ON A NEW SUBCLASS OF HARMONIC UNIVALENT FUNCTIONS DEFINED BY FRACTIONAL CALCULUS OPERATOR

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Abstract. The purpose of the present paper is to establish some results involving coefficient conditions, distortion bounds, extreme points, convolution, convex combinations and neighborhoods for a new class of harmonic univalent functions in the open unit disc. We also discuss a class preserving integral operator. Relevant connections of the results presented here with various known results are briefly indicated.

1. Introduction

A continuous complex-valued function $f = u + iv$ is said to be harmonic in a simply connected domain $D$ if both $u$ and $v$ are real harmonic in $D$. In any simply connected domain we can write $f = h + g$ where $h$ and $g$ are analytic in $D$. We call $h$ the analytic part and $g$ the co-analytic part of $f$. A necessary and sufficient condition for $f$ to be locally univalent and sense-preserving in $D$ is that $|h'(z)| > |g'(z)|$, $z \in D$. See Clunie and Sheil-Small [4], (see also [7], [12], [13]).

Denote by $S_H$ the class of functions $f = h + g$ that are harmonic univalent and sense-preserving in the open unit disk $U = \{ z : |z| < 1 \}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + g \in S_H$ we may express the analytic functions $h$ and $g$ as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad |b_1| < 1.$$  \hspace{1cm} (1)

Note that $S_H$ reduces to the class $S$ of normalized analytic univalent functions if the co-analytic part of its member is zero. For this class the function $f(z)$ may be expressed as

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$  \hspace{1cm} (2)

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A function $f$ of the form (1) is said to be harmonic starlike of order $\alpha$, $(0 \leq \alpha < 1)$ for $|z| = r < 1$, if
\[
\frac{\partial}{\partial \theta} (\arg f(re^{i\theta})) = \text{Re} \left\{ \frac{zh'(z) - zg'(z)}{h(z) + g(z)} \right\} > \alpha.
\]

The class of all harmonic starlike functions of order $\alpha$ is denoted by $S^*_H(\alpha)$ and extensively studied by Jahangiri [8]. The case $\alpha = 0$ and $\alpha = b_1 = 0$ were studied by Silverman and Silvia [17] and Silverman [16], (see also [3]). In [8] Jahangiri proved that the coefficient condition
\[
\sum_{k=2}^{\infty} \frac{k - \alpha}{1 - \alpha} |a_k| + \sum_{k=1}^{\infty} \frac{k + \alpha}{1 - \alpha} |b_k| \leq 1
\]
is sufficient condition for functions $f = h + \overline{g}$ to be harmonic starlike of order $\alpha$. If we put $\alpha = 0$ in above inequalities then we obtain sufficient condition for function $f = h + \overline{g}$ belonging to the class $S^*_H$ of harmonic starlike functions.

Further, we denote by $V_H$ the subclass of $S_H$ consisting of functions of form $f = h + \overline{g}$, where
\[
h(z) = z + \sum_{k=2}^{\infty} |a_k| z^k, \quad g(z) = (-1)^n \sum_{k=1}^{\infty} |b_k| z^k, \quad |b_1| < 1.
\]

2. Fractional Calculus

Let $L(a,b)$ consists of Lebesgue measurable real or complex valued function $f(x)$ on $[a, b]$:
\[
L(a, b) = \left \{ f : ||f||_1 = \int_a^b |f(t)|dt < +\infty \right \}.
\]

**Definition 1** (see [10], page 79). Let $f(x) \in L(a, b)$, $\alpha \in C$, $\text{Re}(\alpha) > 0$, then
\[
a_D^\alpha f(x) = aD^{-\alpha}_x f(x) = I^{-\alpha}_a f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a,
\]
is called the Riemann-Liouville left-sided fractional integral of order $\alpha$.

