Journal of Fractional Calculus and Applications,
Vol. 4(2) July 2013, pp. 224- 233.
ISSN: 2090-5858.
http://www.fcaj.webs.com/

CERTAIN CLASSES OF INFINITE SERIES ASSOCIATED WITH q-DIGAMMA FUNCTION BY MEANS OF FRACTIONAL q-CALCULUS

AHMED SALEM

Abstract. The main purpose of this paper is to establish certain families of infinite series involving the q-digamma function summable by means of the fractional q-calculus techniques based upon the Riemann-Liouville fractional q-integral and the fractional q-Leibniz rule. We also consider relevant connections of the results presented here with various results which were derived earlier with or without the use of fractional calculus operators when $q \to 1$.

1. Introduction

The sums of several interesting families of infinite series were expressed in terms of the digamma function by several authors who used the technique of applying such operators of fractional calculus as the familiar Riemann-Liouville fractional differ-integral operator. For a reasonably detailed historical account of these sums and its generalizations, one may refer to a work on the subject by Nishimoto and Srivastava [16], who also furnished a number of relevant earlier references on summation of infinite series by means of fractional calculus. Many further developments on this subject are reported (among others) by Srivastava [19], Al-Saqabi et al. [6], Nishimoto and Saxena [15], Aular de Duran et al. [8], Choi [10], Wu et al. [21] and Chen et al. [9]. The main purpose of this paper is to establish certain families of infinite series involving the q-digamma function which appeared in the work of Krattenthaler and Srivastava [13] when they studied the summations for basic hypergeometric series.

We first show a list of various definitions and notations in q-calculus which are useful to understand the subject of this paper and will be taken from the well known books in this field [7, 11], unless otherwise stated.

2000 *Mathematics Subject Classification.* 05A30, 33D05, 33D15.

Key words and phrases. Fractional q-integral; Fractional q-Leibniz rule; q-Digamma function; Infinite series.

Submitted Jan. 26, 2013.

For any complex number a, the basic number and the q-factorial are defined as

$$[a]_q = \frac{1-q^a}{1-q},\ q \neq 1;\ [n]_q! = [n]_q[n-1]_q \cdots [1]_q,\ n \in \mathbb{N};\ [0]_q! = 1 \tag{1.1}$$

and the scalar q-shifted factorials are defined as

$$(a;q)_0 = 1, \qquad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad n \in \mathbb{N} \tag{1.2}$$

and

$$(a_1, a_2, \cdots, a_k; q)_n = (a_1;q)_n(a_2;q)_n \cdots (a_k;q)_n \qquad n, k \in \mathbb{N}. \tag{1.3}$$

The limit, $\lim_{n\to\infty}(a;q)_n$, is denoted by $(a;q)_\infty$ provided $|q| < 1$. This implies that

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \qquad n \in \mathbb{N}_0, \quad |q| < 1 \tag{1.4}$$

and, for any complex number α

$$(a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty}, \qquad |q| < 1, \tag{1.5}$$

where the principal value of q^α is taken.
The q-binomial coefficient is defined for positive integers n, k as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} = \begin{bmatrix} n \\ n-k \end{bmatrix}_q. \tag{1.6}$$

This definition can be generalized in the following way. For arbitrary complex α we have

$$\begin{bmatrix} \alpha \\ k \end{bmatrix}_q = \frac{(q^{-\alpha};q)_k}{(q;q)_k}(-1)^k q^{\alpha k - \binom{k}{2}} = \frac{\Gamma_q(\alpha+1)}{\Gamma_q(k+1)\Gamma_q(\alpha-k)}, \tag{1.7}$$

where $\Gamma_q(z)$ is the q-gamma function defined by the representation

$$\Gamma_q(z) = \frac{(q;q)_\infty}{(q^z;q)_\infty}(1-q)^{1-z}, \qquad z \neq 0, -1, -2, \cdots;\ |q| < 1. \tag{1.8}$$

The basic hypergeometric series is defined as

$${}_r\phi_s\left[\begin{matrix} a_1, a_2, \cdots, a_r \\ b_1, b_2, \cdots, b_s \end{matrix}; q, z\right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \cdots, a_r; q)_n}{(q, b_1, b_2, \cdots, b_s; q)_n} \left((-1)^n q^{\binom{n}{2}}\right)^{s-r+1} z^n \tag{1.9}$$

for all complex variable z if $r \leq s, 0 < |q| < 1$ and for $|z| < 1$ if $r = s+1$.
The q-derivative $D_q f(z)$ of a function f is given as

$$(D_q f)(z) = \frac{f(z) - f(zq)}{(1-q)z}, \quad q \neq 1, \quad z \neq 0, \quad (D_q f)(0) = f'(0) \tag{1.10}$$

provided $f'(0)$ exists. If f is differentiable then $D_q f(z)$ tends to $f'(z)$ as $q \to 1$. The Jackson q-integral from 0 to z is defined as

$$\int_0^z f(t) d_q t = z(1-q)\sum_{k=0}^{\infty} q^k f(zq^k) \tag{1.11}$$

provided the sum converges absolutely.

