COEFFICIENT BOUND FOR A NEW CLASS OF ANALYTIC AND BI-UNIVALENT FUNCTIONS

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Abstract. In the present investigation we consider a new class of bi-univalent functions in the unit disk \( \Delta \) using subordination and obtain estimates for the Taylor-Maclaurin coefficients \( |a_2| \) and \( |a_3| \).

1. Introduction and Preliminaries

Let \( \mathcal{A} \) denote the class of functions of the form
\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k
\]
which are analytic in the open unit disk \( \Delta = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \} \) and \( \mathcal{S} \) denote the subclass of class \( \mathcal{A} \) consisting functions in \( \mathcal{A} \) which are also univalent in \( \Delta \). A domain \( D \subset \mathbb{C} \) is convex if the line segment joining any two points in \( D \) lies entirely in \( D \), while a domain is starlike with respect to a point \( w_0 \in D \) if the line segment joining any point of \( D \) to \( w_0 \) lies inside \( D \). A function \( f \in \mathcal{A} \) is starlike if \( f(\Delta) \) is a starlike domain with respect to origin, and convex if \( f(\Delta) \) is convex. Analytically, \( f \in \mathcal{A} \) is starlike if and only if \( \Re \left( \frac{zf'(z)}{f(z)} \right) > 0 \) in \( \Delta \), whereas \( f \in \mathcal{A} \) is convex if and only if \( \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > 0 \). The classes consisting of starlike and convex functions are denoted by \( \mathcal{S}^* \) and \( \mathcal{K} \) respectively. The classes \( \mathcal{S}^*(\alpha) \) and \( \mathcal{K}(\alpha) \) of starlike and convex functions of order \( \alpha \), \( 0 \leq \alpha < 1 \), are respectively characterized by \( \Re \left( \frac{zf'(z)}{f(z)} \right) > \alpha \) and \( \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \) in \( \Delta \), \( f \in \mathcal{A} \). Also let \( \mathcal{P} \) denote the family of analytic functions \( p(z) \) in \( \Delta \) such that \( p(0) = 1 \) and \( \Re (p(z)) > 0 \) in \( \Delta \).

An analytic function \( f \) is subordinate to an analytic function \( g \), written as \( f(z) \preceq g(z) \) \( (z \in U) \), if there is an analytic function \( w \) defined on \( \Delta \) with \( w(0) = 0 \) and \( |w(z)| < 1 \), \( z \in \Delta \) such that \( f(z) = g(w(z)) \). In particular, if \( g \) is univalent in \( \Delta \) then we have the following equivalence:
\[
f(z) \preceq g(z) \iff f(0) = g(0) \text{ and } f(\Delta) \subset g(\Delta).
\]
It is well known by the Koebe one quarter theorem \(5\) that the image of \(\Delta\) under every function \(f \in \mathcal{S}\) contains a disk of radius 1/4. Thus every univalent function \(f\) has an inverse \(f^{-1}\) satisfying \(f^{-1}(f(z)) = z\), \((z \in \Delta)\) and

\[
f(f^{-1}(w)) = w \left(|w| < r_0(f), \ r_0(f) \geq 1/4\right).
\]

The inverse of \(f(z)\) has a series expansion in some disk about the origin of the form

\[
f^{-1}(w) = w + \gamma_2 w^2 + \gamma_3 w^3 + \ldots.
\]

(2)

It was shown early \(11\ 14\) that the inverse of the Koebe function provides the best bound for all \(|\gamma_k|\). New proofs of the latter along with unexpected and unusual behavior of the coefficients \(\gamma_k\) for various subclasses of \(\mathcal{S}\) have generated further interest in this problem \(7\) \(8\) \(9\) \(16\).

