EXISTENCE AND UNIQUENESS RESULT OF SOLUTIONS TO INITIAL VALUE PROBLEMS OF FRACTIONAL DIFFERENTIAL EQUATIONS OF VARIABLE-ORDER

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ABSTRACT. In this work, an initial value problem is discussed for a fractional differential equation of variable-order. By means of some analysis techniques and Arzela-Ascoli theorem, existence result of solution is obtained; Using the upper solutions and lower solutions and monotone iterative method, uniqueness existence results of solutions are obtained.

1. INTRODUCTION

The fractional operators (fractional derivatives and integrals) refer to the differential and integral operators of arbitrary order, and fractional differential equations refer to those containing fractional derivatives. The former are the generalization of integer-order differential and integral operators and the latter, the generalization of differential equations of integer order. The fractional operators of variable order, which fall into a more complex category, are the derivatives and integrals whose order is the function of certain variables. In recent years, fractional operators and fractional differential equations of variable order have been applied in engineering more and more frequently. For the examples and details, see [1]-[14] and the references therein. Their extensive applications urgently call for systematic studies on the existence, uniqueness of solutions to initial value problems of these equations. Research in this area is at the trailblazing stage and has so far but produced a very limited number of published papers dealing with relatively simple problems with limited methods, such as [15], [16].
The following are several definitions of fractional integral and fractional derivatives of variable-order, which can be founded in [14]:

\[ I^{p(t)}_{a+} f(t) = \int_{a}^{t} \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))} f(s) ds, \quad p(t) > 0, t > a, \]  

(1)

where \( \Gamma(\cdot) \) denotes the Gamma function, \( -\infty < a < +\infty \), provided that the right-hand side is pointwise defined.

\[ I^{p(t)}_{a+} f(t) = \int_{a}^{t} \frac{(t-s)^{p(s)-1}}{\Gamma(p(s))} f(s) ds, \quad p(t) > 0, t > a, \]  

(2)

provided that the right-hand side is pointwise defined.

\[ I^{p(t)}_{a+} f(t) = \int_{a}^{t} \frac{(t-s)^{p(t-s)-1}}{\Gamma(p(t-s))} f(s) ds, \quad p(t) > 0, t > a, \]  

(3)

provided that the right-hand side is pointwise defined.

\[ D^{p(t)}_{a+} f(t) = \frac{d^n}{dt^n} I^{n-p(t)}_{a+} f(t) = \frac{d^n}{dt^n} \int_{a}^{t} \frac{(t-s)^{n-1-p(t)}}{\Gamma(n-p(t))} f(s) ds, \quad t > a, \]  

(4)

where \( n-1 < p(t) < n, t > a, n \in N \), provided that the right-hand side is pointwise defined.

\[ D^{p(t)}_{a+} f(t) = \frac{d^n}{dt^n} I^{n-p(t)}_{a+} f(t) = \frac{d^n}{dt^n} \int_{a}^{t} \frac{(t-s)^{n-1-p(s)}}{\Gamma(n-p(s))} f(s) ds, \quad t > a, \]  

(5)

where \( n-1 < p(t) < n, t > a, n \in N \), provided that the right-hand side is pointwise defined.

\[ D^{p(t)}_{a+} f(t) = \frac{d^n}{dt^n} I^{n-p(t)}_{a+} f(t) = \frac{d^n}{dt^n} \int_{a}^{t} \frac{(t-s)^{n-1-p(s)}}{\Gamma(n-p(t-s))} f(s) ds, \quad t > a, \]  

(6)

where \( n-1 < p(t) < n, t > a, n \in N \), provided that the right-hand side is pointwise defined.

In particular, when \( p(t) \) is a constant function, i.e. \( p(t) \equiv q(q \) is a finite positive constant), then \( I^{p(t)}_{a+} ; D^{p(t)}_{a+} \) are the usual Riemann-Liouville fractional calculus [\( ? \)]. It is well known that fractional calculus \( D^{q}_{a+} ; I^{q}_{a+} \) have some very important properties, which play very important role in considering existence of solutions of fractional differential equation denoted by \( D^{q}_{a+} \), by means of nonlinear analysis method. Such as, the following some properties, which can be founded in [17]:

**Proposition 1.1.** ([17]) The equality \( I^{\gamma}_{a+} D^{\delta}_{a+} f(t) = I^{\gamma+\delta}_{a+} f(t) \), \( \gamma > 0, \delta > 0 \) holds for \( f \in L(a, b) \).

**Proposition 1.2.** ([17]) The equality \( D^{\gamma}_{a+} I^{\delta}_{a+} f(t) = f(t) \), \( \gamma > 0 \) holds for \( f \in L(a, b) \).

**Proposition 1.3.** ([17]) Let \( \alpha > 0 \). Then the differential equation

\[ D^{\alpha}_{a+} u = 0 \]

has unique solution

\[ u(t) = c_{1}(t-a)^{\alpha-1} + c_{2}(t-a)^{\alpha-2} + \cdots + c_{n}(t-a)^{\alpha-n}, \]

\( c_{i} \in R, i = 1, 2, \cdots, n \), here \( n-1 < \alpha \leq n \).

**Proposition 1.4.** ([17]) Let \( \alpha > 0, u \in L(a, b) \), \( D^{\alpha}_{a+} u \in L(a, b) \). Then the following equality holds

\[ I^{\alpha}_{a+} D^{\alpha}_{a+} u(t) = u(t) + c_{1}(t-a)^{\alpha-1} + c_{2}(t-a)^{\alpha-2} + \cdots + c_{n}(t-a)^{\alpha-n}, \]
where \( I_{a+}^p \) denote one of fractional integrals defined by (1)-(3).

**Example 1.1.** Let \( p(t) = t, 0 \leq t \leq 6, q(t) = \begin{cases} 2, & 0 \leq t \leq 2 \\ 1, & 2 < t \leq 3 \\ t, & 3 < t \leq 6, \end{cases} \)

We calculate \( I_{0+}^p f(t) \) and \( I_{0+}^{p(t)+q(t)} f(t) \) defined by (1).

\[
I_{0+}^p I_{0+}^q f(t) = \int_0^t \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))} \int_0^s \frac{(s-\tau)^{q(s)-1}}{\Gamma(q(s))} f(\tau) d\tau ds
\]

\[
= \int_0^2 \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))} \int_0^s \frac{(s-\tau)^{q(s)-1}}{\Gamma(q(s))} d\tau ds + \int_2^t \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))} \int_0^s \frac{(s-\tau)^{q(s)-1}}{\Gamma(q(s))} d\tau ds
\]

\[
= \int_0^2 \frac{(t-s)^{p(t)-1}}{2\Gamma(p(t))} d\tau ds + \int_2^t \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))} \int_0^s \frac{(s-\tau)^{q(s)-1}}{\Gamma(q(s))} d\tau ds,
\]

\[
I_{0+}^{p(t)+q(t)} f(t) = \int_0^t \frac{(t-s)^{p(t)+q(t)-1}}{\Gamma(p(t)+q(t))} f(s) ds,
\]

we see that

\[
I_{0+}^p I_{0+}^q f(t) \big|_{t=3} = \int_0^2 \frac{(3-s)^{3-1}}{2\Gamma(3)} d\tau ds + \int_2^3 \frac{(3-s)^{3-1}}{\Gamma(3)} \int_0^s \frac{1}{\Gamma(1)} d\tau ds = \frac{8}{5} + \frac{9}{24} = 1,
\]

\[
I_{0+}^{p(t)+q(t)} f(t) \big|_{t=3} = \frac{3}{2} \frac{(3-s)^{3+1-1}}{\Gamma(3+1)} f(s) ds = \frac{3}{2} \frac{(3-s)^{3+1-1}}{\Gamma(3+1)} ds = \frac{27}{8},
\]

we see easily that

\[
I_{0+}^p I_{0+}^q f(t) \big|_{t=3} \neq I_{0+}^{p(t)+q(t)} f(t) \big|_{t=3}.
\]