It is worth mentioning that the q-gamma function is related to many q-special functions. The most important of these functions are the q-digamma function $\psi_q(z)$ which is defined as the logarithmic derivative of the q-gamma function [13] (for more details on the q-digamma function, see [17])

$$\psi_q(z) = \frac{d}{dz}\left(\ln \Gamma_q(z)\right) = \frac{\Gamma_q'(z)}{\Gamma_q(z)}, \qquad z \neq 0, -1, -2, \cdots \tag{1.12}$$

and the q-beta function which is defined as

$$B_q(\alpha, \lambda) = \int_0^1 t^{\lambda-1} \frac{(tq;q)_\infty}{(tq^\alpha;q)_\infty} d_q t = \frac{\Gamma_q(\alpha)\Gamma_q(\lambda)}{\Gamma_q(\alpha+\lambda)},$$
$$(\Re(\lambda) > 0, \alpha \neq 0, -1, -2, \cdots). \tag{1.13}$$

As tools to accomplish our work, we need to provide the definition of the Riemann-Liouville fractional q-integral [5]

$$\begin{aligned} I_q^\alpha f(z) &= \frac{z^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^z (tq/z;q)_{\alpha-1} f(t) d_q t \\ &= z^\alpha (1-q)^\alpha \sum_{n=0}^\infty \frac{(q^\alpha;q)_n}{(q;q)_n} q^n f(zq^n) \end{aligned} \tag{1.14}$$

which is considered the usual starting point for a definition of fractional operators and its applications in q-calculus taken in [3, 4, 5, 18] and the q-extension of the Leibniz rule for the fractional q-integrals for a product of two functions which is defined by Agarwal [2] in the following manner

$$I_q^\alpha\{f(z)g(z)\} = \sum_{k=0}^\infty \begin{bmatrix} -\alpha \\ k \end{bmatrix}_q D_q^k f(z) I_q^{\alpha+k} g(zq^k), \tag{1.15}$$

where $f(z)$ and $g(z)$ are two regular functions such that

$$f(z) = \sum_{r=0}^\infty a_r z^r,\ (|z| < R_1) \text{ and } g(z) = \sum_{r=0}^\infty b_r z^r,\ (|z| < R_2) \tag{1.16}$$

then for the result (1.15), $|z| < R = \min\{R_1, R_2\}$.

We are seeking in this paper to investigate rather systematically general families of infinite series relations which are considered series expansions of the q-digamma function (1.12). We also consider relevant connections of the results presented here with various results which were derived earlier with or without the use of fractional calculus operators when $q \to 1$. From now on, we will consider $|q| < 1$.

2. The main result

In this section, we apply the fractional q-calculus techniques based upon the aforementioned Riemann-Liouville fractional q-integral (1.14) and the fractional q-Leibniz rule (1.15) to establish the general functional relation involving the basic hypergeometric series (1.9) and the q-digamma function (1.12). Before we present our main theorem, we provide the following lemmas to help in proving this theorem:

Lemma 2.1. Let $\alpha, \lambda, z \in \mathbb{C}$. Then, we have

$$I_q^\alpha\{z^{\lambda-1}\} = \frac{z^{\alpha+\lambda-1}\Gamma_q(\lambda)}{\Gamma_q(\alpha+\lambda)}, \qquad (\Re(\lambda) > 0, \alpha \neq 0, -1, -2\cdots). \tag{2.1}$$

Proof. From the definitions of fractional q-integral (1.14) and the q-beta function (1.13), we get

$$\begin{aligned} I_q^\alpha\{z^{\lambda-1}\} &= \frac{z^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^z (tq/z;q)_{\alpha-1}t^{\lambda-1}d_qt \\ &= \frac{z^{\alpha+\lambda-1}}{\Gamma_q(\alpha)}\int_0^1 \frac{(tq;q)_\infty}{(tq^\alpha;q)_\infty}t^{\lambda-1}d_qt = \frac{z^{\alpha+\lambda-1}}{\Gamma_q(\alpha)}B_q(\alpha,\lambda) \\ &= \frac{z^{\alpha+\lambda-1}\Gamma_q(\lambda)}{\Gamma_q(\alpha+\lambda)}, \qquad (\Re(\lambda) > 0, \alpha \neq 0, -1, -2\cdots). \end{aligned}$$

Lemma 2.2. Let $\alpha, \lambda, z \in \mathbb{C}$ such that $\Re(\lambda) > 0$ and $\alpha \neq 0, -1, -2\cdots$. Then, we have

$$I_q^\alpha\{z^{\lambda-1}\ln z\} = \frac{z^{\alpha+\lambda-1}\Gamma_q(\lambda)}{\Gamma_q(\alpha+\lambda)}\left[\ln z + \frac{\ln q}{1-q}\sum_{k=1}^{\infty}\frac{(q^\alpha;q)_k q^{\lambda k}}{[k]_q(q^{\alpha+\lambda};q)_k}\right]. \tag{2.2}$$