A function \(f(z)\) univalent in a neighborhood of the origin and its inverse satisfy the condition \(f(f^{-1}(w)) = w\). Using \(1\), we have

\[
w = f^{-1}(w) + a_2(f^{-1}(w))^2 + a_3(f^{-1}(w))^3 + \ldots
\]

(3)

Now using \(2\) we get the following result

\[
g(w) = f^{-1}(w) - a_2w^2 + (2a_2^2 - a_3)w^3 + \cdots
\]

(4)

A function \(f \in \mathcal{A}\) is said to be bi-univalent in \(\Delta\) if both \(f\) and \(f^{-1}\) are univalent in \(\Delta\). Let \(\Sigma\) denote the class of all bi-univalent functions defined in the unit disk \(\Delta\) given by the Taylor-Maclaurin series expansion \(1\). Note the familiar Koebe function is not a member of \(\Sigma\) because it maps unit disk univalently onto entire complex plane minus slit along \(-1/4\) to \(-\infty\). Hence the image domain does not contain unit disk.

Lewin \(10\) investigated the class \(\Sigma\) of bi-univalent functions and showed that \(|a_2| < 1.51\). Subsequently, Brannan and Clunie \(2\) conjectured that \(|a_2| \leq \sqrt{2}\). Netanyahu \(13\), on the other hand, showed that \(\max_{f \in \Sigma} |a_2| = 4/3\). The coefficient estimate problem i.e. bound of \(|a_n|\) \((n \in \mathbb{N} \setminus \{1, 2\})\) for each \(f \in \Sigma\) given by \(1\) is still an open problem. Several authors have subsequently studied similar problems in this direction. In \(3\) (see also \(4\) \(18\) \(19\)), certain subclasses of the bi-univalent function class \(\Sigma\) were introduced, and non-sharp estimates on the first two coefficients \(|a_2|\) and \(|a_3|\) were found. In the present investigation, estimates on the initial coefficients of a new class of bi-univalent functions are obtained. Several related classes are also considered and a connection to earlier known result are made. The classes introduced in this paper are motivated by the corresponding classes investigated in \(6\) \(12\) \(15\).

Let \(\varphi\) be an analytic function with positive real part on \(\Delta\), satisfying \(\varphi(0) = 1\), \(\varphi'(0) > 0\), and \(\varphi(\Delta)\) is symmetric with respect to the real axis. Such a function has a series expansion of the form

\[
\varphi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \cdots (B_1 > 0).
\]

(5)

We now introduce the following class of functions:

**Definition 1.1.** Let \(0 \leq \gamma \leq 1\), \(\tau \in \mathbb{C} \setminus\{0\}\). A function \(f \in \Sigma\) is in the class \(\Sigma S_\gamma^\tau(\varphi)\), if the following subordinations hold:

\[
1 + \frac{1}{\tau} \left[(1 - \gamma) \frac{f(z)}{z} + \gamma f'(z) - 1\right] < \varphi(z) \ (z \in \Delta)
\]

(6)
and
\[ 1 + \frac{1}{\tau} \left[ (1 - \gamma) \frac{g(w)}{w} + \gamma g'(w) - 1 \right] < \varphi(w) \quad (w \in \Delta), \tag{7} \]
where \( g(w) = f^{-1}(w) \).

We list few particular cases of this class discussed in the literature

[1] If we set \( \gamma = 1 \) and \( \tau = 1 \) in \( \Sigma \) we obtain the class introduced in [1].

[2] If we set \( \gamma = 1 \) and \( \tau = 1 \) and \( \varphi(z) = 1 + (1 - 2\beta)z \) \( (0 \leq \beta < 1) \) we obtain the class introduced in [17, p. 1191].

[3] If we set \( \gamma = 1 \) and \( \tau = 1 \) and \( \varphi(z) = \left( \frac{1 + z}{1 - z} \right)^{\alpha} \) \( (0 < \alpha \leq 1) \) we obtain the class introduced in [17, p. 1190].

For more details about these classes see the corresponding references.

Further if we set \( \tau = 1, \gamma = 0 \) and \( \varphi(z) = 1 + Az + Bz \) \( (-1 \leq B < A \leq 1; z \in \Delta) \) in Definition 1.1, we obtain a new class \( \Sigma_{A,B} \) defined in the following way.