According to (7), we can see that Proposition 1.1 – 1.4 don’t hold for \( D_{a+}^p \) and \( I_{a+}^p \) defined by (1)-(6).

**Example 1.2.** Let \( 0 < p(t) < 1, t > 0 \). By calculating, we have that

\[
I_{a+}^p = \frac{1}{\Gamma(p(t))} \int_0^t (t-s)^{p(t)-1} ds = \frac{t^{p(t)}}{p(t)\Gamma(p(t))}, t > 0,
\]

\[
D_{a+}^p I_{a+}^p = \frac{d}{dt} \int_0^t (t-s)^{p(t)-1} ds = \frac{1}{\Gamma(1-p(t))} \int_0^t (t-s)^{-p(t)} s^{p(s)} ds
\]

\[
\neq 1,
\]

\[
D_{a+}^p = \frac{d}{dt} I_{a+}^p.
\]
and
\[ I_{0+}^{p(t)} I^{p(t)}_{0+} = I_{0+}^{p(t)} \frac{d}{dt} I^{1-p(t)}_{0+} = I_{0+}^{p(t)} \left( \frac{d}{dt} \frac{t^{1-p(t)}}{(1-p(t))\Gamma(1-p(t))} \right) \neq 1. \]

**Remark 1.1.** For fractional integral of variable-order defined by (5)-(6), we can’t easily calculate out fractional integral \( I_{a+}^{p(t)} \) of some functions \( f(t) \), for example, we don’t know that what \( I_{a+}^{p(t)} = \int_{a}^{t} (t-s)^{p(s)-1} ds \) and \( I_{a+}^{p(t)} = \int_{a}^{t} (t-s)^{p(s)-1} ds \) equal.

Therefore, without those properties like as Propositions 1.1, 12, 1.3 and 1.4, ones can not transform a fractional differential equation of variable-order into an equivalent integral equation, so that one can consider existence of solutions of a fractional differential equation of variable-order, by means of nonlinear functional analysis method.

There also has more complex fractional calculus of variable-order, whose order function \( p(t) \) of (1)-(6) are replaced by \( p(t, f(t)) \), please see [1], [15], [16]. In [15], authors considered the solution existence of the following variable order fractional differential equations

\[
\begin{cases}
  D_{c+}^{p(t,x(t))} x(t) = f(t, x(t)), & c < t \leq b \\
  x(c) = x_0,
\end{cases}
\]  

(8)

where \( D_{c+}^{p(t,x(t))} \) is a fractional derivative of variable-order defined by

\[ D_{c+}^{p(t,x(t))} x(t) = \frac{d}{dt} \int_{c}^{t} \frac{(t-s)^{-p(s,x(s))}}{\Gamma(1-p(s,x(s)))} x(s)ds, t > c. \]

(9)

In [15], authors transformed (8) into one equivalent integral equation

\[ x(t) = I_{c+}^{p(t,x(t))} f(t, x(t)) = \int_{c}^{t} \frac{(t-s)^{p(s,x(s))-1}}{\Gamma(p(s,x(s)))} f(s, x(s))ds, c \leq t \leq b, \]

(10)

and then, using approximated method, authors obtained existence result of solution for problem (8) with \( \frac{1}{2} \leq q(t, x) \leq 1, c \leq t \leq b, x \in R. \)

In my opinion, the problem and analysis techniques are interesting and meaning, but, there had a critical wrong, that is, the transformation( from (8) to (10)) is unsuitable, because fractional calculus \( D_{c+}^{p(t,x(t))} I_{c+}^{p(t,x(t))} \) don’t usually have properties like Propositions 1.3 and 1.4. Among of those analysis, the sequence (6) has little mistakes, since ones can’t know whether such sequence exists. As well, the initial value condition \( x(c) = x_0 \) isn’t suitable when \( x_0 \neq 0 \), because, when \( p(t, x(t)) \) is a constant function, i.e. \( p(t) \equiv q(0 < q < 1 \text{ is a finite positive constant}) \), then \( D_{c+}^{p(t,x(t))} \) is the usual Riemann-Liouville fractional derivative \( D_{c+}^{q} \). From [17], we know that Riemann-Liouville fractional derivative of order \( 0 < q < 1 \) of constant \( x_0 \) is not zero, but is \( x_0 (t-c)^{-q} \), \( t > c \), as a result, fractional differential equation(involving Riemann-Liouville fractional derivative) can not have such \( x(c) = x_0 \) initial value condition, but is initial value condition \( (t-c)^{-q} x(t) \mid_{t=c} \) or \( I_{c+}^{1-q} x(t) \mid_{t=c} (t-c)^{-q} x(t) \mid_{t=c} \) and \( I_{c+}^{1-q} x(t) \mid_{t=c} \) can transform each other, see [17]). Hence, in some degree, problem (8) is not a suitable problem(expect \( x_0 = 0 \)).

In this paper, based on characters of fractional derivative of variable-order, by means of some analysis techniques, we will consider existence and uniqueness of
solution to initial value problems for fractional differential equation of variable-order (IVP for short)

\[
\begin{cases}
    D^{q(t,x(t))}_{0+} x(t) = f(t,x), & 0 < t \leq T, 0 < T < +\infty \\
    x(0) = 0,
\end{cases}
\]  

(11)

where \(D^{q(t,x(t))}_{0+}\) denotes fractional derivative of variable-order defined by (9), \(0 < q(t,x(t)) \leq q^* < 1, 0 \leq t \leq T, x \in R\), and \(f : [0,T] \times R \rightarrow R\) is a continuous function.

The rest of this paper is organized as follows. In Section 2, we give some preparation results. In Section 3, the existence results of solutions for IVP (11) are presented. In section 4, uniqueness existence result of solution for a particular case of IVP(11).

## 2. Preliminaries

We assume that

(H1) \(q : [0,T] \times R \rightarrow (0,q^*], 0 < q^* < 1,\) is a continuous function;

(H2) \(f : [0,T] \times R \rightarrow R\) is a continuous function.

It follows from the continuity of compose functions that \(\Gamma(q(t,x(t)))\) is continuous on \([0,T]\), when \(q\) satisfies assumption condition \((H_1)\). The following results will play very important role in proving our existence result of solution to problems (11).