**Definition 2** (see [10], page 84). The left-sided Riemann-Liouville fractional derivative of order $\alpha \in C$, $\text{Re}(\alpha) \geq 0$ of the function $f(x)$ is defined by
\[
(aD^\alpha_x f)(x) = (D^\alpha_{a+} f)(x) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dx} \right)^n \int_a^x \frac{f(t)}{(x-t)^{n+1}} dtd, \quad n = [\text{Re}(\alpha)] + 1; \quad x > a,
\]
where $[\text{Re}(\alpha)]$ means the integral part of $\text{Re}(\alpha)$.

The following definitions of fractional derivatives and fractional integrals are due to Owa [11] and Srivastava and Owa [18].

**Definition 3.** The fractional integral of order $\lambda$ is defined for a function $f(z)$ of the form (2) by
\[
D^{-\lambda}_z f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\xi)}{(z-\xi)^{1-\lambda}} d\xi,
\]
where $\lambda > 0$, $f(z)$ is an analytic functions in a simply connected region of the $z$-plane containing the origin and the multiplicity of $(z-\xi)^{\lambda-1}$ is removed by requiring $\log(z-\xi)$ to be real when $z > 0$.

It is easy to see that the Definition 3 is a particular case of Definition 1 for $a = 0$. 

Definition 4. The fractional derivative of order $\lambda$ is defined for a function $f(z)$ of the form (2) by

$$D_\lambda^z f(z) = \frac{1}{\Gamma(1 - \lambda)} \frac{d}{dz} \int_0^z \frac{f(\xi) (z - \xi)^{-\lambda}}{d\xi},$$

where $0 \leq \lambda < 1$, $f(z)$ is an analytic functions in a simply connected region of the $z$-plane containing the origin and the multiplicity of $(z - \xi)^{-\lambda}$ is removed as in Definition 3 above.

It is easy to see that the Definition 4 is a particular case of Definition 2 for $a = 0$ and $0 \leq \alpha < 1$.

Very recently, Dixit and Porwal [5] introduce a new fractional derivative operator for function of the form (2) as follows

$$\Omega^n f(z) = \left(\Omega^{n-1} f(z)\right).$$

Thus, we note that

$$\Omega^n f(z) = z + \sum_{k=2}^{\infty} [\phi(k, \lambda)]^n a_k z^k,$$

where

$$\phi(k, \lambda) = \frac{\Gamma(k + 1)\Gamma(1 - \lambda)}{\Gamma(k - \lambda)}.$$

It is interesting to note that for $\lambda = 0$, $\Omega^n f(z)$ reduces to familiar Salagean operator introduced by Salagean in [15].

From the motivation of the definition of modified Salagean operator defined by Jahangiri et al. [9] for function of the form $f = h + \bar{g}$, where $h$ and $g$ are the form (1) as follows

$$D^n f(z) = D^n h(z) + (-1)^n \overline{D^n g(z)}.$$

Now, we define

$$\Omega^n f(z) = \Omega^n h(z) + (-1)^n \overline{\Omega^n g(z)}$$

where

$$\Omega^n h(z) = z + \sum_{k=2}^{\infty} [\phi(k, \lambda)]^n a_k z^k$$

and

$$\Omega^n g(z) = \sum_{k=1}^{\infty} [\phi(k, \lambda)]^n b_k z^k.$$

Now, we let $R_H(n, \beta, \lambda)$ denote the subclass $S_H$ consisting of functions $f = h + \bar{g}$ of the form (1) that satisfy the condition

$$\text{Re} \left\{ \frac{\Omega^n h(z) + (-1)^n \overline{\Omega^n g(z)}}{z} \right\} < \beta,$$

for some $\beta(1 < \beta \leq 2)$, $0 \leq \lambda \leq 1$, $n \in N$ and $z \in U$.

We further let $R_H(n, \beta, \lambda)$ denote the subclass of $R_H(n, \beta, \lambda)$ consisting of functions $f = h + \bar{g} \in S_H$ such that $h$ and $g$ are of the form (3).
We note that for \( n = 1, \lambda = 0 \) and \( g \equiv 0 \) the class \( R_H(n, \beta, \lambda) \) reduces to the class \( R(\beta) \) studied by Uralegaddi et al. [19], (see also [6]).

