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STABILITY OF SOLUTION FOR RAO-NAKRA SANDWICH BEAM WITH BOUNDARY DISSIPATION OF FRACTIONAL DERIVATIVE TYPE

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Abstract. This paper deals with stability for a one-dimensional model of Rao-Nakra sandwich beam with a boundary dissipation of fractional derivative type. Fractional derivative can applied in several real life situations, [15, 26, 35, 37]. We show the polynomial stability of the system by using the semigroup theory together with a sharp result given by Borichev and Tomilov.

1. Introduction

In this article we are interested in studying the stabilization of Rao-Nakra sandwich beam with a boundary dissipation of fractional derivative type given by

$$\begin{cases} EQNS \begin{cases} \varrho_1 h_1 u_{tt} - E_1 h_1 u_{xx} - k(-u + v + \gamma w_x) = 0, \\ \varrho_3 h_3 v_{tt} - E_3 h_3 v_{xx} + k(-u + v + \gamma w_x) = 0, \\ \varrho h w_{tt} + EI w_{xxxx} - k\gamma(-u + v + \gamma w_x)_x = 0, \end{cases} \\ \\ IC \begin{cases} u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \\ v(x,0) = v_0(x),\ v_t(x,0) = v_1(x), \\ w(x,0) = w_0(x),\ w_t(x,0) = w_1(x), \end{cases} \\ \\ BC \begin{cases} u(0,t) = 0, \quad v(0,t) = 0, \text{ in } (0,+\infty), \\ w_x(0,t) = w_x(L,t) = w_{xxx}(0,t) = 0, \text{ in } (0,+\infty), \\ E_1 h_1 u_x(L,t) = -\partial_t^{\alpha,\eta} u(L,t), \\ E_3 h_3 v_x(L,t) = -\partial_t^{\alpha,\eta} v(L,t), \\ EI w_{xxx}(L,t) = \partial_t^{\alpha,\eta} w(L,t), \end{cases} \end{cases} \tag{1}$$

with $0 < x < L$ and $t > 0$. In (1): EQNS = Equations, IC = Initial Condition and BC = Boundary Condition.

A sandwich beam is an engineering model for a beam consisting of three-layer stiff: Botton and top faces, and a compliant inner more core layer. The Rao-Nakra [33] system consists of three layers and a no-slip assumption along in the

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interfaces of contacts. The top and bottom layers are are wave equations for the longitudinal displacements under Euler-Bernoulli beam assumptions. The core layer is one equation that describes the transversal displacement under Timoshenko beam assumptions.

For the physical origin of problem of the hinged beam which is either stretched or compressed by an axial force see Burgreen [5] and Eisley [7] for instance. From the mathematical point of view, we cite the pioneer works of Kirchhoff [17], Woinowsky-Krieger [38] and Berger [3].

S. P. Timoshenko [36] presented in 1921 a system that describes the dynamics of a beam, given by

$$\varrho_1 u_{tt} - k(u_x + \psi)_x = 0, \text{ in } (0, L) \times \mathbb{R}^+, \tag{2}$$

$$\varrho_2 \psi_{tt} - b\psi_{xx} + k\,(u_x + \psi) = 0, \text{ in } (0, L) \times \mathbb{R}^+, \tag{3}$$

where $u(x,t)$, $\psi(x,t)$ model the transverse displacement of the beam and the angular direction of the filament of the beam respectively and $\varrho_1, \varrho_2, k, b$ are positive real numbers. From them, (2)-(3) has been widely studied by several authors in different contexts. See, for instance [27] and references therein.

