SUBORDINATION RESULT FOR A CLASS OF ANALYTIC FUNCTIONS WITH MISSING COEFFICIENTS

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Abstract. In the present investigation we consider a new class of functions $R_n(\alpha, \gamma)$ and prove some subordination result. Our result gives result of Owa and Ma [4].

1. Introduction and Preliminaries

Let $A_n$ denote the class of functions of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \quad (n \in \mathbb{N})$$

which are analytic in the open unit disk $\Delta = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \}$. A domain $D \subset \mathbb{C}$ is convex if the line segment joining any two points in $D$ lies entirely in $D$. A univalent function $f \in A_n$ is convex if $f(\Delta)$ is convex. Analytically, a univalent function $f \in A_n$ is convex if and only if

$$\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > 0.$$  \hspace{1cm} (2)

Further, a function $f(z)$ in the class $A_n$ is said to be close-to-convex of order $0 \leq \alpha < 1$ in the unit disk $\Delta$ if there exists a convex function $g(z) \in A_n$ such that

$$\Re \left\{ \frac{f'(z)}{g'(z)} \right\} > \alpha \quad (z \in \Delta)$$  \hspace{1cm} (3)

The concept of close-to-convex functions was introduced by Kaplan [2].

Definition 1.1. Let $0 \leq \gamma \leq 1$, $0 \leq \alpha < 1$. A function $f(z) \in A_n$ is said to be in the class $R_n(\alpha, \gamma)$ if and only if satisfies

$$\left| (1 - \gamma) \frac{f(z)}{z} + \gamma f'(z) - 1 \right| < 1 - \alpha \quad (z \in \Delta).$$  \hspace{1cm} (4)

Note that $f(z) \in R_n(\alpha, \gamma)$ gives

$$\Re \left\{ (1 - \gamma) \frac{f(z)}{z} + \gamma f'(z) \right\} > \alpha.$$  \hspace{1cm} (5)

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Obviously \( f(z) = z \) belongs to the class \( R_n(\alpha, \gamma) \). The class \( R_n(\alpha, \gamma) \) is a subclass of class \( P_\gamma^\beta \) defined by Swaminathan [6]. If \( \gamma = 1 \), we get the class \( R_n(\alpha) \) defined in Owa and Ma [4]. For \( \gamma = 0 \) we get the class \( A_n(\alpha) \) defined by Owa and Hu [5].

An analytic function \( f \) is subordinate to an analytic function \( g \), written as \( f(z) \prec g(z) \) \((z \in \mathbb{U})\), if there is an analytic function \( w \) defined on \( \Delta \) with \( w(0) = 0 \) and \( |w(z)| < 1, z \in \Delta \) such that \( f(z) = g(w(z)) \). In particular, if \( g \) is univalent in \( \Delta \) then we have the following equivalence:

\[
f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(\Delta) \subset g(\Delta).
\]

In order to prove our main result we need following lemma due to Miller and Mocanu [3], see also Jack [1].

**Lemma 1.2.** Let the function \( w(z) = b_n z^n + b_{n+1} z^{n+1} + \cdots (n \in \mathbb{N}) \) be analytic in \( \Delta \) with \( w(z) \) is not identically zero. If \( z_0 = r_0 e^{i\theta_0} \) \((r_0 < 1)\) and

\[
|w(z_0)| = \max \{|w(z); |z| \leq |z_0|\},
\]

then

\[
z_0 w'(z_0) = mw(z_0).
\]

where \( m \) is real and \( m \geq n \geq 1 \).