In the present paper, we study the coefficient bounds, distortion bounds, extreme points, convolution condition, convex combinations, neighborhood problems and discuss a class preserving integral operator.

3. Main Results

First, we give a sufficient coefficient condition for functions in \( R_H(n, \beta, \lambda) \).

**Theorem 1.** Let \( f = h + g \) be such that \( h \) and \( g \) are given by (1). Furthermore, let

\[
\sum_{k=2}^{\infty} |\phi(k, \lambda)|^n |a_k| + \sum_{k=1}^{\infty} |\phi(k, \lambda)|^n |b_k| \leq \beta - 1.
\]

Then \( f \) is sense-preserving, harmonic univalent in \( U \) and \( f \in R_H(n, \beta, \lambda) \).

**Proof.** If \( z_1 \neq z_2 \), then

\[
\left| \frac{f(z_1) - f(z_2)}{h(z_1) - h(z_2)} \right| \geq 1 - \frac{|g(z_1) - g(z_2)|}{|h(z_1) - h(z_2)|} \geq 1 - \frac{1}{1 - \sum_{k=1}^{\infty} k|a_k|} \geq 1 - \frac{1}{1 - \sum_{k=2}^{\infty} k|a_k|} \geq 1 - \frac{1}{1 - \sum_{k=2}^{\infty} \frac{|\phi(k, \lambda)|^n}{\beta - 1}|a_k|} \geq 0,
\]

which proves univalence.
Note that $f$ is sense-preserving in $U$. This is because
\[
|h'(z)| \geq 1 - \sum_{k=2}^{\infty} k|a_k||z|^{k-1} \\
> 1 - \sum_{k=2}^{\infty} k|a_k| \\
\geq 1 - \sum_{k=2}^{\infty} \frac{[\phi(k, \lambda)]^n}{\beta - 1}|a_k| \\
\geq \sum_{k=1}^{\infty} \frac{[\phi(k, \lambda)]^n}{\beta - 1}|b_k| \\
\geq \sum_{k=1}^{\infty} k|b_k| \\
> \sum_{k=1}^{\infty} k|b_k||z|^{k-1} \\
\geq |g'(z)|.
\]

Now, we show that $f \in R_H(n, \beta, \lambda)$. Using the fact that $Re \omega < \beta$, if and only if, $|\omega - 1| < |\omega + 1 - 2\beta|$, it suffices to show that
\[
\frac{\Omega^n h(z) + (-1)^n \Omega^n g(z)}{z} - 1 < 1, \quad z \in U.
\]

We have
\[
\frac{z + \sum_{k=2}^{\infty} [\phi(k, \lambda)]^n a_k z^k + (-1)^n \sum_{k=1}^{\infty} [\phi(k, \lambda)]^n b_k z^k - 1}{z + \sum_{k=2}^{\infty} [\phi(k, \lambda)]^n a_k z^k + (-1)^n \sum_{k=1}^{\infty} [\phi(k, \lambda)]^n b_k z^k - (2\beta - 1)}
\]
\[
= \frac{\sum_{k=2}^{\infty} [\phi(k, \lambda)]^n a_k z^{k-1} + (-1)^n \frac{\sum_{k=1}^{\infty} [\phi(k, \lambda)]^n b_k z^{k-1}}{\sum_{k=2}^{\infty} [\phi(k, \lambda)]^n a_k z^k} - (2\beta - 1)}{2(\beta - 1) - \sum_{k=2}^{\infty} [\phi(k, \lambda)]^n a_k z^{k-1} + (-1)^n \frac{\sum_{k=1}^{\infty} [\phi(k, \lambda)]^n b_k z^{k-1}}{\sum_{k=2}^{\infty} [\phi(k, \lambda)]^n a_k z^k}}
\]
\[
\leq \frac{\sum_{k=2}^{\infty} [\phi(k, \lambda)]^n a_k |z|^{k-1} + \sum_{k=1}^{\infty} [\phi(k, \lambda)]^n |b_k| |z|^{k-1}}{2(\beta - 1) - \sum_{k=2}^{\infty} [\phi(k, \lambda)]^n a_k |z|^{k-1} + \sum_{k=1}^{\infty} [\phi(k, \lambda)]^n |b_k| |z|^{k-1}}
\]
\[
= \frac{\sum_{k=2}^{\infty} [\phi(k, \lambda)]^n a_k |x_k| + \sum_{k=1}^{\infty} [\phi(k, \lambda)]^n |y_k|}{2(\beta - 1) - \sum_{k=2}^{\infty} [\phi(k, \lambda)]^n a_k |x_k| + \sum_{k=1}^{\infty} [\phi(k, \lambda)]^n |y_k|}
\]
which is bounded above by 1 by using (6) and so the proof is complete.