Proof. It is not difficult, by applying (1.10), to see that

$$D_q^k \ln z = \frac{\ln q}{1-q}(-1)^k q^{-k(k-1)/2}[k-1]_q! z^{-k}, \qquad k \in \mathbb{N}$$

which can be used with the fractional q-Leibniz rule (1.15) to obtain

$$I_q^\alpha\{z^{\lambda-1}\ln z\} = \frac{z^{\alpha+\lambda-1}\Gamma_q(\lambda)}{\Gamma_q(\alpha+\lambda)}\left[\ln z + \ln q\sum_{k=1}^{\infty}\begin{bmatrix}-\alpha\\ k\end{bmatrix}_q (-1)^k q^{\binom{k}{2}}(q;q)_{k-1}\frac{q^{k(\alpha+\lambda)}}{(q^{\alpha+\lambda};q)_k}\right].$$

When (1.7) is substituted in the previous equation and after simplification, we obtain the desired result.

Theorem 2.3. Let $r, s \in \mathbb{N}$ and $a_i, b_j \in \mathbb{C}$ such that $i = 1, 2, \cdots, r$ and $j = 1, 2, \cdots, s$. Let also $\nu, \lambda, a, z, q \in \mathbb{C}$ such that $|q| < 1$. Then, we have

$$\frac{\ln q}{1-q}\sum_{k=1}^{\infty}\frac{(q^\nu;q)_k q^{k(\lambda-\nu)}}{[k]_q(q^\lambda;q)_k}\ {}_{r+1}\phi_{s+1}\left[\begin{matrix}a_1, a_2, \cdots, a_r, q^\lambda\\ b_1, b_2, \cdots, b_s, q^{\lambda+k}\end{matrix}; q, \frac{zq^k}{a}\right] \tag{2.3}$$

$$= \sum_{n=0}^{\infty}\frac{(a_1, a_2, \cdots, a_r;q)_n}{(q, b_1, b_2, \cdots, b_s;q)_n}\left((-1)^n q^{\binom{n}{2}}\right)^{s-r+1}\left(\frac{z}{a}\right)^n\left[\psi_q(\lambda-\nu+n) - \psi_q(\lambda+n)\right],$$

$(\Re(\lambda-\nu) > 0; \lambda \neq 0, -1, -2, \cdots; |z| < \infty$ if $r \leq s$; $|z| < |a|$ if $r = s+1)$.

Proof. In order to prove the functional relation (2.3), we begin with the following relation

$$\begin{aligned} &I_q^\alpha\left\{z^{\beta-1}\ {}_r\phi_s\left[\begin{matrix}a_1, a_2, \cdots, a_r\\ b_1, b_2, \cdots, b_s\end{matrix}; q, \frac{z}{a}\right]\right\} \\ &\quad = \frac{z^{\alpha+\beta-1}\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)}\ {}_{r+1}\phi_{s+1}\left[\begin{matrix}a_1, a_2, \cdots, a_r, q^\beta\\ b_1, b_2, \cdots, b_s, q^{\alpha+\beta}\end{matrix}; q, \frac{z}{a}\right], \end{aligned} \tag{2.4}$$

$(\Re(\beta) > 0; \alpha+\beta \neq 0, -1, -2, \cdots; |z| < \infty$ if $r \leq s$; $|z| < |a|$ if $r = s+1)$

which can be obtained immediately from the definition of fractional q-integral (1.14) with permuting the q-integral and the summation, due to the convergence of basic hypergeometric series (1.9), and the relation (2.1). On q-differentiating both sides of (2.4) with respect to β, we get

$$
\begin{aligned}
&I_q^\alpha \left\{ z^{\beta-1} \ln z \; {}_r\phi_s \left[\begin{matrix} a_1, a_2, \cdots, a_r \\ b_1, b_2, \cdots, b_s \end{matrix}; q, \frac{z}{a} \right] \right\} \\
&= \frac{z^{\alpha+\beta-1}\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)} \sum_{n=0}^{\infty} \frac{(a_1, a_2, \cdots, a_r, q^\beta; q)_n}{(q, b_1, b_2, \cdots, b_s, q^{\alpha+\beta}; q)_n} \\
&\times \left((-1)^n q^{\binom{n}{2}} \right)^{s-r+1} \left(\frac{z}{a} \right)^n \left[\ln z + \psi_q(\beta+n) - \psi_q(\alpha+\beta+n) \right], \qquad (2.5) \\
&(\Re(\beta) > 0; \alpha+\beta \neq 0, -1, -2, \cdots; |z| < \infty \text{ if } r \leq s; |z| < |a| \text{ if } r = s+1).
\end{aligned}
$$