Let \(\delta\) is an arbitrary small positive constant.

**Lemma 2.1.** Let \((H_1)\) hold. And let \(x_n, x \in C[0,T]\), assume that \(x_n(t) \rightarrow x(t), t \in [0,T]\) as \(n \rightarrow \infty\), then

\[
\int_{0}^{t-\delta} \frac{(t-s)^{-q(s,x_n(s))}}{\Gamma(1 - q(s,x_n(s)))} x_n(s)ds \rightarrow \int_{0}^{t-\delta} \frac{(t-s)^{-q(s,x(s))}}{\Gamma(1 - q(s,x(s)))} x(s)ds, t \in [\delta,T],
\]  

(12)

as \(n \rightarrow \infty\).

**Proof.** We see that

\[
\begin{align*}
    &\text{if } 0 < T \leq 1, \text{ then } T^{-q(s,x(s))} \leq T^{-q^*}, \\
    &\text{if } 1 < T < +\infty, \text{ then } T^{-q(s,x(s))} < 1.
\end{align*}
\]

Thus, for \(0 < T < +\infty\), we let

\[T^* = \max\{T^{-q^*}, 1\}.
\]

(15)

We let

\[M = \max_{0 \leq t \leq T} |x(t)| + 1, M_1 = \max_{0 \leq t \leq T} |x_n(t)| + 1, L = \max_{0 \leq t \leq T, \|x_n\| \leq M_1} \frac{1}{\Gamma(1 - q(t,x_n(t)))} + 1.\]

By the convergence of \(x_n\), for \((1-q^*)\varepsilon\) \((\varepsilon\) is arbitrary small positive number), there exists \(N_0 \in N\) such that

\[|x_n(t) - x(t)| < \frac{(1-q^*)\varepsilon}{3MLT}, t \in [0,T], n \geq N_0.
\]

(16)

Since \((t-s)^{-q(s,x(s))}, \delta \leq t - s \leq T,\) is continuous with respect to its exponent \(-q(s,x(s))\), for \(\frac{\varepsilon}{3MLT}\), when \(n \geq n_0\), it holds

\[|(t-s)^{-q(s,x(s))} - (t-s)^{-q(s,x(s))}| < \frac{\varepsilon}{3MLT}, \delta \leq t - s \leq T,
\]

(17)
also, by continuity of \( \frac{1}{\Gamma(1-q(s,x_n(s)))} \), for \((1-q^*)\varepsilon\), when \( n \geq n_0 \), it holds

\[
\left| \frac{1}{\Gamma(1-q(s,x_n(s)))} - \frac{1}{\Gamma(1-q(s,x(s)))} \right| < \frac{(1-q^*)\varepsilon}{3MT^* T}, 0 \leq s \leq T. \tag{18}
\]

Hence, from (13)-(18), we have that

\[
\begin{align*}
&\int_0^{t-\delta} (t-s)^{-q(s,x_n(s))} x_n(s) ds - \int_0^{t-\delta} \frac{(t-s)^{-q(s,x(s))}}{\Gamma(1-q(s,x(s)))} x(s) ds \\
&\leq \int_0^{t-\delta} \left| \frac{(t-s)^{-q(s,x_n(s))}}{\Gamma(1-q(s,x_n(s)))} \right| |x_n(s) - x(s)| ds \\
&+ \int_0^{t-\delta} \left| \frac{(t-s)^{-q(s,x_n(s))} - (t-s)^{-q(s,x(s))}}{\Gamma(1-q(s,x_n(s)))} \right| |x(s)| ds \\
&+ \int_0^{t-\delta} \left| (t-s)^{-q(s,x(s))} \right| \frac{1}{\Gamma(1-q(s,x_n(s)))} - \frac{1}{\Gamma(1-q(s,x(s)))} ||x(s)|| ds \\
&\leq \frac{L(1-q^*)\varepsilon}{3MT^* T} \int_0^{t-\delta} (t-s)^{-q(s,x_n(s))} ds + \frac{ML\varepsilon}{3MT^* T} \int_0^{t-\delta} ds \\
&+ \frac{M(1-q^*)\varepsilon}{3MT^* T} \int_0^{t-\delta} (t-s)^{-q(s,x(s))} ds \\
&= \frac{(1-q^*)\varepsilon}{3T^* T} \int_0^{t-\delta} T^{-q(s,x_n(s))} \left( \frac{t-s}{T} \right)^{-q(s,x_n(s))} ds + \varepsilon \int_0^{t-\delta} ds \\
&+ \frac{(1-q^*)\varepsilon}{3T^* T} \int_0^{t-\delta} T^{-q(s,x(s))} \left( \frac{t-s}{T} \right)^{-q(s,x(s))} ds \\
&\leq \frac{(1-q^*)\varepsilon}{3T^* T} \int_0^{t-\delta} T^* \left( \frac{t-s}{T} \right)^{-q^*} ds + \varepsilon \frac{(1-q^*)\varepsilon}{3T^* T} \int_0^{t-\delta} T^* \left( \frac{t-s}{T} \right)^{-q^*} ds \\
&= \frac{(1-q^*)\varepsilon}{3T^* T} \int_0^{t-\delta} (t-s)^{-q^*} ds + \varepsilon \frac{(1-q^*)\varepsilon}{3T^* T} \int_0^{t-\delta} (t-s)^{-q^*} ds \\
&= \frac{\varepsilon}{3T^* T} \left( t^{1-q^*} - \delta^{1-q^*} \right) + \frac{\varepsilon}{3T^* T} \left( t^{1-q^*} - \delta^{1-q^*} \right) \\
&< \frac{\varepsilon T^{1-q^*}}{3T^* T} + \frac{T\varepsilon}{3T^* T} + \frac{\varepsilon T^{1-q^*}}{3T^* T} \\
&= \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,
\end{align*}
\]

which implies that (12) holds.

By the similar arguments, we can know that

**Lemma 2.2.** Let \((H_2)\) hold. And let \( x_n, x \in C[0,T] \), assume that \( x_n(t) \to x(t), t \in [0,T] \) as \( n \to \infty \), then

\[
\int_0^{t-\delta} f(s,x_n(s)) ds \to \int_0^{t-\delta} f(s,x(s)) ds, t \in [\delta,T], \tag{19}
\]
as \( n \to \infty \).

**Proof.** By the convergence of \( x_n \), for \( \zeta > 0 \), there exists \( N_0 \in \mathbb{N} \) such that

\[
|x_n(t) - x(t)| < \zeta, t \in [0,T], n \geq N_0,
\]
by the continuity of $f$, for $\frac{\varepsilon}{T}$ (where $\varepsilon$ is arbitrary small number), when $n \geq N_0$, it holds

$$|f(s, x_n(s)) - f(s, x(s))| < \frac{\varepsilon}{T}, s \in [0, T].$$

Thus, we have that

$$\int_0^{t-\delta} |f(s, x_n(s)) - f(s, x(s))|ds \leq \frac{\varepsilon}{T} \int_0^{t-\delta} ds = \frac{\varepsilon}{T} (t-\delta) \leq \varepsilon T,$$

which implies that (19) holds.