The harmonic univalent functions of the form
\[
f(z) = z + \sum_{k=2}^{\infty} \frac{\beta - 1}{\phi(k, \lambda)} |x_k| z^k + \sum_{k=1}^{\infty} \frac{\beta - 1}{\phi(k, \lambda)} |y_k| z^k,
\]
where $1 < \beta \leq 2$, $0 \leq \lambda \leq 1$, $n \in N$ and $\sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 1$, show that the coefficient bound given by (6) is sharp. It is worthy to note that the function
of the form (7) belongs to the class $R_H(n, \beta, \lambda)$ for all \( \sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| \leq 1 \) because coefficient inequality (6) holds.

**Theorem 2.** Let \( f_n \) be given by (3). Then \( f_n \in R_H(n, \beta, \lambda) \) if and only if
\[
\sum_{k=2}^{\infty} [\phi(k, \lambda)]^n |a_k| + \sum_{k=1}^{\infty} [\phi(k, \lambda)]^n |b_k| \leq \beta - 1.
\]

**Proof.** Since \( \overline{R_H(n, \beta, \lambda)} \subset R_H(n, \beta, \lambda) \), we only need to prove the "only if" part of the theorem. To this end, for functions \( f_n \) of the form (3), we notice that the condition
\[
\text{Re}\left\{ \Omega^n h(z) + (-1)^n \Omega^n g(z) \right\} < \beta
\]
is equivalent to
\[
\text{Re}\left\{ 1 + \sum_{k=2}^{\infty} [\phi(k, \lambda)]^n a_k z^{k-1} + (-1)^n \frac{1}{z} \sum_{k=1}^{\infty} [\phi(k, \lambda)]^n b_k z^{k-1} \right\}
\]
\[
\leq 1 + \sum_{k=2}^{\infty} [\phi(k, \lambda)]^n |a_k| |z|^{k-1} + \sum_{k=1}^{\infty} [\phi(k, \lambda)]^n |b_k| |z|^{k-1} < \beta, \quad z \in U.
\]

The above condition must hold for all values of \( z, |z| = r < 1 \). Upon choosing the values of \( z \) to be real and let \( z \to 1^- \), we obtain
\[
\sum_{k=2}^{\infty} [\phi(k, \lambda)]^n |a_k| + \sum_{k=1}^{\infty} [\phi(k, \lambda)]^n |b_k| \leq \beta - 1,
\]
which is the required condition.

The harmonic univalent functions of the form
\[
f_n(z) = z + \sum_{k=2}^{\infty} \frac{\beta - 1}{\phi(k, \lambda)} x_k z^k + (-1)^n \sum_{k=1}^{\infty} \frac{\beta - 1}{\phi(k, \lambda)} y_k \overline{z^k},
\]
where \( 1 < \beta \leq 2, 0 \leq \lambda \leq 1, n \in \mathbb{N}, x_k \geq 0, y_k \geq 0 \) and \( \sum_{k=2}^{\infty} x_k + \sum_{k=1}^{\infty} y_k \leq 1 \) belongs to the class \( \overline{R_H(n, \beta, \lambda)} \).