Applying the fractional q-integral (1.14) to the left hand side of the functional relation (2.5) with inserting the relation (2.2) would yield

$$
\begin{aligned}
&I_q^\alpha \left\{ z^{\beta-1} \ln z \; {}_r\phi_s \left[\begin{matrix} a_1, a_2, \cdots, a_r \\ b_1, b_2, \cdots, b_s \end{matrix}; q, \frac{z}{a} \right] \right\} \\
&= \frac{z^{\alpha+\beta-1}\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)} \; {}_{r+1}\phi_{s+1} \left[\begin{matrix} a_1, a_2, \cdots, a_r, q^\beta \\ b_1, b_2, \cdots, b_s, q^{\alpha+\beta} \end{matrix}; q, \frac{z}{a} \right] \ln z + \frac{z^{\alpha+\beta-1}\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)} \frac{\ln q}{1-q} \\
&\times \sum_{k=1}^{\infty} \frac{(q^\alpha; q)_k q^{k\beta}}{[k]_q (q^{\alpha+\beta}; q)_k} \; {}_{r+1}\phi_{s+1} \left[\begin{matrix} a_1, a_2, \cdots, a_r, q^\beta \\ b_1, b_2, \cdots, b_s, q^{\alpha+\beta+k} \end{matrix}; q, \frac{zq^k}{a} \right], \qquad (2.6) \\
&(\Re(\beta) > 0; \alpha+\beta \neq 0, -1, -2, \cdots; |z| < \infty \text{ if } r \leq s; |z| < |a| \text{ if } r = s+1)
\end{aligned}
$$

By comparing (2.5) and (2.6), we formally get

$$
\begin{aligned}
&\frac{\ln q}{1-q} \sum_{k=1}^{\infty} \frac{(q^\alpha; q)_k q^{k\beta}}{[k]_q (q^{\alpha+\beta}; q)_k} \; {}_{r+1}\phi_{s+1} \left[\begin{matrix} a_1, a_2, \cdots, a_r, q^\beta \\ b_1, b_2, \cdots, b_s, q^{\alpha+\beta+k} \end{matrix}; q, \frac{zq^k}{a} \right] \\
&= \sum_{n=0}^{\infty} \frac{(a_1, a_2, \cdots, a_r, q^\beta; q)_n}{(q, b_1, b_2, \cdots, b_s, q^{\alpha+\beta}; q)_n} \left((-1)^n q^{\binom{n}{2}} \right)^{s-r+1} \left(\frac{z}{a} \right)^n \\
&\times \left[\psi_q(\beta+n) - \psi_q(\alpha+\beta+n) \right], \qquad (2.7) \\
&(\Re(\beta) > 0; \alpha+\beta \neq 0, -1, -2, \cdots; |z| < \infty \text{ if } r \leq s; |z| < |a| \text{ if } r = s+1).
\end{aligned}
$$

Replacing α and $\alpha+\beta$ by ν and λ, respectively, we get

$$
\begin{aligned}
&\frac{\ln q}{1-q} \sum_{k=1}^{\infty} \frac{(q^\nu; q)_k q^{k(\lambda-\nu)}}{[k]_q (q^\lambda; q)_k} \; {}_{r+1}\phi_{s+1} \left[\begin{matrix} a_1, a_2, \cdots, a_r, q^{\lambda-\nu} \\ b_1, b_2, \cdots, b_s, q^{\lambda+k} \end{matrix}; q, \frac{zq^k}{a} \right] \\
&= \sum_{n=0}^{\infty} \frac{(a_1, a_2, \cdots, a_r, q^{\lambda-\nu}; q)_n}{(q, b_1, b_2, \cdots, b_s, q^\lambda; q)_n} \left((-1)^n q^{\binom{n}{2}} \right)^{s-r+1} \left(\frac{z}{a} \right)^n \\
&\times \left[\psi_q(\lambda-\nu+n) - \psi_q(\lambda+n) \right], \qquad (2.8) \\
&(\Re(\lambda-\nu) > 0; \lambda \neq 0, -1, -2, \cdots; |z| < \infty \text{ if } r \leq s; |z| < |a| \text{ if } r = s+1).
\end{aligned}
$$

Now, multiplying both sides of the functional relation (2.8) by $z^{\lambda-1}$ followed by taking the fractional q-operator $I_q^{-\nu}$ with using the relation (2.1) we obtain the functional relation (2.3) which is the desired result.

3. Special Cases

This section is devoted to introduce some of special cases which are considered q-extension of some ordinary results obtained by some authors.