**Lemma 2.3.** Assume that $(H_1)$ hold. Then for arbitrary fixed $x \in C[0, T]$, the following expression holds

$$\lim_{\delta \to 0} \int_0^{t-\delta} \frac{(t-s)^{-q(s,x(s))}}{\Gamma(1-q(s,x(s)))} x(s)ds = \int_0^t \frac{(t-s)^{-q(s,x(s))}}{\Gamma(1-q(s,x(s)))} x(s)ds. \quad (20)$$

**Proof.** For arbitrary fixed $x \in C[0, T]$, we let

$$M = \max_{0 \leq t \leq T} |x(t)| + 1, L = \max_{0 \leq t \leq T, \|x\| \leq M} \frac{1}{\Gamma(1-q(t,x(t)))} + 1.$$

Thus, for arbitrary fixed function $x \in C[0, T]$, for $\forall \varepsilon > 0$, take $\delta_0 = \left(\frac{\varepsilon(1-q^*)}{MLT^q} \right)^{1-q^*}$, then, when $0 < \delta < \delta_0$, by (14), (15), (16), we have that

$$\int_0^{t-\delta} \frac{(t-s)^{-q(s,x(s))}}{\Gamma(1-q(s,x(s)))} x(s)ds \leq \int_0^t \frac{(t-s)^{-q(s,x(s))}}{\Gamma(1-q(s,x(s)))} x(s)ds$$

$$= \int_0^t \frac{(t-s)^{-q(s,x(s))}}{\Gamma(1-q(s,x(s)))} x(s)ds$$

$$\leq ML \int_{t-\delta}^t T^{q^*(\frac{t-s}{T})} x(s)ds$$

$$= \frac{MLT^q x^*}{1-q^*} \delta^{1-q^*}$$

$$< \frac{MLT^q x^* \delta_0^{1-q^*}}{1-q^*} = \varepsilon,$$

which implies that (20) holds.

**Remark 2.1.** Assume that $(H_1), (H_2)$ hold. Obviously, we can know that: for the arbitrary fixed function $x \in C[0, T]$,

$$\lim_{\delta \to 0} \int_0^{t-\delta} x(s)ds = \int_0^t x(s)ds, \lim_{\delta \to 0} \int_0^{t-\delta} f(s, x(s))ds = \int_0^t f(s, x(s))ds. \quad (21)$$
Lemma 2.4. ([17]) Let \([a, b](\infty < a < b < +\infty)\) be a finite interval and let \(AC[a, b]\) be the space of functions which are absolutely continuous on \([a, b]\). It is known that \(AC[a, b]\) coincides with the space of primitives of Lebesgue summable functions:

\[
f(t) \in AC[a, b] \Leftrightarrow f(t) = c + \int_0^t \varphi(s)ds, \varphi \in L(a, b), c \in R,
\]
and therefore an absolutely continuous function \(f(t)\) has a summable derivative \(f'(t) = \varphi(t)\) almost everywhere on \([a, b]\).

3. Existence result

In this section, we will consider the existence of solution for IVP \((11)\), by means of some analysis techniques and Arzela-Ascoli theorem. By the definition of fractional derivative defined by \((9)\), we see that problem \((11)\) is equivalent to \(x(0) = 0\) and the following expression

\[
\int_0^t (t - s)^{-q(s,x(s))} x(s)ds = c + \int_0^t f(s, x(s))ds, t \in [0, T],
\]
where \(c \in \mathbb{R}\).

Theorem 3.1. Assume that \((H_1), (H_2)\) hold. Then IVP \((11)\) exists one solution \(x^* \in C[0, T]\).

Proof. In order to obtain the existence result of solution IVP \((11)\), we firstly verify the following sequence has convergent subsequence,

\[
x_k(t) = \begin{cases} 
0, & 0 \leq t \leq \delta, \\
x_{k-1}(t) + \int_0^t \frac{(t - s)^{-q(s,x_{k-1}(s))}}{\Gamma(1 - q(s,x_{k-1}(s)))} x_{k-1}(s)ds \\
- \int_0^t \frac{f(s, x_{k-1}(s))ds}{\Gamma(1 - q(s,x_{k-1}(s)))}, & \delta < t \leq T,
\end{cases}
\]

\(k = 1, 2, \ldots\), where \(x_0(t) = 0, t \in [\delta, T]\), \(\delta\) is arbitrary small number.

In order to apply Arzela-Ascoli theorem to consider the existence of convergent subsequence of sequence \(x_k\) defined by \((24)\), firstly, we prove the uniformly bounded of sequence \(x_k\) on \([0, T]\).

We find that \(x_k\) is uniformly bounded on \([0, \delta]\). Now, we will verify sequence \(x_k\) is uniformly bounded on \([\delta, T]\). Let \(M = \max_{0 \leq t \leq T} |f(s, 0)| + 1\). Since \(x_0 = 0\) is uniformly bounded on \([0, T]\), then, for \(t \in [\delta, T]\), we have that

\[
|x_1(t)| = |x_0(t) + \int_0^t \frac{(t - s)^{-q(s,x_0(s))}}{\Gamma(2 - q(s,x_0(s)))} x_0(s)ds - \int_0^t f(s, 0)ds|
\]

\[
\leq | \int_0^t f(s, 0)ds |
\]

\[
\leq M \int_0^t ds
\]

\[
\leq MT = M_1,
\]
which implies that \(x_1\) is uniformly bounded on \([\delta, T]\), together with \(x_1(t) = 0\) for \(t \in [0, \delta]\), we obtain that \(x_1\) is uniformly bounded on \([0, T]\).
Let $M_f = \max_{0 \leq t \leq T} ||x||_1 \leq M_1, |f(t, x_1)| + 1, L = \max_{0 \leq t \leq T} ||x||_2 \leq M_1, |\frac{1}{1 - q(t, x_1(t))}| + 1$. From (13), (14), (15), for $t \in [\delta, T]$, we have that

$$|x_2(t)| \leq |x_1(t)| + \int_0^{t-\delta} |(t-s)^{-q(s, x_1(s))}| ||x_1(s)|| ds + \int_0^{t-\delta} |f(s, x_1(s))| ds$$

$$\leq M_1 + M_1 L \int_0^{t-\delta} T^{-q(s, x_1(s))} \left(\frac{t-s}{T}\right)^{-q(s, x_1(s))} ds + M_f (T - \delta)$$

$$= M_1 + M_1 L \int_0^{t-\delta} T^s \left(\frac{t-s}{T}\right)^{-q^*} ds + M_f T$$

$$\leq M_1 + M_1 L T^s \frac{1 - q^*}{1 - q^*} (1 - q^* - \delta^{1-q^*}) + M_f T$$

$$= M_2,$$

which implies that $x_2$ is uniformly bounded on $[\delta, T]$, together with $x_2(t) = 0$ for $t \in [0, \delta]$, we obtain that $x_2$ is uniformly bounded on $[0, T]$. Continue this process, we can obtain that sequence $x_k$ is uniformly bounded on $[0, T]$.