**Theorem 3.** If \( f \in \overline{R_H(n, \beta, \lambda)} \), then
\[
|f(z)| \leq (1 + |b_1|)r + \left( \frac{1 - \lambda}{2} \right)^n (\beta - 1 - |b_1|) r^2, \quad |z| = r < 1
\]
and
\[
|f(z)| \geq (1 - |b_1|)r - \left( \frac{1 - \lambda}{2} \right)^n (\beta - 1 - |b_1|) r^2, \quad |z| = r < 1.
\]
Proof. Let \( f \in \overline{R_H}(n, \beta, \lambda) \). Taking the absolute value of \( f \), we have

\[
|f(z)| \leq (1 + |b_1|)r + \sum_{k=2}^{\infty}(|a_k| + |b_k|)r^k
\]

\[
\leq (1 + |b_1|)r + \sum_{k=2}^{\infty}(|a_k| + |b_k|)r^2
\]

\[
\leq (1 + |b_1|)r + (\frac{1-\lambda}{2})^{\infty} \sum_{k=2}^{\infty} \left( \frac{2}{1-\lambda} \right)^n (|a_k| + |b_k|)r^2
\]

\[
\leq (1 + |b_1|)r + (\frac{1-\lambda}{2})^{\infty} \sum_{k=2}^{\infty} [\phi(k, \lambda)]^n(|a_k| + |b_k|)r^2
\]

\[
\leq (1 + |b_1|)r + (\frac{1-\lambda}{2})^{\infty} (\beta - 1 - |b_1|)r^2
\]

and

\[
|f(z)| \geq (1 - |b_1|)r - \sum_{k=2}^{\infty}(|a_k| + |b_k|)r^k
\]

\[
\geq (1 - |b_1|)r - \sum_{k=2}^{\infty}(|a_k| + |b_k|)r^2
\]

\[
\geq (1 - |b_1|)r - (\frac{1-\lambda}{2})^{\infty} \sum_{k=2}^{\infty} \left( \frac{2}{1-\lambda} \right)^n (|a_k| + |b_k|)r^2
\]

\[
\geq (1 - |b_1|)r - (\frac{1-\lambda}{2})^{\infty} \sum_{k=2}^{\infty} [\phi(k, \lambda)]^n(|a_k| + |b_k|)r^2
\]

\[
\geq (1 - |b_1|)r - (\frac{1-\lambda}{2})^{\infty} (\beta - 1 - |b_1|)r^2.
\]

Theorem 4. Let \( f \in \text{cloc} \overline{R_H}(n, \beta, \lambda) \), if and only if

\[
f(z) = \sum_{k=1}^{\infty} (\lambda_k h_k(z) + \gamma_k g_k(z)),
\]

where \( h_1(z) = z \)

\[
h_k(z) = z + \frac{\beta - 1}{|\phi(k, \lambda)|^n} z^k, \quad (k = 2, 3, \ldots)
\]

\[
g_k(z) = z + (-1)^n \frac{\beta - 1}{|\phi(k, \lambda)|^n} z^k, \quad (k = 1, 2, 3, \ldots)
\]

and \( \sum_{k=1}^{\infty} (\lambda_k + \gamma_k) = 1, \lambda_k \geq 0 \) and \( \gamma_k \geq 0 \).

In particular the extreme points of \( \overline{R_H}(n, \beta, \lambda) \) are \( \{h_k\} \) and \( \{g_k\} \).