Setting $r = s + 1 = 2$ and replacing a_1, a_2 and b_1 by α, β and γ, respectively, in the functional relation (2.3) would yield

$$\frac{\ln q}{1-q}\sum_{k=1}^{\infty}\frac{(q^{\nu};q)_k q^{k(\lambda-\nu)}}{[k]_q(q^{\lambda};q)_k}\ {}_3\phi_2\left[\begin{matrix}\alpha,\beta,q^{\lambda}\\ \gamma,q^{\lambda+k}\end{matrix};q,\frac{zq^k}{a}\right]$$
$$=\sum_{n=0}^{\infty}\frac{(\alpha,\beta;q)_n}{(q,\gamma;q)_n}\left(\frac{z}{a}\right)^n\left[\psi_q(\lambda-\nu+n)-\psi_q(\lambda+n)\right], \tag{3.1}$$
$$(\Re(\lambda-\nu)>0;\lambda\neq 0,-1,-2,\cdots;|z|<|a|).$$

In the case of $\beta = \gamma$, we get

$$\frac{\ln q}{1-q}\sum_{k=1}^{\infty}\frac{(q^{\nu};q)_k q^{k(\lambda-\nu)}}{[k]_q(q^{\lambda};q)_k}\ {}_2\phi_1\left[\begin{matrix}\alpha,q^{\lambda}\\ q^{\lambda+k}\end{matrix};q,\frac{zq^k}{a}\right]$$
$$=\sum_{n=0}^{\infty}\frac{(\alpha;q)_n}{(q;q)_n}\left(\frac{z}{a}\right)^n\left[\psi_q(\lambda-\nu+n)-\psi_q(\lambda+n)\right],$$
$$(\Re(\lambda-\nu)>0;\lambda\neq 0,-1,-2,\cdots;|z|<|a|). \tag{3.2}$$

If α and a are replaced by $q^{-\mu}$ and $-a$, respectively, with letting $q \to 1$, we get

$$\sum_{k=1}^{\infty}\frac{(\nu)_k}{k(\lambda)_k}\ {}_2F_1(-\mu,\lambda;\lambda+k;-z/a)$$
$$=\sum_{n=0}^{\infty}\frac{(-\mu)_n}{n!}\left(-\frac{z}{a}\right)^n\left[\psi(\lambda+n)-\psi(\lambda-\nu+n)\right], \tag{3.3}$$
$$(\Re(\lambda-\nu)>0;\lambda\neq 0,-1,-2,\cdots;|z|<|a|)$$

which was derived by Kalla and Al-Saqabi [12]. Here ${}_2F_1(a,b;c;z)$ denotes the Gauss hypergeometric series and $\psi(z)$ is the digamma function defined as

$$\psi(z)=\frac{d}{dz}(\ln\Gamma(z))=\frac{\Gamma'(z)}{\Gamma(z)}\qquad\text{or}\qquad \ln\Gamma(z)=\int_1^z\psi(\xi)d\xi,$$

where $\Gamma(z)$ is the classical gamma function.

Putting $z/a = 1/\alpha = q^{\mu}, \Re(\mu) > 0$ in the relation (3.2), we get

$$\frac{\ln q}{1-q}\sum_{k=1}^{\infty}\frac{(q^{\nu};q)_k q^{k(\lambda-\nu)}}{[k]_q(q^{\lambda};q)_k}\ {}_2\phi_1\left[\begin{matrix}q^{-\mu},q^{\lambda}\\ q^{\lambda+k}\end{matrix};q,q^{\mu+k}\right]$$
$$=\sum_{n=0}^{\infty}\frac{(q^{-\mu};q)_n q^{\mu n}}{(q;q)_n}\left[\psi_q(\lambda-\nu+n)-\psi_q(\lambda+n)\right], \tag{3.4}$$
$$(\Re(\lambda-\nu)>0;\lambda\neq 0,-1,-2,\cdots;\Re(\mu)>0).$$

Using the well known basic Gauss hypergeometric identity [11]

$${}_2\phi_1(a,b;c;q,c/(ab))=\frac{(c/a;q)_\infty(c/b;q)_\infty}{(c;q)_\infty(c/(ab);q)_\infty},\qquad |c/(ab)|<1 \tag{3.5}$$

and the identities (1.5) and (1.8) we obtain

$$\frac{\Gamma_q(\lambda)\Gamma_q(\mu)}{\Gamma_q(\lambda+\mu)}\left({}_2\phi_1\left[\begin{matrix} q^{\mu}, q^{\nu} \\ q^{\lambda+\mu} \end{matrix}; q, q^{\lambda-\nu}\right] - 1\right)$$
$$= \frac{1-q}{\ln q}\sum_{n=0}^{\infty}\frac{(q^{-\mu};q)_n q^{\mu n}}{(q;q)_n}\left[\psi_q(\lambda-\nu+n)-\psi_q(\lambda+n)\right], \tag{3.6}$$
$$(\Re(\lambda-\nu)>0; \lambda\neq 0,-1,-2,\cdots; \Re(\mu)>0).$$

Again, using (3.5) and (1.8) give

$$\Gamma_q(\mu)\left(\frac{\Gamma_q(\lambda-\nu)}{\Gamma_q(\lambda+\mu-\nu)} - \frac{\Gamma_q(\lambda)}{\Gamma_q(\lambda+\mu)}\right)$$
$$= \frac{1-q}{\ln q}\sum_{n=0}^{\infty}\frac{(q^{-\mu};q)_n q^{\mu n}}{(q;q)_n}\left[\psi_q(\lambda-\nu+n)-\psi_q(\lambda+n)\right], \tag{3.7}$$
$$(\Re(\lambda-\nu)>0; \lambda\neq 0,-1,-2,\cdots; \Re(\mu)>0).$$