Now, we consider the equicontinuous of sequence $x_k$ on $[0, T]$. Obviously, $x_0$ is equicontinuous on $[0, T]$. Firstly, we can know that function $k(t) = a^t - b^t$, $t \in (-1, 0)$, $0 < a < b < 1$, is decreasing. Indeed, since $\ln a < 0 < b$, and $a^t > b^t > 0$, we have that

$$k'(t) = a^t \ln a - b^t \ln b < b^t \ln a - b^t \ln b = b^t (\ln a - \ln b) < 0,$$

which implies that $k(t)$ is a decreasing. Thus, for $l(s) = \frac{t^s - x(s)}{s^2}$, we may look $l(s)$ as the same type as $k(s)$, then $l(s)$ is decreasing with respect to its exponent $-q(s, x(s))$.

As well, in the next analysis, we will use the Minkowskis’s inequality: for $a, b$ non negative, and any $R \geq 0$, it holds

$$(a + b)^R \leq c_R (a^R + b^R), \quad c_R = \max\{1, 2^{R-1}\}. \quad (25)$$

As a result, for $a, b$ non negative, and any $0 < r < 1$, it follows from (25) that

$$(a + b)^r \leq c_r (a^r + b^r) = \max\{1, 2^{r-1}\} (a^r + b^r) = a^r + b^r. \quad (26)$$

We let $M = \max_{0 \leq t \leq T} |f(s, 0)| + 1$. For $\forall \varepsilon > 0, \forall t_1, t_2 \in [0, T], t_1 < t_2$, we consider result in two cases.

Case I: $0 \leq t_1 \leq \delta < t_2 \leq T$. We take $\eta_{1,t} = \frac{\varepsilon}{M}$, when $t_2 - t_1 < \eta_{1,t}$, we have that

$$|x_1(t_2) - x_1(t_1)| = \int_0^{t_2-\delta} |f(s, 0)| ds$$

$$\leq M \int_0^{t_2-\delta} ds$$

$$= M (t_2 - \delta)$$

$$\leq M (t_2 - t_1)$$

$$< M \eta_{1,t}$$

$$= \varepsilon.$$

Case II: \( \delta \leq t_1 < t_2 \leq T \). We take \( \eta_{1,II} = \frac{\varepsilon}{M} \), when \( t_2 - t_1 < \eta_{1,II} \), we have that

\[
|x_1(t_2) - x_1(t_1)| = \left| \int_0^{t_1-\delta} f(s,0)ds - \int_0^{t_2-\delta} f(s,0)ds \right| \\
\leq \int_{t_1-\delta}^{t_2-\delta} |f(s,0)|ds \\
\leq M \int_{t_1-\delta}^{t_2-\delta} ds \\
= M(t_2 - t_1) \\
< M\eta_{1,II} \\
= \varepsilon.
\]

These imply that \( x_1(t) \) is equicontinuous on \([0, T]\), the same result can be obtained when \( t_2 < t_1 \).

We let \( M_f = \max_{0 \leq t \leq T, \|x_1\| \leq M_1} |f(s, x_1)| + 1 \), \( L = \max_{0 \leq t \leq T, \|x_1\| \leq M_1} \left| \frac{1}{1-q(s, x_1(s))} \right| + 1 \).

For \( \forall \frac{\varepsilon}{2} > 0 \) (\( \varepsilon \) is arbitrary small number), \( \forall t_1, t_2 \in [0, T], t_1 < t_2 \), we consider result in two cases.

Case I: \( 0 \leq t_1 \leq \delta < t_2 \leq T \). We take \( \eta_{2,I} = \min \{ \eta_{1,I}, (\frac{1-q^*}{4M_I LT^{*} T^*} \frac{1}{T})^{\frac{1}{q^*}}, \frac{\varepsilon}{4M_I} \} \), when \( t_2 - t_1 < \eta_{2,I} \), by (13), (14), (15), (26), we have that

\[
|x_1(t_2) - x_1(t_1)| \\
= |x_1(t_2) + \int_0^{t_2-\delta} (t_2 - s)^{-q(s, x_1(s))} \frac{1}{\Gamma(1-q(s, x_1(s)))} x_1(s)ds - \int_0^{t_2-\delta} f(s, x_1(s))ds| \\
\leq |x_1(t_2)| + M_1 L \int_0^{t_2-\delta} (t_2 - s)^{-q(s, x_1(s))} ds + M_f \int_0^{t_2-\delta} ds \\
\leq |x_1(t_2)| + M_1 L \int_0^{t_2-\delta} T^{-q(s, x_1(s))} \left( \frac{t_2 - s}{T} \right)^{-q(s, x_1(s))} ds + M_f (t_2 - \delta) \\
\leq |x_1(t_2)| + M_1 L \int_0^{t_2-\delta} T^{*} (\frac{t_2 - s}{T})^{-q^*} ds + M_f (t_2 - \delta) \\
= |x_1(t_2)| + \frac{M_1 L T^{*} T^*}{1-q^*} (t_2^{-q^*} - \delta^{1-q^*}) + M_f (t_2 - \delta) \\
\leq |x_1(t_2) - x_1(t_1)| + \frac{M_1 L T^{*} T^*}{1-q^*} ((t_2 - \delta + \delta)^{1-q^*} - \delta^{1-q^*}) + M_f (t_2 - \delta) \\
\leq |x_1(t_2) - x_1(t_1)| + \frac{M_1 L T^{*} T^*}{1-q^*} ((t_2 - \delta)^{1-q^*} + \delta^{1-q^*} - \delta^{1-q^*}) + M_f (t_2 - \delta) \\
= |x_1(t_2) - x_1(t_1)| + \frac{M_1 L T^{*} T^*}{1-q^*} (t_2 - t_1)^{1-q^*} + M_f (t_2 - t_1) \\
< |x_1(t_2) - x_1(t_1)| + \frac{M_1 L T^{*} T^*}{1-q^*} \eta_{2,I}^{1-q^*} + M_f \eta_{2,I} \\
\leq \varepsilon + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\
= \frac{3\varepsilon}{2}.
\]
\[
|x_1(t_2) - x_1(t_1)| \leq |x_1(t_2) - x_1(t_1)| + \int_{0}^{t_2 - \delta} \frac{(t_2 - s)^{-q(s, x_1(s))}}{\Gamma(1 - q(s, x_1(s)))} x_1(s) ds \\
- \int_{0}^{t_1 - \delta} \frac{(t_1 - s)^{-q(s, x_1(s))}}{\Gamma(1 - q(s, x_1(s)))} x_1(s) ds - \int_{0}^{t_2 - \delta} f(s, x_1) ds + \int_{0}^{t_1 - \delta} f(s, x_1) ds\]

\[
\leq |x_1(t_2) - x_1(t_1)| + M_1 L \int_{t_1 - \delta}^{t_2 - \delta} (t_2 - s)^{-q(s, x_1(s))} ds + M_f \int_{t_1 - \delta}^{t_2 - \delta} ds
\]