Proof. For functions \( f \) of the form (9) we may write

\[
f(z) = \sum_{k=1}^{\infty} \{\lambda_k h_k(z) + \gamma_k g_k(z)\}
\]

\[
= z + \sum_{k=2}^{\infty} \left( \frac{\beta - 1}{|\phi(k, \lambda)|^n} \right) \lambda_k z^k + (-1)^n \sum_{k=1}^{\infty} \left( \frac{\beta - 1}{|\phi(k, \lambda)|^n} \right) \gamma_k z^k.
\]
Then
\[
\sum_{k=2}^{\infty} \frac{[\phi(k, \lambda)]^n}{\beta - 1} \left( \frac{\beta - 1}{[\phi(k, \lambda)]^{n-1}} \lambda_k \right) + \sum_{k=1}^{\infty} \frac{[\phi(k, \lambda)]^n}{\beta - 1} \left( \frac{\beta - 1}{[\phi(k, \lambda)]^{n-1}} \gamma_k \right)
\]
\[
= \sum_{k=2}^{\infty} \lambda_k + \sum_{k=1}^{\infty} \gamma_k
\]
\[
= 1 - \lambda_1 \leq 1,
\]
and so \( f \in \text{clco} \overline{R_H(n, \beta, \lambda)} \).

Conversely, suppose that \( f \in \text{clco} \overline{R_H(n, \beta, \lambda)} \).

Set
\[
\lambda_k = \frac{[\phi(k, \lambda)]^n}{\beta - 1} |a_k|, \quad (k = 2, 3, 4, ...)
\]
and
\[
\gamma_k = \frac{[\phi(k, \lambda)]^n}{\beta - 1} |b_k|, \quad (k = 1, 2, 3, ...).
\]

Then note that by Theorem 2,
\[
0 \leq \lambda_k \leq 1, \quad (k = 2, 3, 4, ...)
\]
and
\[
0 \leq \gamma_k \leq 1, \quad (k = 1, 2, 3, ...).
\]

We define \( \lambda_1 = 1 - \sum_{k=2}^{\infty} \lambda_k - \sum_{k=1}^{\infty} \gamma_k \) and note that by Theorem 2, \( \lambda_1 \geq 0 \).

Consequently, we obtain \( f(z) = \sum_{k=1}^{\infty} \{ \lambda_k h_k(z) + \gamma_k g_k(z) \} \) as required.

**Theorem 5.** \( \overline{R_H(n, \beta, \lambda)} \subseteq S_H^* \) where \( n \in N, 1 < \beta \leq 2, 0 \leq \lambda < 1 \).

**Proof.** Let \( f \in \overline{R_H(n, \beta, \lambda)} \).

Then by Theorem 2, we have
\[
\sum_{k=2}^{\infty} \frac{[\phi(k, \lambda)]^n}{\beta - 1} |a_k| + \sum_{k=1}^{\infty} \frac{[\phi(k, \lambda)]^n}{\beta - 1} |b_k| \leq 1. \tag{10}
\]

Now
\[
\sum_{k=2}^{\infty} k|a_k| + \sum_{k=1}^{\infty} k|b_k|
\]
\[
\leq \sum_{k=2}^{\infty} \frac{[\phi(k, \lambda)]^n}{\beta - 1} |a_k| + \sum_{k=1}^{\infty} \frac{[\phi(k, \lambda)]^n}{\beta - 1} |b_k|
\]
\[
\leq 1, \quad \text{(Using (10)).}
\]

Thus \( f \in S_H^* \).

This completes the proof of the Theorem 5.

For our next theorem, we need to define the convolution of two harmonic functions. For harmonic function of the form
\[
f(z) = z + \sum_{k=2}^{\infty} |a_k| z^k + (-1)^n \sum_{k=1}^{\infty} |b_k| z^k
\]
and
\[ F(z) = z + \sum_{k=2}^{\infty} |A_k| z^k + (-1)^n \sum_{k=1}^{\infty} |B_k| z^k \]
we define their convolution
\[ (f \ast F)(z) = f(z) \ast F(z) = z + \sum_{k=2}^{\infty} |a_k A_k| z^k + (-1)^n \sum_{k=1}^{\infty} |b_k B_k| z^k, \] (11)
using this definition, we show that the class \( \overline{R}_H(n, \beta, \lambda) \) is closed under convolution.