## 2. Main Results

**Theorem 2.1.** Let the function \( f(z) \) defined by (1) be in the class \( R_n(\alpha, \gamma) \). Then

\[
\frac{f(z)}{z} \prec 1 + \frac{(1-\alpha)z}{1+\gamma n}.
\]

**Proof.** It is clear that the result is true if \( f(z) = z \). Then, we assume that \( f(z) \neq z \). Define the analytic function \( w(z) \) in the unit disk \( \Delta \) by

\[
\frac{f(z)}{z} = 1 + \frac{(1-\alpha)w(z)}{1+\gamma n},
\]

then we see that

\[
w(z) = b_n z^n + b_{n+1} z^{n+1} + \cdots (n \in \mathbb{N}).
\]

Obviously \( w(0) = 0 \) and \( w(z) \) is not identically zero since \( f(z) \) is not identically equal to \( z \). Now, we need only to prove that \( |w(z)| < 1 \) for all \( z \in \Delta \). If not so, there exists a point \( z_0 \in \Delta \) such that

\[
\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1.
\]

Therefore, applying our Lemma 1.2, we have

\[
z_0 w'(z_0) = mw(z_0),
\]

where \( m \) is real and \( m \geq n \geq 1 \). Using (9),

\[
f'(z) = 1 + \frac{(1-\alpha)[zw'(z) + w(z)]}{1+\gamma n}.
\]

Now using (9) and (12) we see that

\[
(1-\gamma)\frac{f(z_0)}{z_0} + \gamma f'(z_0) - 1 = \frac{1-\alpha}{1+\gamma n} [w(z_0) + \gamma z_0 w'(z_0)].
\]
Applying (11), we have
\[
(1 - \gamma) \frac{f(z_0)}{z_0} + \gamma f'(z_0) - 1 = \frac{(1 - \alpha)w(z_0)}{1 + \gamma n} [1 + m\gamma].
\]
Thus
\[
\left| (1 - \gamma) \frac{f(z_0)}{z_0} + \gamma f'(z_0) - 1 \right| = \frac{(1 - \alpha)(1 + m\gamma)}{1 + \gamma n} \geq 1 - \alpha.
\]
This contradicts that \( f(z) \) belongs to the class \( \mathcal{R}_n(\alpha, \gamma) \). Therefore, we complete the proof of theorem. \( \square \)

It follows from theorem the following

**Remark 2.2** For \( \gamma = 1 \) in Theorem 2.1, we get the result obtained by Owa and Ma [4] in Theorem 1.

**Corollary 2.3.** If the function \( f(z) \) defined by (1) is in the class \( \mathcal{R}_n(\alpha, \gamma) \), then
\[
\left| \text{Arg} \left( \frac{f(z)}{z} \right) \right| \leq \sin^{-1} \left( \frac{1 - \alpha}{1 + \gamma n} \right).
\]
The bound is best possible for the function \( f(z) \) defined by
\[
f(z) = z + \frac{(1 - \alpha)}{1 + \gamma n} z^{n+1} \in \mathcal{R}_n(\alpha, \gamma).
\]

**Theorem 2.4.** Let the function \( f(z) \) defined by (1) be in the class \( \mathcal{R}_n(\alpha, \gamma) \). Then
\[
|a_k| \leq \frac{1 - \alpha}{1 - \gamma(1 - k)} \quad k > n, \tag{13}
\]
and
\[
\sum_{k=n+1}^{\infty} |a_k|^2 < 1. \tag{14}
\]

**Proof.** Using (1) we can write the condition (4) as follows
\[
\left| \sum_{k=n+1}^{\infty} \frac{1 - \gamma(1-k)}{1 - \alpha} a_k z^{k-1} \right| < 1 \quad (z \in \Delta) \tag{15}
\]
then we see that
\[
\sum_{k=n+1}^{\infty} \frac{1 - \gamma(1-k)}{1 - \alpha} a_k z^{k-1}
\]
is the bounded function, hence it has the coefficients bounded by 1. Therefore, we have
\[
\left| \frac{1 - \gamma(1-k)}{1 - \alpha} a_k \right| \leq 1 \quad k > n,
\]
and we immediately obtain the estimation (13), while (14) also follows immediately from another known property of bounded functions. \( \square \)

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