Based in the Timoshenko's theory, S. Hansen [10] proposed a model for a two-layer laminated beam given by

$$\varrho w_{tt} + G(\psi - w_x)_x = 0, \text{ in } (0, L) \times \mathbb{R}^+, \tag{4}$$

$$I_\varrho(3s_{tt} - \psi_{tt}) - D(3S_{xx} - \psi_{xx}) - G(\psi - u_x) = 0, \text{ in } (0, L) \times \mathbb{R}^+, \tag{5}$$

$$3I_\varrho s_{tt} - 3Ds_{xx} + 3G(\psi - w_x) + 4\mu s + 4\delta s_t = 0, \text{ in } (0, L) \times \mathbb{R}^+, \tag{6}$$

where $\varrho, G, I_\varrho, D, \gamma$ and δ are positive constants and represent density, shear stiffness, mass moment of inertia, flexural rigidity, adhesive stiffness, and adhesive damping parameter, respectively. The function $w(x,t)$ denotes the transversal displacement, $\psi(x,t)$ represents the rotational displacement, and $s(x,t)$ is proportional to the amount of slip along the interface at time t and longitudinal spatial variable x. This model has received a lot of attention of several authors in the last years. See, for instance [8], where was considered the dynamics of laminated Timoshenko beams.

A Rao-Nakra sandwich beam was derived of the general three-layer laminated beam and plate models developed in 1999 by Liu-Trogdon-Yong [21]

$$\varrho_1 h_1 u_{tt} - E_1 h_1 u_{xx} - \tau = 0, \tag{7}$$

$$\varrho_3 h_3 v_{tt} - E_3 h_3 v_{xx} + \tau = 0, \tag{8}$$

$$\varrho h w_{tt} + EI w_{xxxx} - G_1 h_1 (w_x + \phi_1)_x - G_3 h_3 (w_x + \phi_3)_x - h_2 \tau_x = 0, \tag{9}$$

$$\varrho_1 I_1 \phi_{1,tt} - E_1 I_1 \phi_{1,xx} - \frac{h_1}{2}\tau + G_1 h_1 (w_x + \phi_1) = 0, \tag{10}$$

$$\varrho_3 I_3 \phi_{3,tt} - E_3 I_3 \phi_{3,xx} - \frac{h_3}{2}\tau + G_3 h_3 (w_x + \phi_3) = 0. \tag{11}$$

The physical parameters $h_i, \rho_i, E_i, G_i, I_i > 0$ are the thickness, density, Young's modulus, shear modulus, and moments of inertia of the i-th layer for $i = 1, 2, 3$, from bottom to top, respectively. In addition, $\varrho h = \varrho_1 h_1 + \varrho_2 h_2 + \varrho_3 h_3$ and $EI = E_1 I_1 + E_3 I_3$.

The Rao-Nakra system

$$\begin{aligned}\varrho_1 h_1 u_{tt} - E_1 h_1 u_{xx} - k(-u + v + \gamma w_x) &= 0, \text{ in } (0, L) \times \mathbb{R}^+,\\ \varrho_3 h_3 v_{tt} - E_3 h_3 v_{xx} + k(-u + v + \gamma w_x) &= 0, \text{ in } (0, L) \times \mathbb{R}^+,\\ \varrho h w_{tt} + EI w_{xxxx} - \alpha k(-u + v + \gamma w_x)_x &= 0, \text{ in } (0, L) \times \mathbb{R}^+,\end{aligned} \tag{12}$$

is obtained from (7)-(11) when we consider the core material to be linearly elastic, i.e., $\tau = 2G_2\varsigma$ with the shear strain

$$\varsigma = \frac{1}{2h_2}(-u + v + \gamma w_x) \text{ and } \gamma = h_2 + \frac{1}{2}(h_1 + h_3),$$

where $k := \dfrac{G_2}{h_2}$, the shear modulus $G_2 = \dfrac{E_2}{2(1+\nu)}$, and $-1 < \nu < \dfrac{1}{2}$ is the Poisson ratio.

When the extensional motion of the bottom and top layers is neglected, we obtain from (12) the two-layer laminated beam model proposed by Hansen. System (4)-(6) reduces to the Timoshenko system when $s(x, t) = 0$.