**Theorem 6.** For \( 1 < \beta \leq \alpha \leq 2 \), let \( f \in \overline{R}_H(n, \beta, \lambda) \) and \( F \in \overline{R}_H(n, \alpha, \lambda) \).

Then \( (f \ast F)(z) \in \overline{R}_H(n, \beta, \lambda) \subseteq \overline{R}_H(n, \alpha, \lambda) \).

**Proof.** Let \( f(z) = z + \sum_{k=2}^{\infty} |a_k| z^k + (-1)^n \sum_{k=1}^{\infty} |b_k| z^k \) be in \( \overline{R}_H(n, \beta, \lambda) \) and \( F(z) = z + \sum_{k=2}^{\infty} |A_k| z^k + (-1)^n \sum_{k=1}^{\infty} |B_k| z^k \) be in \( \overline{R}_H(n, \alpha, \lambda) \). Then the convolution \((f \ast F)(z)\) is given by (11). We wish to show that the coefficients of \( f \ast F \) satisfy the required condition given in Theorem 2. For \( F(z) \in \overline{R}_H(n, \alpha, \lambda) \), we note that \( |A_k| \leq 1 \) and \( |B_k| \leq 1 \). Now, for the convolution function \((f \ast F)(z)\) we have
\[
\sum_{k=2}^{\infty} \frac{[\phi(k, \lambda)]^n}{\beta - 1} |a_k A_k| + \sum_{k=1}^{\infty} \frac{[\phi(k, \lambda)]^n}{\beta - 1} |b_k B_k| \\
\leq \sum_{k=2}^{\infty} \frac{[\phi(k, \lambda)]^n}{\beta - 1} |a_k| + \sum_{k=1}^{\infty} \frac{[\phi(k, \lambda)]^n}{\beta - 1} |b_k| \\
\leq 1, \text{ (since } f \in \overline{R}_H(n, \beta, \lambda).)\]

Therefore \((f \ast F)(z) \in \overline{R}_H(n, \beta, \lambda) \subseteq \overline{R}_H(n, \alpha, \lambda)\).

**Theorem 7.** The class \( \overline{R}_H(n, \beta, \lambda) \) is closed under convex combination.

**Proof.** For \( i = 1, 2, 3... \) let \( f_i(z) \in \overline{R}_H(n, \beta, \lambda) \) where \( f_i(z) \) is given by
\[ f_i(z) = z + \sum_{k=2}^{\infty} |a_{k_i}| z^k + (-1)^n \sum_{k=1}^{\infty} |b_{k_i}| z^k. \]

Then by Theorem 2, we have
\[
\sum_{k=2}^{\infty} \frac{[\phi(k, \lambda)]^n}{\beta - 1} |a_{k_i}| + \sum_{k=1}^{\infty} \frac{[\phi(k, \lambda)]^n}{\beta - 1} |b_{k_i}| \leq 1.
\]

For \( \sum_{i=1}^{\infty} t_i = 1, 0 \leq t_i \leq 1 \), the convex combination of \( f_i \) may be written as
\[
\sum_{i=1}^{\infty} t_i f_i(z) = z + \sum_{k=2}^{\infty} \left( \sum_{i=1}^{\infty} t_i |a_{k_i}| \right) z^k + (-1)^n \sum_{k=1}^{\infty} \left( \sum_{i=1}^{\infty} t_i |b_{k_i}| \right) z^k.
\]
Then by Theorem 2, we have
\[
\sum_{k=2}^{\infty} \frac{[\phi(k, \lambda)]^n}{\beta - 1} \left( \sum_{i=1}^{\infty} t_i |a_k| \right) + \sum_{k=1}^{\infty} \frac{[\phi(k, \lambda)]^n}{\beta - 1} \left( \sum_{i=1}^{\infty} t_i |b_k| \right)
= \sum_{i=1}^{\infty} t_i \left( \sum_{k=2}^{\infty} \frac{[\phi(k, \lambda)]^n}{\beta - 1} |a_k| \right) + \sum_{k=1}^{\infty} \frac{[\phi(k, \lambda)]^n}{\beta - 1} \left( \sum_{i=1}^{\infty} t_i |b_k| \right)
\leq \sum_{i=1}^{\infty} t_i = 1.
\]
Therefore
\[
\sum_{i=1}^{\infty} t_i f_i(z) \in \overline{R_H}(n, \beta, \lambda).
\]
The \(\delta\)-neighborhood of \(f\) is the set, (see \([2], [14]\))
\[
N_\delta(f) = \left\{ F : F(z) = z + \sum_{k=2}^{\infty} |A_k| z^k + (-1)^n \sum_{k=1}^{\infty} |B_k| z^k \text{ and } \sum_{k=1}^{\infty} k(|a_k - A_k| + |b_k - B_k|) \leq \delta \right\}.
\]