Notice that the limit of the left hand side as $\mu\to 0$ tends to $\frac{1-q}{\ln q}[\psi_q(\lambda-\nu)-\psi_q(\lambda)]$ which is the first term of the right hand side.
Now taking $\nu=-m,\ m\in\mathbb{N}$ with using the well-known identity [7]

$$\Gamma_q(z+m) = \frac{(q^z;q)_m}{(1-q)^m}\Gamma_q(z) \tag{3.8}$$

and its logarithmic derivative

$$\psi_q(z+m) = \psi_q(z) - \frac{\ln q}{1-q}\sum_{k=0}^{m-1}\frac{q^{k+z}}{[k+z]_q}, \quad m\in\mathbb{N}, \tag{3.9}$$

after simplification by (1.5), (1.8) and (3.5) would yield

$$\frac{\Gamma_q(\lambda)\Gamma_q(\mu)}{\Gamma_q(\lambda+\mu)}\left(1-\frac{(q^{\lambda};q)_m}{(q^{\lambda+\mu};q)_m}\right) = \frac{\Gamma_q(\lambda+1)\Gamma_q(\mu+1)}{\Gamma_q(\lambda+\mu+1)}\sum_{k=0}^{m-1}\frac{(q^{\lambda+1};q)_k}{(q^{\lambda+\mu+1};q)_k}\frac{q^{\lambda+k}}{[\lambda+k]_q},$$
$$(\Re(\lambda)>-m; \lambda\neq 0,-1,-2,\cdots,1-m; m\in\mathbb{N}; \Re(\mu)>0) \tag{3.10}$$

which can be rewritten as

$$\sum_{k=0}^{m-1}\frac{(q^{\lambda+1};q)_k}{(q^{\lambda+\mu+1};q)_k}\frac{q^{\lambda+k}}{[\lambda+k]_q} = \frac{[\lambda+\mu]_q}{[\lambda]_q[\mu]_q}\left(1-\frac{(q^{\lambda};q)_m}{(q^{\lambda+\mu};q)_m}\right), \tag{3.11}$$
$$(\Re(\lambda)>-m; \lambda\neq 0,-1,-2,\cdots,1-m; m\in\mathbb{N}; \Re(\mu)>0).$$

If $m\to\infty$, we get

$$\sum_{k=0}^{\infty}\frac{(q^{\lambda+1};q)_k}{(q^{\lambda+\mu+1};q)_k}\frac{q^{\lambda+k}}{[\lambda+k]_q} = \frac{[\lambda+\mu]_q}{[\lambda]_q[\mu]_q}\left(1-\frac{(q^{\lambda};q)_\infty}{(q^{\lambda+\mu};q)_\infty}\right), \tag{3.12}$$
$$(\lambda\neq 0,-1,-2,\cdots; \Re(\mu)>0),$$

or equivalently

$$\sum_{k=0}^{\infty}\frac{(q^{\lambda+1};q)_k}{(q^{\lambda+\mu+1};q)_k}\frac{q^{\lambda+k}}{[\lambda+k]_q} = \frac{[\lambda+\mu]_q}{[\lambda]_q[\mu]_q} - \frac{(1-q)^{\mu}\Gamma_q(\lambda+\mu+1)}{[\mu]_q\Gamma_q(\lambda+1)}, \tag{3.13}$$
$$(\lambda\neq 0,-1,-2,\cdots; \Re(\mu)>0).$$

When letting $q \to 1$, we have

$$\sum_{k=0}^{\infty} \frac{(\lambda+1)_k}{(\lambda+\mu+1)_k} \frac{1}{\lambda+k} = \frac{\lambda+\mu}{\lambda\mu}, \quad (\lambda \neq 0, -1, -2, \cdots; \Re(\mu) > 0). \tag{3.14}$$

This combinatorial series identity is shown to be new.

Returning to functional relation (3.7) with setting $\nu = -1$ and using (1.7) and (1.13), we get

$$\sum_{n=0}^{\infty} (-1)^n \begin{bmatrix} \mu \\ n \end{bmatrix}_q \frac{q^{\frac{n(n+1)}{2}}}{[\lambda+n]_q} = B_q(\mu+1, \lambda), \quad (\Re(\lambda) > -1; \lambda \neq 0; \Re(\mu) > 0). \tag{3.15}$$

The q-combinatorial series identity (3.15) tends to the combinatorial series identity ([20], p.102, Eq.8) when letting $q \to 1$.