The following Rao-Nakra model with internal damping and Kelvin-Voigt damping was considered in [18]

$$\varrho_1 h_1 u_{tt} - E_1 h_1 u_{xx} - k(-u + v + \gamma w_x) - a_1 u_{txx} + a_2 u_t = 0, \tag{13}$$

$$\varrho_3 h_3 v_{tt} - E_3 h_3 v_{xx} + k(-u + v + \gamma w_x) - b_1 u_{txx} + b_2 u_t = 0, \tag{14}$$

$$\varrho h w_{tt} + EI w_{xxxx} - \gamma k(-u + v + \gamma w_x)_x - c_1 w_{txxxx} + c_2 u_t = 0, \tag{15}$$

where $a_i, b_i, c_i \geq 0$, $i = 1, 2$. Authors showed that (13)-(15) is unstable if one damping is only imposed on the beam equation, beyond this, the exponential stability holds when all three displacements are damped while polynomial stability holds when just two of the three equations are damped.

Liu-Rao-Zhang [20] studied the Rao-Nakra system with a internal damping given by

$$\begin{aligned}\varrho_1 h_1 u_{tt} - E_1 h_1 u_{xx} - k(-u + v + \gamma w_x) + a_0 u_t &= 0, \text{ in } (0, 1) \times \mathbb{R}^+,\\ \varrho_3 h_3 v_{tt} - E_3 h_3 v_{xx} + k(-u + v + \gamma w_x) + a_1 v_t &= 0, \text{ in } (0, 1) \times \mathbb{R}^+,\\ \varrho h w_{tt} + EI w_{xxxx} - \gamma k(-u + v + \gamma w_x)_x + a_2 w_t &= 0, \text{ in } (0, 1) \times \mathbb{R}^+.\end{aligned}$$

They proved that the polynomial stability occurs when there is only one viscous damping acting either on the beam equation or one of the wave equations.

Now we present a brief literature review on Rao-Nakra system. Exact controllability results for the multilayer Rao-Nakra plate system with locally distributed control in a neighborhood of a portion of the boundary was obtained in [11, 12].

Exact controllability of a multilayer plate system with free boundary conditions was obtained by the method of Carleman estimates in [11]. The multilayer plate system is a natural multilayer generalization of a three-layer "sandwich plate" system due to Rao and Nakra. This paper is the sequel to [12] in which only clamped and hinged boundary conditions are considered.

In [13] was considered the problem of boundary control using bending moment and lateral force control at one end. Authors proved that the space of exact controllability has finite co-dimension and provide sufficient conditions for exact controllability to a zero energy state. Boundary controllability for the Rao-Nakra beam equation have been studied also in [14, 28, 29, 32].

As far as we know, this is the first time that Rao-Nakra system with fractional derivative type is analysed. For the past three decades or maybe so, a growing interest in the study of fractional calculus has been shown by a great many number of scientists. Several point views of engineering, applied sciences, and mathematical physics benefited greatly from this ascending wave of applications growing in this area. Space sciences, fluids mechanics, porous media flows, viscoelastic and biological processes, are but few areas in which fractional order differential equations have become a favored tool to tread new path.

On the appearance of the fractional derivative in the behavior of real materials see [26, 35, 37] and references therein. Many problems in several scientific applied areas, including analysis of viscoelastic materials, heat conduction in materials with memory, electrodynamics with memory, signal processing, among others, can be modeled with fractional differential calculus, this because of that many investigations have shown that models involving fractional derivatives are more realistic to represent some natural phenomena that models involving classical derivatives.

For instance, [15] collects review articles surveying areas of physics in which applications of fractional calculus have recently become prominent: Fractional kinetics of Hamiltonian chaotic systems. Problems in polymer physics and rheology. Problems in biophysics. Regular variation in thermodynamics. In [1], Three real life applications for fractional calculus are given: Nuclear (strong) interactions, earthquake prediction and epidemics.