\[
= |x_1(t_2) - x_1(t_1)| + M_1 L \int_{0}^{t_2 - \delta} T^{-q(s, x_1(s))}(s) ((\frac{t_1 - s}{T})^{-q(s, x_1(s))} - (\frac{t_2 - s}{T})^{-q(s, x_1(s))}) ds \\
+ M_1 L \int_{0}^{t_2 - \delta} T^{-q(s, x_1(s))}(s) \frac{t_2 - s}{T}^{-q(s, x_1(s))} ds + M_f (t_2 - t_1)
\]

\[
\leq |x_1(t_2) - x_1(t_1)| + M_1 L \int_{0}^{t_2 - \delta} T^{s}((\frac{t_1 - s}{T})^{-q^*} - (\frac{t_2 - s}{T})^{-q^*}) ds \\
+ M_1 L \int_{0}^{t_2 - \delta} T^{s}(\frac{t_2 - s}{T})^{-q^*} ds + M_f (t_2 - t_1)
\]

\[
= |x_1(t_2) - x_1(t_1)| + \frac{M_1 L T^s}{1 - q^*} (t_1^{-q^*} - 2(2(t_2 - t_1) + \delta)1^{-q^*} - t_2^{-q^*} - \delta 1^{-q^*}) \\
+ M_f (t_2 - t_1)
\]

\[
\leq |x_1(t_2) - x_1(t_1)| + \frac{M_1 L T^s}{1 - q^*} (t_2^{-q^*} - 2(2(t_2 - t_1)1^{-q^*} + 2\delta 1^{-q^*} - t_2^{-q^*}) \\
+ M_f (t_2 - t_1)
\]

\[
= |x_1(t_2) - x_1(t_1)| + \frac{2M_1 L T^s}{1 - q^*} (t_2 - t_1)1^{-q^*} + M_f (t_2 - t_1)
\]

\[
< |x_1(t_2) - x_1(t_1)| + \frac{2M_1 L T^s}{1 - q^*} \eta_2, II + M_f \eta_2, II
\]

\[
< \varepsilon + \frac{\varepsilon}{4} + \frac{\varepsilon}{4}
\]

\[
= \frac{3\varepsilon}{4}.
\]
These imply that \( x_2(t) \) is equicontinuous on \([0, T]\), the same result can be obtained when \( t_2 < t_1 \). Continue these process, we can obtain that \( x_k, k = 1, 2 \cdots, \) is equicontinuous on \([0, T]\).

As well, by the arguments of equicontinuity of \( x_k \), we can know that \( x_k \in C[0, T], k = 1, 2, \cdots \). Then, from Arzela-Ascoli theorem, sequence \( x_k \) exists a convergent subsequence \( x_{m_k} \). From (24), \( x_{m_k} \) should satisfy

\[
x_{m_k}(t) = \begin{cases} 0, & 0 \leq t \leq \delta, \\ x_{m_k-1}(t) + \int_{0}^{t-\delta} \frac{(t-s)-q(x_{m_k-1}(s))}{1-q(s,x_{m_k-1}(s))} x_{m_k-1}(s)ds \\ -\int_{0}^{t-\delta} f(s, x_{m_k-1}(s))ds, & \delta < t \leq T. \end{cases}
\]

(27)

Now, we will prove that the continuous limit of \( x_{m_k} \), denoted by \( x^* \) is one solution of IVP (11).

Let \( \delta \to +\infty \) in (27), by Lemmas 2.1, 2.2, we have that

\[
x^*(t) = \begin{cases} 0, & 0 \leq t \leq \delta, \\ x^*(t) + \int_{0}^{t-\delta} \frac{(t-s)-q(s,x^*(s))}{1-q(s,x^*(s))} x^*(s)ds \\ -\int_{0}^{t-\delta} f(s, x^*(s))ds, & \delta < t \leq T. \end{cases}
\]

(28)

Thus, we find that,

\[
x^*(t) = 0, 0 \leq t \leq \delta; \quad \int_{0}^{t-\delta} \frac{(t-s)-q(s,x^*(s))}{1-q(s,x^*(s))} x^*(s)ds - \int_{0}^{t-\delta} f(s, x^*(s))ds = 0, \quad \delta < t \leq T.
\]

(29)

In order to verify \( x^* \) is one solution of IVP (11), we let \( \delta \to 0 \) in (29), by (20), (21), we obtain that

\[
x^*(0) = 0; \quad \int_{0}^{t} \frac{(t-s)-q(s,x^*(s))}{1-q(s,x^*(s))} x^*(s)ds = \int_{0}^{t} f(s, x^*(s))ds, 0 < t \leq T.
\]

(30)

It follows from the continuity of \( f \) and Lemma 2.4 that \( \int_{0}^{t} f(s, x^*(s))ds \in AC[0, T] \), consequently, from (30), we get

\[
\int_{0}^{t} f(s, x^*(s))ds = \int_{0}^{t} \frac{(t-s)-q(s,x^*(s))}{1-q(s,x^*(s))} x^*(s)ds \in AC[0, T].
\]

As a result, differential on both sides of the second expression in (30), we get

\[
D_{0+}^{\delta t} x^*(t) = f(t, x^*), 0 < t \leq T,
\]

(31)

together with \( x^*(0) = 0 \), we see that \( x^* \) is one solution of IVP (11). Thus, we complete this proof.

4. Unique results

In this section, using the monotone iterative method, we will consider the existence and unique result of solution to the following particular case of IVP (11),

\[
\begin{cases} D_{0+}^{p(t)} x(t) = f(t, x), & 0 < t \leq T, 0 < T < +\infty \\ x(0) = 0, \end{cases}
\]

(32)

where \( D_{0+}^{p(t)} \) denotes fractional derivative of variable-order defined by (5), where \( p : [0, T] \to (0, p^*], 0 < p^* < 1, \) is a continuous function.

We assume that

\[
(H_3) \quad p : [0, T] \to (0, p^*] \text{ is continuous, here } 0 < p^* < 1.
\]
The following result will play very important role in our next analysis.

**Lemma 4.1.** Let \((H_3)\) hold. If \(x \in C[0, T]\) and satisfies the relations

\[
\begin{cases}
D_{0+}^{\rho(t)} x + d_0 x \geq 0, t \in (0, T] \\
x(0) \geq 0
\end{cases}
\]

(33)

where \(D_{0+}^{\rho(t)}\) denotes fractional derivative of variable-order defined by (5), \(d_0 > 0\) is a constant. Then \(x \geq 0\) for \(t \in [0, T]\).

**Proof.** We assume that \(x(t) \geq 0, t \in [0, T]\) is false. Then from \(x(0) \geq 0\), there exists points \(t_0 \in [0, T], t_0' \in (0, T]\) such that, \(x(t_0) = 0, x(t_0') < 0\); and that \(x(t) \geq 0\) for \(t \in [0, t_0]\), \(x(t) < 0\) for \(t \in (t_0, t_0']\), and assume that \(t_1\) is the first minimal point of \(x(t)\) on \([t_0, t_0']\).