A function \( f \in \Sigma \) is in the class \( \Sigma_{A,B} \), if the following subordinations hold:
\[ \frac{f(z)}{z} < \frac{1 + Az}{1 + Bz} \quad \text{and} \quad \frac{g(w)}{w} < \frac{1 + Aw}{1 + Bw} \quad (z,w \in \Delta), \tag{8} \]
where \( g(w) = f^{-1}(w) \).

To prove our main result we need following Lemma:

**Lemma 1.1** (see [5]). Let the function \( p \in \mathcal{P} \) be given by the series
\[ p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \quad (z \in \Delta), \tag{9} \]
then, the sharp estimate
\[ |c_n| \leq 2 \quad (n \in \mathbb{N}), \tag{10} \]
holds.

2. **Main Results**

For functions in the class \( \Sigma_{A,B}^\gamma(\varphi) \), the following result is obtained.

**Theorem 2.1.** Let \( f(z) \in \Sigma_{A,B}^\gamma(\varphi) \) is of the form (1), then
\[ |a_2| \leq \frac{|\tau| B_1^{3/2}}{\sqrt{\tau B_1^2 (1 + 2\gamma) + (1 + \gamma)^2 (B_1 - B_2)}} \tag{11} \]
and
\[ |a_3| \leq B_1 |\tau| \left( \frac{1}{1 + 2\gamma} + \frac{B_1 |\tau|}{(1 + \gamma)^2} \right) \tag{12} \]

**Proof.** Let \( f \in \Sigma_{A,B}^\gamma(\varphi) \) and \( g = f^{-1} \). Then there are analytic functions \( u, v : \Delta \to \Delta \), with \( u(0) = v(0) = 0 \), satisfying
\[ 1 + \frac{1}{\tau} \left[ (1 - \gamma) \frac{f(z)}{z} + \gamma f'(z) - 1 \right] = \varphi(u(z)) \quad (z \in \Delta) \tag{13} \]
and
\[ 1 + \frac{1}{\tau} \left[ (1 - \gamma) \frac{g(w)}{w} + \gamma g'(w) - 1 \right] = \varphi(v(w)) \quad (w \in \Delta). \tag{14} \]
Define the functions $p_1$ and $p_2$ by

$$p_1(z) = \frac{1 + u(z)}{1 - u(z)} = 1 + c_1 z + c_2 z^2 + \cdots \quad \text{and} \quad p_2(z) = \frac{1 + v(z)}{1 - v(z)} = 1 + b_1 z + b_2 z^2 + \cdots.$$ 

Then $p_1$ and $p_2$ are analytic in $\Delta$ with $p_1(0) = 1 = p_2(0)$. Since $u, v : \Delta \to \Delta$, the functions $p_1$ and $p_2$ have a positive real part in $\Delta$, and in view of Lemma 1.1

$$|b_n| \leq 2 \quad \text{and} \quad |c_n| \leq 2 \quad (n \in \mathbb{N}). \quad (15)$$

Solving for $u(z)$ and $v(z)$ we have

$$u(z) = \frac{p_1(z) - 1}{p_1(z) + 1} = \frac{1}{2} \left( c_1 z + \left( c_2 - \frac{c_1^2}{2} \right) z^2 + \cdots \right) \quad (z \in \Delta). \quad (16)$$

and

$$v(z) = \frac{p_2(z) - 1}{p_2(z) + 1} = \frac{1}{2} \left( b_1 z + \left( b_2 - \frac{b_1^2}{2} \right) z^2 + \cdots \right) \quad (z \in \Delta). \quad (17)$$

In view of (5) and (13)-(17), clearly

$$1 + \frac{1}{\tau} \left[ (1 - \gamma) \frac{f(z)}{z} + \gamma f'(z) - 1 \right] = \varphi \left( \frac{p_1(z) - 1}{p_1(z) + 1} \right) = 1 + \frac{1}{2} B_1 c_1 z + \left( \frac{1}{2} B_1 \left( c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4} B_2 c_1^2 \right) z^2 + \cdots. \quad (18)$$

and

$$1 + \frac{1}{\tau} \left[ (1 - \gamma) \frac{g(w)}{w} + \gamma g'(w) - 1 \right] = \varphi \left( \frac{p_2(w) - 1}{p_2(w) + 1} \right) = 1 + \frac{1}{2} B_1 b_1 w + \left( \frac{1}{2} B_1 \left( b_2 - \frac{b_1^2}{2} \right) + \frac{1}{4} B_2 b_1^2 \right) w^2 + \cdots. \quad (19)$$