**Theorem 8.** Let \(f \in \overline{R_H}(n, \beta, \lambda)\) and \(\delta \leq 2 - \beta\). If \(F \in N_\delta(f)\), then \(F\) is harmonic starlike function.

**Proof.** Let \(F(z) = z + \sum_{k=2}^{\infty} |A_k| z^k + (-1)^n \sum_{k=1}^{\infty} |B_k| z^k\) belong to \(N_\delta(f)\). We have
\[
\sum_{k=2}^{\infty} k|A_k| + \sum_{k=1}^{\infty} k|B_k|
\leq \sum_{k=2}^{\infty} k(|a_k - A_k| + |b_k - B_k|) + \sum_{k=1}^{\infty} k(|a_k| + |b_k|) + |b_1 - B_1| + |b_1|
\leq \delta + \beta - 1
\leq 1.
\]
Hence, \(F(z)\) is harmonic starlike function.

4. **A Family of Class Preserving Integral Operator**

Let \(f(z) = h(z) + \overline{g(z)} \in S_H\) be given by (1) then \(F(z)\) defined by relation
\[
F(z) = \frac{e + 1}{z^c} \int_{z}^{e} t^{c-1}h(t)dt + \frac{e + 1}{z^c} \int_{0}^{z} t^{c-1}g(t)dt, \ (c > -1). \tag{12}
\]

**Theorem 9.** Let \(f(z) = h(z) + \overline{g(z)} \in S_H\) be given by (3) and \(f(z) \in \overline{R_H}(n, \beta, \lambda)\) then \(F(z)\) be defined by (12) also belong to \(\overline{R_H}(n, \beta, \lambda)\).

**Proof.** Let
\[
f(z) = z + \sum_{k=2}^{\infty} |a_k| z^k + (-1)^n \sum_{k=1}^{\infty} |b_k| z^k
\]
be in \(\overline{R_H}(n, \beta, \lambda)\) then by Theorem 2, we have
\[
\sum_{k=2}^{\infty} \frac{[\phi(k, \lambda)]^n}{\beta - 1} |a_k| + \sum_{k=1}^{\infty} \frac{[\phi(k, \lambda)]^n}{\beta - 1} |b_k| \leq 1. \tag{13}
\]
By definition of $F(z)$ we have

$$F(z) = z + \sum_{k=2}^{\infty} \frac{c+1}{c+k} |a_k| z^k + (-1)^n \sum_{k=1}^{\infty} \frac{c+1}{c+k} |b_k| z^k.$$

Now

$$\sum_{k=2}^{\infty} \frac{[\phi(k,\lambda)]^n}{\beta-1} \left( \frac{c+1}{c+k} |a_k| \right) + \sum_{k=1}^{\infty} \frac{[\phi(k,\lambda)]^n}{\beta-1} \left( \frac{c+1}{c+k} |b_k| \right) \leq \sum_{k=2}^{\infty} \frac{[\phi(k,\lambda)]^n}{\beta-1} |a_k| + \sum_{k=1}^{\infty} \frac{[\phi(k,\lambda)]^n}{\beta-1} |b_k| \leq 1.$$

Thus $F(z) \in \mathcal{R}_H(n, \beta, \lambda)$.

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