It follows from the inequality of (33) that

\[ D_{0+}^{\rho(t)} x(t) \geq 0, \quad t \in [t_0, t_0'], \]

hence, we have

\[ \int_{t_0}^{t} D_{0+}^{\rho(s)} x(s) ds \geq 0, \quad t \in [t_0, t_0'], \]

from the definition of fractional derivative of variable order defined by (5), we can obtain that

\[ \int_{t_0}^{t} \frac{d}{ds} (I_{0+}^{1-p(s)} x(s)) ds = I_{0+}^{1-p(t)} x(t) - I_{0+}^{1-p(t)} x(t_0) \geq 0, \quad t \in [t_0, t_0']. \]

(34)

On the other hand, for \(t \in [t_0, t_0']\), we have

\[
I_{0+}^{1-p(t)} x(t) - I_{0+}^{1-p(t)} x(t_0) = \int_{0}^{t} \frac{(t-s)^{1-p(s)} x(s)}{\Gamma(1-p(s))} ds - \int_{0}^{t_0} \frac{(t_0-s)^{1-p(s)} x(s)}{\Gamma(1-p(s))} ds
\]

\[ = \int_{0}^{t} \frac{(t-s)^{1-p(s)} - (t_0-s)^{1-p(s)} x(s)}{\Gamma(1-p(s))} ds + \int_{t_0}^{t} \frac{(t-s)^{1-p(s)} x(s)}{\Gamma(1-p(s))} ds \]

\[ < 0 + 0 = 0, \]

which contradicts (34). Therefore, we obtain that \(x(t) \geq 0, t \in [0, T]\). Thus we complete this proof.

For problem (32), we have the following definitions of upper and lower solutions.

**Definition 4.1.** A function \(\alpha \in C[0, T]\) is called a upper solution of problem (32), if it satisfies

\[
\begin{cases}
D_{0+}^{\rho(t)} \alpha(t) \geq f(t, \alpha), t \in (0, T] \\
\alpha(0) \geq 0
\end{cases}
\]

(35)

Analogously, function \(\beta \in C[0, T]\) is called a lower solution of problem (32), if it satisfies

\[
\begin{cases}
D_{0+}^{\rho(t)} \beta(t) \leq f(t, \beta), t \in (0, T] \\
\beta(0) \leq 0
\end{cases}
\]

(36)

In what follows we assume that

\[ \alpha(t) \geq \beta(t), \quad t \in [0, T], \]

(37)

and define that sector

\[ \langle \alpha, \beta \rangle = \{ u \in C[0, T]; \alpha(t) \leq u(t) \leq \beta(t), t \in [0, T] \}. \]
We also assume that \( f \) satisfies the following condition
\[
f(t, x_1) - f(t, x_2) \geq -d_0(x_1 - x_2), \quad \alpha \leq x_2 \leq x_1 \leq \beta,
\]
where \( d_0 \geq 0 \) is a constant and \( \alpha, \beta \in C[0, T] \) are lower and upper solutions of problem (32). Clearly this condition is satisfied with \( d_0 = 0 \), when \( f \) is monotone nondecreasing in \( u \). In view of (38), the function
\[
F(t, x) = d_0x + f(t, x)
\]
is monotone nondecreasing in \( x \) for \( x \in (\alpha, \beta) \).

We also suppose that there exists a constant \( d_1 \leq 0 \), such that
\[
f(t, x_1) - f(t, x_2) \leq d_1(x_1 - x_2), \quad \alpha \leq x_2 \leq x_1 \leq \beta,
\]
where \( \alpha, \beta \in C[0, T] \) are lower and upper solutions of problem (32).

The following is existence and uniqueness theorem of solution for (32).

**Theorem 4.1.** Let \( (H_2), (H_3) \) hold. Assume that \( \alpha, \beta \in C[0, T] \) are lower and upper solutions of problem (32), such that (37) holds, \( f \) also satisfies (38). Then problem (32) exists one solution in the sector \( (\alpha, \beta) \), as well, if condition (40) holds, then (32) exists one unique solution in the sector \( (\alpha, \beta) \).

**proof.** We see that (32) is equivalent to the following problem
\[
\begin{align*}
D_0^{p(t)} v + d_0 v &= f(t, v), \quad t \in (0, T], \\
v(0) &= 0,
\end{align*}
\]
where \( d_0 \) is the constant in (38). This proof consists of six steps.

**Step 1.** Constructing sequences \( \{v^{(k)}\}, k = 1, 2, \ldots \) as following
\[
\begin{align*}
D_0^{p(t)} v^{(k)}(t) + d_0 v^{(k)} &= f(t, v^{(k)}), \quad t \in (0, T], \\
v^{(k)}(0) &= 0.
\end{align*}
\]
Theorem 3.1 assures that the sequence \( \{v^{(k)}\} \) is well defined, since for each \( k \), we can obtain result from theorem 3.1 with \( f(t, v^{(k)}) = d_0 v^{(k-1)} + f(t, v^{(k-1)}) - d_0 v^{(k)} \) and \( q(t, v^{(k)}) = p(t) \). Of particular interest is the sequence obtained from (42) with a upper solution or lower solution of problem (32) as the initial iteration. Denote the sequence with the initial iteration \( v^{(0)} = \beta \) by \( \{v^{(k)}\} \) and the sequence with \( v^{(0)} = \alpha \) by \( \{v^{(k)}\} \).

**Step 2.** Monotone property of the two sequences.

The sequences \( \{v^{(k)}\}, \{v^{(k)}\} \) constructed by (41) process the monotone property
\[
\alpha \leq v^{(k)} \leq v^{(k+1)} \leq \beta, \quad t \in (0, T],
\]
for every \( k = 1, 2, \ldots \).