Fractional derivative models are widely used to easily characterise more complex damping behaviour than the viscous one, although the underlying properties are not trivial. The studies the properties of structural systems whose damping is represented by a fractional model from the point of view of a mechanical engineer was considered in [39]. See also [30].

For 1D wave equation with a boundary viscoelastic damper of the fractional derivative type, see [24]. Author showed that the system is well-posed in the sense of semigroup and proved that the associated semigroup is not exponentially stable, but only strongly asymptotically. For more information we refer to [16, 25, 34] and references therein.

The present manuscript is organized in the following way: in Section 2, we introduce the basic spaces, the norms, properties, and notations which we are going to work on within the subsequent sections. The augmented model is presented. In Section 3, by using the semigroup theory of linear operators we obtain the existence, uniqueness, and smoothness theorem for the augmented model. In section 4 by using a general criteria due to Arendt-Batty (see [2]) we prove the strong stability of the C_0-semigroup $e^{t\mathcal{A}}$ associated to the system (33) in the absence of the compactness of the resolvent of $\mathcal{A}$. In the section 5, by using Borichev-Tomilov Theorem (see [4])we show that the C_0-semigroup is polynomially stable. Finally, we present a short conclusion where an interesting open problem is collocated.

2. Preliminary

Throughout this paper, we will use the following standard $L^2(0, L)$space, we are the scalar product and the norm are denoted by

$$\langle f, g\rangle_{L^2(0,L)} = \int_0^L f\,\overline{g}\,dx, \quad \|f\|^2_{L^2(0,L)} = \int_0^L |f|^2 dx.$$

In a similar way, let $L^2(\mathbb{R})$ be the Hilbert space of all measurable square integrable functions on the real line with the inner product

$$\langle f,\, g\rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} f\,\overline{g}\,d\xi, \quad f,\, g \in L^2(\mathbb{R}).$$

As we are interested in the stability of the solution, we will start proving that the full energy of the system (1), defined by

$$\begin{aligned} E(t) = \Big[&\varrho_1 h_1 \|u_t\|^2_{L^2(0,\,L)} + \varrho_3 h_3 \|v_t\|^2_{L^2(0,\,L)} + \varrho h \|w_t\|^2_{L^2(0,\,L)} \\ &+ E_1 h_1 \|u_x\|^2_{L^2(0,\,L)} + E_3 h_3 \|v_x\|^2_{L^2(0,\,L)} + EI \|w_{xx}\|^2_{L^2(0,\,L)} \\ &+ k\| -u + v + \gamma w_x\|^2_{L^2(0,\,L)}\Big], \end{aligned} \tag{16}$$

is nonincreasing.

Lemma 2.1. *The energy functional $E(t)$, satisfies*

$$\frac{d}{dt}E(t) = -u_t(L,\,t)\,\partial_t^{\alpha,\,\eta}u(L,\,t) - v_t(L,\,t)\partial_t^{\alpha,\,\eta}v(L,\,t) - w_t(L,\,t)\partial_t^{\alpha,\,\eta}w(L,\,t). \tag{17}$$

Proof. Multiplying $(1)_{EQNS_{1,2,3}}$ by u_t, v_t and w_t respectively, integrating on $(0,\,L)$ and using integration by parts, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\Big[\varrho_1 h_1\|u_t\|^2_{L^2(0,\,1)} + E_1 h_1 \|u_x\|^2_{L^2(0,\,1)}\Big] - k\int_0^L (-u + v + \gamma w_x)\,u_t dx \\ = -E_1 h_1 u_t(L,\,t)\,u_x(L,\,t), \end{aligned} \tag{18}$$

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\Big[\varrho_3 h_3\|v_t\|^2_{L^2(0,\,1)} + E_3 h_3 \|v_x\|^2_{L^2(0,\,1)}\Big] + k\int_0^L (-u + v + \gamma w_x) v_t dx \\ = -E_3 h_3 v_t(L,\,t)\,v_x(L,\,t), \end{aligned} \tag{19}$$