Another special case can be obtained by setting $\gamma = q^{\lambda}, \alpha = q^{\eta}$ and $\beta = q^{\lambda-\eta}$ in (3.1) with taking the limit as $z \to a$ in the resulting equation as

$$\begin{aligned} &\frac{\ln q}{1-q} \sum_{k=1}^{\infty} \frac{(q^{\nu}; q)_k q^{k(\lambda-\nu)}}{[k]_q (q^{\lambda}; q)_k} \, {}_2\phi_1 \begin{bmatrix} q^{\eta}, q^{\lambda-\eta} & \\ & ; q, q^k \\ q^{\lambda+k} & \end{bmatrix} \\ &= \sum_{n=0}^{\infty} \frac{(q^{\eta}, q^{\lambda-\eta}; q)_n}{(q, q^{\lambda}; q)_n} \left[\psi_q(\lambda-\nu+n) - \psi_q(\lambda+n)\right], \\ &(\Re(\lambda-\nu) > 0; \lambda \neq 0, -1, -2, \cdots). \end{aligned} \tag{3.16}$$

Using the well-known basic Gauss hypergeometric identity (3.5) we obtain

$$\begin{aligned} &\frac{\Gamma_q(\lambda) \ln q}{\Gamma_q(\eta)\Gamma_q(\lambda-\eta)} \sum_{k=1}^{\infty} \frac{(q^{\nu}; q)_k (q; q)_{k-1} q^{k(\lambda-\nu)}}{[k]_q (q^{\eta}; q)_k (q^{\lambda-\eta}; q)_k} \\ &= \sum_{n=0}^{\infty} \frac{(q^{\eta}, q^{\lambda-\eta}; q)_n}{(q, q^{\lambda}; q)_n} \left[\psi_q(\lambda-\nu+n) - \psi_q(\lambda+n)\right], \\ &(\Re(\lambda-\nu) > 0; \lambda \neq 0, -1, -2, \cdots). \end{aligned} \tag{3.17}$$

The derivative of a function can, in principle, be computed from the definition by considering the difference quotient, and computing its limit. Therefore, we can deduce that

$$\psi'_q(\lambda) = \lim_{\nu \to 0} \frac{\psi_q(\lambda) - \psi_q(\lambda-\nu)}{\nu}. \tag{3.18}$$

Due to differentiability of the q-digamma function, the limit in the right hand side of the previous equation exists for $\lambda \neq 0, -1, -2, \cdots$. Also, it is easy to show that

$$\lim_{\nu \to 0} \frac{(q^{\nu}; q)_k}{\nu} = -(q; q)_{k-1} \ln q, \qquad (k \in \mathbb{N}). \tag{3.19}$$

Dividing the functional relation (2.3) by ν followed by taking the limit as $\nu \to 0$ with using (3.18) and (3.19) we get

$$\left(\frac{\ln q}{1-q}\right)^2 \sum_{k=1}^{\infty} \frac{(q;q)_k q^{\lambda k}}{[k]_q^2 (q^\lambda;q)_k} \; {}_{r+1}\phi_{s+1}\left[\begin{matrix} a_1, a_2, \cdots, a_r, q^\lambda \\ b_1, b_2, \cdots, b_s, q^{\lambda+k} \end{matrix}; q, \frac{zq^k}{a}\right]$$
$$= \sum_{n=0}^{\infty} \frac{(a_1, a_2, \cdots, a_r; q)_n}{(q, b_1, b_2, \cdots, b_s; q)_n} \left((-1)^n q^{\binom{n}{2}}\right)^{s-r+1} \left(\frac{z}{a}\right)^n \psi_q'(\lambda+n), \qquad (3.20)$$
$$(\Re(\lambda) > 0; |z| < \infty \text{ if } r \le s; |z| < |a| \text{ if } r = s+1).$$

When putting $\alpha = 1$ in (3.2), then we have

$$\frac{\ln q}{1-q} \sum_{n=1}^{\infty} \frac{(q^\nu;q)_n}{[n]_q (q^\lambda;q)_n} q^{(\lambda-\nu)n} = \psi_q(\lambda-\nu) - \psi_q(\lambda), \qquad (3.21)$$
$$(\Re(\lambda-\nu) > 0,\ \lambda \ne 0, -1, -2\cdots,\ |q| < 1).$$

which is considered the q-analogue of the well-known result in the theory of the digamma function [1]

$$\sum_{n=1}^{\infty} \frac{(\nu)_n}{n(\lambda)_n} = \psi(\lambda) - \psi(\lambda-\nu), \quad (\Re(\lambda-\nu) > 0, \lambda \ne 0, -1, -2\cdots). \quad (3.22)$$

All of the previous special cases mentioned in this section can also be used to establish certain classes of infinite series associated with the derivative of the q-digamma function. The functional relation (3.21) can give the closed form expression for derivative of the q-digamma function which is so-called q-trigamma function

$$\psi_q'(\lambda) = \left(\frac{\ln q}{1-q}\right)^2 \sum_{k=1}^{\infty} \frac{(q;q)_k q^{\lambda k}}{[k]_q^2 (q^\lambda;q)_k}, \qquad (\Re(\lambda) > 0,\ |q| < 1). \qquad (3.23)$$