In fact, let \( r = v^{(0)} - \beta \). By (42), (36), (37), (38), and \( v^{(0)} = \beta \), there has
\[
D_0^{p(t)} r + d_0 r = (D_0^{p(t)} + d_0) r - (d_0 + f(t, \beta)) = D_0^{p(t)} r - f(t, \beta) \geq 0, \quad t \in (0, T],
\]
\[
r(0) \geq 0 - 0 = 0.
\]
In view of Lemma 4.1, \( r \geq 0 \) for \( t \in [0, T] \), which leads to \( v^{(1)} \leq v^{(0)} = \beta \), \( t \in [0, T] \). A similar argument using the property of a lower solution of (32) gives
\[ v^{(1)}(t) \geq v^{(0)}(t) = \alpha, \ t \in [0, T]. \] Let \( r^{(1)} = \overline{v}^{(1)} - \underline{v}^{(1)} \). By (42), (36), (37), (38), there has
\[
D_{0^+}^{\alpha(t)} r^{(1)} + d_0 r^{(1)} = \begin{align*}
&d_0 \overline{v}^{(0)} + f(t, \overline{v}^{(0)}) - (d_0 \underline{v}^{(0)} + f(t, \underline{v}^{(0)})) \\
&= d_0 (\beta - \alpha) + f(t, \beta) - f(t, \alpha) \\
&\geq 0, \ t \in (0, T],
\end{align*}
\]
\[ r^{(1)}(0) = \overline{v}^{(1)}(0) - \underline{v}^{(1)}(0) = 0, \]
Again, in view of Lemma 4.1, \( r^{(1)} \geq 0 \) for \( t \in [0, T] \), the above conclusion shows that
\[ \alpha = \overline{v}^{(0)} \leq \underline{v}^{(1)} \leq \overline{v}^{(1)} \leq \beta, \ t \in [0, T] \]
Assume, by induction
\[ \alpha \leq \underline{v}^{(k-1)} \leq \overline{v}^{(k)} \leq \overline{v}^{(k-1)} \leq \beta, \ t \in (0, T]. \tag{44} \]
Then by (42), (44), (38), the function \( \overline{v}^{(k)} = \overline{v}^{(k)} - \overline{v}^{(k+1)} \) satisfies the relations
\[
D_{0^+}^{\alpha(t)} \overline{v}^{(k)}(t) + d_0 \overline{v}^{(k)}(t) = \begin{align*}
&d_0 \overline{v}^{(k-1)} + f(t, \overline{v}^{(k-1)}) - (d_0 \overline{v}^{(k)} + f(t, \overline{v}^{(k)})) \\
&\geq 0, \ t \in (0, T],
\end{align*}
\]
\[ \overline{v}^{(k)}(0) = 0, \]
In view of Lemma 4.1, \( r^{(k)} \geq 0 \), that is \( \overline{v}^{(k+1)} \leq \overline{v}^{(k)} \) for \( t \in [0, T] \). Similar reasoning gives \( \underline{v}^{(k)} \leq \underline{v}^{(k+1)} \) and \( \overline{v}^{(k+1)} \leq \overline{v}^{(k)} \) for \( t \in [0, T] \). Hence, the monotone property (43) follows from the principle of induction.

**Step 3.** The two sequences constructed by (42) have pointwise limits and satisfy some relations, that is
\[ \lim_{k \to \infty} \overline{v}^{(k)}(t) = v(t), \ \lim_{k \to \infty} \underline{v}^{(k)}(t) = w(t), t \in [0, T] \tag{45} \]
exists and satisfy the relation
\[ \alpha \leq \underline{v}^{(k)} \leq \underline{v}^{(k+1)} \leq w \leq \overline{v}^{(k+1)} \leq \overline{v}^{(k)} \leq \beta, t \in [0, T] \tag{46} \]
for every \( k = 1, 2, \ldots \).

In fact, By (43), we see that the upper sequence \( \{ \overline{v}^{(k)} \} \) is monotone nonincreasing and is bounded from below and that the lower sequence \( \{ \underline{v}^{(k)} \} \) is monotone nondecreasing and is bounded from above. Therefore the pointwise limits exist and these limits are denoted by \( v \) and \( w \) as in (45). Moreover, by (43), the limits \( v, w \) satisfy (46).

**Step 4.** To prove that \( v \) and \( w \) are solutions of initial value problem (32).

Let \( \overline{v}^{(k)} \) be either \( \overline{v}^{(k)} \) or \( \underline{v}^{(k)} \). From the definition of fractional derivative \( D_{0^+}^{\alpha(t)} \) defined by (5), we see that the equation of (42) may be expressed as and
\[ I_{0^+}^{1-p(t)} \overline{v}^{(k)}(t) + d_0 I_{0^+}^{1} \overline{v}^{(k)}(t) = c + d_0 I_{0^+}^{1} v^{(k-1)}(t) + I_{0^+}^{1} f(t, v^{(k-1)}(t)), \]
where \( c \in R \), that is
\[ \int_0^t \frac{(t-s)^{-p(s)}}{\Gamma(1-p(s))} \overline{v}^{(k)}(s)ds + d_0 \int_0^t \overline{v}^{(k)}(s)ds = c + d_0 \int_0^t \overline{v}^{(k-1)}(s)ds + \int_0^t f(s, v^{(k-1)}(s))ds. \tag{47} \]
Now, we consider the expression
\[ \int_0^{t-\delta} \frac{(t-s)^{-p(s)}}{\Gamma(1-p(s))} \overline{v}^{(k)}(s)ds + d_0 \int_0^{t-\delta} \overline{v}^{(k)}(s)ds = c + d_0 \int_0^{t-\delta} \overline{v}^{(k-1)}(s)ds + \int_0^{t-\delta} f(s, v^{(k-1)}(s))ds. \tag{48} \]
where $\delta > 0$ is arbitrary small number. By Lemma 2.1, we know that $\int_0^t (t-s)^{-\frac{\alpha}{1-p(s)}} v(s) ds \in C[\delta, T]$ with $v(0) = \alpha$ or $v(0) = \beta$.

Let $k \to \infty$ in (48) and apply the Lemmas 2.2, 2.3 and the dominated convergence theorem, $v$ satisfies the integral equation

$$\int_0^t (t-s)^{-\frac{\alpha}{1-p(s)}} v(s) ds + \int_0^t f(s, v(s)) ds = c + d_0 \int_0^t \frac{(t-s)^{-\frac{\alpha}{1-p(s)}}}{s} v(s) ds + \int_0^t f(s, v(s)) ds,$$

that is

$$\int_0^t (t-s)^{-\frac{\alpha}{1-p(s)}} v(s) ds = c + \int_0^t f(s, v(s)) ds. \quad (49)$$

Now let $\delta \to 0$ in (49), by (20), (21), we get

$$\int_0^t (t-s)^{-\frac{\alpha}{1-p(s)}} v(s) ds = c + \int_0^t f(s, v(s)) ds. \quad (50)$$

It follows from the continuity of $f$ and Lemma 2.4 that $c + \int_0^t f(s, v(s)) ds \in AC[0, T]$, consequently, from (50), we get

$$c + \int_0^t f(s, v(s)) ds = \int_0^t (t-s)^{-\frac{\alpha}{1-p(s)}} v(s) ds \in AC[0, T].$$

As a result, differential on two sides of (50), we have that

$$D_{0+}^{p(t)} v(t) = f(t, v(t)), t \in (0, T]. \quad (51)$$

We also Let $k \to \infty$ in the second expression of (42), it holds $v(0) = 0$, this together with (51), we know that $v(t)$ is a solution of (32). This proves that the upper sequence $\{w^{(k)}\}$ converges to a solution $v$ of problem (32), the lower sequence $\{v^{(k)}\}$ converges to a solution $w$ of problem (32), and satisfies relation $v(t) \geq w(t), t \in [0, T]$.

Step 5 If condition (40) holds, then $v = w$ is unique solution of problem (32).

It is sufficient to prove $v(t) \leq w(t), t \in [0, T]$, by $v(t) \geq w(t), t \in [0, T]$ obtained in Step 4. In fact, by (32) and (40), the function $r = w - v$ satisfies the relations

$$\begin{cases} D_{0+}^{p(t)} r = -(f(t, v) - f(t, w)) \geq d_1 r, t \in (0, T], \\ r(0) = 0, \end{cases}$$

then, Lemma 4.1 implies that $r(t) \geq 0, t \in [0, T]$, this proves $w \geq v$, therefore, we obtain that $v = w$ is unique solution of problem (32). Thus, we complete this proof.

References


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