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\Big[\varrho h\|w_t\|^2_{L^2(0,\,1)} + EI \|w_{xx}\|^2_{L^2(0,\,1)}\Big] + k\int_0^L (-u + v + \gamma w_x)\,\gamma w_{xt} dx \\ = -EI w_t(L,\,t)\,w_{xxx}(L,\,t). \end{aligned} \tag{20}$$

Adding (18)-(20) and replacing $(1)_{BC_{4,5}}$ the result follows. □

Now, consider the following definitions of fractional integro-differential operators with weight exponential establish by Choi and MacCamy [6].

The exponential fractional integral of order α, $0 < \alpha < 1$, $\eta \geq 0$,

$$J^{\alpha,\,\eta} f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t e^{-\eta(t-\tau)}(t-\tau)^{\alpha-1} f(\tau)\,d\tau, \tag{21}$$

with $f \in L^1(0,t)$ and $t > 0$.

The exponential fractional derivative operator of order α, $0 < \alpha < 1$, $\eta \geq 0$,

$$\partial_t^{\alpha,\,\eta} f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t e^{-\eta\,(t-\tau)}(t-\tau)^{-\alpha}\frac{df(\tau)}{d\tau}d\tau, \tag{22}$$

with $f \in W^{1,\,1}(0,t)$ and $t > 0$. Note that $\partial_t^{\alpha,\,\eta} f(t) = J^{1-\alpha,\,\eta} f'(t)$.

The following results are going to be used some time from now on and are fundamental to the proof of our results:

Theorem 2.2. [24] *Let μ be the function*

$$\mu(\xi) = |\xi|^{(2\,\alpha-1)/2}, \qquad \xi \in \mathbb{R}, \qquad 0 < \alpha < 1. \tag{23}$$

Then, the relation between the Input U and the Output O if the following system

$$\varphi_t(\xi,\, t) + \xi^2\varphi(\xi,\, t) + \eta\varphi(\xi,\, t) - U(t)\mu(\xi) = 0, \quad \xi \in \mathbb{R}, \quad \eta \geq 0, \quad t > 0, \tag{24}$$

$$\varphi(\xi,\, 0) = 0, \tag{25}$$

$$O = [\pi]^{-1}\sin(\alpha\,\pi)\int_{\mathbb{R}} \mu(\xi)\varphi(\xi,\, t)d\xi, \tag{26}$$

is given by $O = I^{1-\alpha,\,\eta}U = D^{\alpha,\,\eta}U$, where

$$[I^{\alpha,\,\eta}f](t) = e^{-\eta t}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}e^{\eta s}f(s)ds. \tag{27}$$

Lemma 2.3. *If $\lambda \in D = \{\lambda \in \mathbb{C}:\ Re\,\lambda + \eta > 0\} \cup \{\lambda \in \mathbb{C}:\ Im\,\lambda \neq 0\}$. Then*

$$\int_{\mathbb{R}} \frac{\mu^2(\xi)d\xi}{\xi^2 + \eta + \lambda} = \frac{\pi}{\sin(\alpha\pi)}(\eta + \lambda)^{\alpha-1}.$$

On the other hand, the strategy for to get our target is related to the elimination of the fractional derivatives in time from the boundary condition in system (1). To this, setting $\mu(\xi) = |\xi|^{(2\alpha-1)/2}$, $\xi \in \mathbb{R}$, $\mathfrak{C} = \pi^{-1}\sin(\alpha\,\pi)$, and exploiting the technique from [9], we transform (1) into a new system. That is, we reformulate system (1) using Theorem 2.2, and the new system can be included into the augmented model