Since $f \in \Sigma$ has the Maclaurin series given by (1), a computation shows that its inverse $g = f^{-1}$ has the expansion

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 + \cdots \quad (20)$$

Using (11) and (20) in (18) and (19), we obtain

$$\frac{(1 + \gamma) a_2}{\tau} = B_1 c_1, \quad (21)$$

$$\frac{(1 + 2\gamma) a_3}{\tau} = \frac{1}{2} B_1 \left( c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4} B_2 c_1^2, \quad (22)$$

$$\frac{-(1 + \gamma) a_1}{\tau} = B_1 b_1, \quad (23)$$

and

$$\frac{(1 + 2\gamma) (2a_2^2 - a_3)}{\tau} = \frac{1}{2} B_1 \left( b_2 - \frac{b_1^2}{2} \right) + \frac{1}{4} B_2 b_1^2. \quad (24)$$
From (21) and (23), it follows that
\[ c_1 = -b_1. \]
Further computation gives
\[ a_2 = \frac{\tau^2 B_1^2 (b_2 + c_2)}{4 \left[ \tau B_1^2 (1 + 2\gamma) + (1 + \gamma)^2 (B_1 - B_2) \right]} \]  
and
\[ a_3 = \frac{B_1^2 \tau B_2^2}{4(1 + \gamma)^2} + \frac{B_1 \tau}{4(1 + 2\gamma)} (c_2 - b_2). \]  
(25)

In view of Lemma 1.1 we get the desired result (11) and (12).

**Remark 2.1.** If we set \( \gamma = 1 \) and \( \tau = 1 \) in Theorem 2.1 we get Theorem 2.1 of [1].

If we set
\[ \varphi(z) = \frac{1 + Az}{1 + Bz} \quad (-1 \leq B < A \leq 1; z \in \Delta) \]  
(27)
in Theorem 2.1, we get the following Corollary:

**Corollary 2.1.** Let \( f(z) \in \Sigma S_\gamma((1 + Az)/(1 + Bz)) \) is of the form (1), then
\[ |a_2| \leq \frac{\tau |A - B|}{\sqrt{\tau (A - B)(1 + 2\gamma) + (1 + \gamma)^2 (1 + B)}} \]  
and
\[ |a_3| \leq (A - B) \left| \frac{1}{1 + 2\gamma} + \frac{(A - B) |\tau|}{(1 + \gamma)^2} \right|. \]  
(28)
(29)

**Remark 2.2.** If we set \( \gamma = 1, \tau = 1, A = 1 - 2\beta \) (0 \( \leq \beta < 1 \)) and \( B = -1 \) in Corollary 2.1 we get the Theorem 2 of [17].

**Corollary 2.2.** Let \( f(z) \in \Sigma S_\gamma\left(\left(\frac{1 + Az}{1 + Bz}\right)^\alpha\right) \) is of the form (1), then
\[ |a_2| \leq \frac{2 |\tau| \alpha}{\sqrt{2\alpha \tau (1 + 2\gamma) + (1 + \gamma)^2 (1 - \alpha)}} \]  
and
\[ |a_3| \leq 2\alpha |\tau| \left( \frac{1}{1 + 2\gamma} + \frac{2\alpha |\tau|}{(1 + \gamma)^2} \right). \]  
(30)
(31)

**Remark 2.3.** Further if we set \( \gamma = 1, \tau = 1 \) in Corollary 2.2, we get Theorem 1 of [17].

Finally setting \( \tau = 1, \gamma = 0 \) in Corollary 2.1, we get the following new result:

**Corollary 2.3.** Let \( f(z) \in \Sigma S(A, B) \) is of the form (1), then
\[ |a_2| \leq \frac{A - B}{\sqrt{A + 1}} \quad \text{and} \quad |a_3| \leq (A - B + 1)(A - B). \]  
(32)

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