) using L-ADM but in ordinary case as follows:

\[
x(t) = 20 - 3 t - 0.045 t^2 + 0.02805 t^3,
\]

\[
y(t) = 15 + 2.7 t + 0.018 t^2 - 0.02817 t^3,
\]

\[
z(t) = 10 + 0.3 t + 0.027 t^2 + 0.00012 t^3.
\]

5. CONCLUSION

In the present work, we have considered a fractional version of the SIR model, describing the spread of an epidemic in a given population. The aim of this work is to use the Laplace-Adomian Decomposition method for obtaining the solution of the fractional epidemic model. The comparison for some different values of \(\alpha\) has been obtained.
Figure 1. The numerical results for $x(t)$, $y(t)$, $z(t)$ at $\alpha_1 = \alpha_3 = \alpha_3 = 1$. The solutions of $x(t)$, $y(t)$ by using L-ADM and ADM completely overlap.

Figure 2. The numerical results for $x(t)$, $y(t)$, $z(t)$ at different values of $\alpha$. 
Table 2. The number of susceptible individuals $x(t)$ at different values of $\alpha$.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$\alpha_1 = 1$</th>
<th>$\alpha_1 = 0.99$</th>
<th>$\alpha_1 = 0.95$</th>
<th>$\alpha_2 = 0.99$</th>
<th>$\alpha_2 = 0.95$</th>
<th>$\alpha_3 = 0.99$</th>
<th>$\alpha_3 = 0.95$</th>
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Table 3. The number of infected individuals $y(t)$ at different values of $\alpha$.

<table>
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<tr>
<th>$t$</th>
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<th>$\alpha_1 = 0.99$</th>
<th>$\alpha_1 = 0.95$</th>
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Table 4. The number of isolated individuals $z(t)$ at different values of $\alpha$.

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References


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