The functional relations (3.21) and (3.23) can be rewritten with $\Re(\lambda-\nu) > 0$ as

$${}_3\phi_2(q^{\nu+1}, q, q; q^{\lambda+1}, q^2; q, q^{\lambda-\nu}) = \begin{cases} \dfrac{1-q}{\ln q} \dfrac{q^{\nu-\lambda}[\lambda]_q}{[\nu]_q} [\psi_q(\lambda-\nu) - \psi_q(\lambda), & \nu \ne 0 \\ \left(\dfrac{1-q}{\ln q}\right)^2 q^{-\lambda}[\lambda]_q \psi_q'(\lambda), & \nu = 0. \end{cases} \qquad (3.24)$$

When letting $q \to 1$, we get

$${}_3F_2(\nu+1, 1, 1; \lambda+1, 2; 1) = \begin{cases} \dfrac{\lambda}{\nu}[\psi(\lambda) - \psi(\lambda-\nu)], & \nu \ne 0 \\ \lambda\psi'(\lambda), & \nu = 0. \end{cases} \qquad (3.25)$$

which was proven, by using a simple technique involving IHôpitals theorem on limits, by Luke ([14], p.111) see also ([6], Eq.1.5, p.362).

Acknowledgement

The author is grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

References

[1] M. Abramowitz and C.A. Stegun, Handbook of Mathematical functions with formulas, Graphs, Mathematical tables 7th printing, Applied Mathematics Series, Vol. 55, Nathional Bureau of standards, Washington, DC, 1964.

[2] R.P. Agarwal, Fractional q-derivatives and q-integrals and certain hypergeometric transformations, Ganita, Vol.27, 25-32, 1976.

[3] R.P. Agarwal, Certain fractional q-integrals and q-derivatives, Proc. Camb. Phil. Soc., Vol.66, 365-370, 1969.

[4] W.A. Al-Salam, A. Verma, A fractional Leibniz q-formula, Pacific Journal of Mathematics, Vol.60, 1-9, 1975.

[5] W.A. Al-Salam, Some fractional q-integrals and q-derivatives, Proceedings of the Edinburgh Mathematical Society, Vol.15, 135-140, 1966.

[6] B.N. Al-Saqabi, S.L. Kalla, H.M. Srivastava, A certain family of infinite series associated with Digamma functions, J. Math. Anal. Appl., Vol.159, 361-372, 1991.

[7] G. Andrews, R. Askey and R. Roy, Special Functions, Cambridge university press, Cambridge, 1999.

[8] J. Aular de Durán, S.L. Kalla, H.M. Srivastava, Fractional calculus and the sums of certain families of infinite series, J. Math. Anal. Appl., Vol.190, 738-754, 1995.

[9] K.Y. Chen and H.M. Srivastava, Some Infinite Series and Functional Relations That Arose in the Context of Fractional Calculus, J. Math. Anal. Appl., Vol.252, 376-388, 2000.

[10] J. Choi, A class of infinite series summable by means of fractional calculus, Comm. Korean Math. Soc., Vol.11, 139-145, 1996.

[11] G. Gasper and M. Rahman, Basic Hypergeometric Series, Cambridge University Press, Cambridge, 2004.

[12] S.L. Kalla, B.N. Al-Saqabi, A functional relation involving ψ-funclion, Rev. Tk. Ing. Univ. Zulia, Vol.29, 31-35, 1985.

[13] C. Krattenthaler, H.M. Srivastava, Summations for basic hypergeometric series involving a q-analogue of the digamma function, Computers Math. Applic., Vol.32(2), 73-91, 1996.

[14] Y.L. Luke, The Special Functions and Their Approximations, Vol.I, Academic Press, New York/San Francisco/London, 1969.

[15] K. Nishimoto, R.K. Saxena, An application of Riemann-Liouville operator in the unification of certain functional relations, J. College Engrg. Nihon Univ. Ser., Vol.32(B), 133-139, 1991.

[16] K. Nishimoto, H.M. Srivastava, Certain classes of infinite series summable by means of fractional calculus, J. College Engrg. Nihon Univ. Ser., Vol.30(B), 97-106, 1989.

[17] A. Salem, Some Properties and Expansions Associated with q-Digamma Function, Quaestiones mathematicae, (To appear).

[18] A. Salem, Some applications of fractional q-calculus and fractional q-Leibniz rule, Journal of Fractional Calculus and Applications, Vol.2(4), 1-11, 2012.

[19] H.M. Srivastava, A simple algorithm for the evaluation of a class of generalized hypergeometric series, Stud. Appl. Math., Vol.86, 79-86, 1992.

[20] T.C. Wu, S.T. TU, H.M. Srivastava, Some Combinatorial Series Identities Associated with the Digamma Function and Harmonic Numbers, Appl. Math. Letters, Vol.13, 101-106, 2000.

[21] T.C. Wu, S.H. Leu, S.T. Tu, H.M. Srivastava, A certain class of infinite sums associated with Digamma functions, Applied Mathematics and Computation, Vol.105, 1-9, 1999.

Ahmed Salem

Department of Mathematics, Faculty of Science, Al Jouf University, Sakaka, Al Jouf, Kingdom of Saudi Arabia.

E-mail address: `ahmedsalem74@hotmail.com`