$$\begin{cases} EQNSA \begin{cases} \varrho_1 h_1 u_{tt} - E_1 h_1 u_{xx} - k(-u + v + \gamma w_x) = 0, \\ \varphi_t(\xi,\, t) + (\xi^2 + \eta)\varphi(\xi,\, t) - u_t(L,\, t)\mu(\xi) = 0, \\ \varrho_3 h_3 v_{tt} - E_3 h_3 v_{xx} + k(-u + v + \gamma w_x) = 0, \\ \phi_t(\xi,\, t) + (\xi^2 + \eta)\phi(\xi,\, t) - v_t(L,\, t)\mu(\xi) = 0, \\ \varrho h w_{tt} + EI w_{xxxx} - k\gamma(-u + v + \gamma w_x)_x = 0, \\ \psi_t(\xi,\, t) + (\xi^2 + \eta)\psi(\xi,\, t) = 0, \end{cases} \\ IC \begin{cases} u(x,\, 0) = u_0(x),\ u_t(x,\, 0) = u_1(x), \\ v(x,\, 0) = v_0(x),\ v_t(x,\, 0) = v_1(x), \\ w(x,\, 0) = w_0(x),\ w_t(x,\, 0) = w_1(x), \\ \varphi(\xi,\, 0) = \varphi_0(\xi) = 0,\ \phi(\xi,\, 0) = \phi_0(\xi) = 0,\ \psi(\xi,\, 0) = \psi_0(\xi) = 0, \end{cases} \\ BC \begin{cases} u(0,\, t) = 0, \quad v(0,\, t) = 0 \quad \text{in} \quad (0,\, +\infty), \\ w_x(0,\, t) = w_x(L,\, t) = w_{xxx}(0,\, t) = 0 \quad \text{in} \quad (0,\, +\infty), \\ E_1 h_1 u_x(L,\, t) = -\mathfrak{C}\int_{\mathbb{R}} \mu(\xi)\varphi(\xi,\, t)d\xi, \\ E_3 h_3 v_x(L,\, t) = -\mathfrak{C}\int_{\mathbb{R}} \mu(\xi)\phi(\xi,\, t)d\xi, \\ EI w_{xxx}(L,\, t) = \mathfrak{C}\int_{\mathbb{R}} \mu(\xi)\psi(\xi,\, t)d\xi, \end{cases} \end{cases} \tag{28}$$

where we denote $\mathfrak{C} = \pi^{-1}\sin(\alpha\,\pi)$.

The dissipative properties of the system (28) is given by the following lemma.

Lemma 2.4. *Let* $(u, u_t, v, v_t, w, w_t, \varphi, \phi, \psi)$ *be a solution of the system* (28). *Then, the energy functional defined by*

$$\mathcal{E}(t) = \frac{1}{2}\Big[\varrho_1 h_1 \|u_t\|^2_{L^2(0,\,L)} + \varrho_3 h_3 \|v_t\|^2_{L^2(0,\,L)} + \varrho h \|w_t\|^2_{L^2(0,\,L)} + E_1 h_1 \|u_x\|^2_{L^2(0,\,L)} + E_3 h_3 \|v_x\|^2_{L^2(0,\,L)} + EI\|w_{xx}\|^2_{L^2(0,\,L)} + k\| - u + v + \gamma w_x\|^2_{L^2(0,\,L)} + \mathfrak{C}\|\varphi\|^2_{L^2(\mathbb{R})} + \mathfrak{C}\|\phi\|^2_{L^2(\mathbb{R})} + \mathfrak{C}\|\psi\|^2_{L^2(\mathbb{R})}\Big] \tag{29}$$

where $\mathcal{E}(t)$ *be the energies associated with the system* (28)*, satisfies*

$$\frac{d}{dt}\mathcal{E}(t) = -\,\mathfrak{C}\int_{\mathbb{R}}(\xi^2+\eta)\varphi^2(\xi,\,t)d\xi - \mathfrak{C}\int_{\mathbb{R}}(\xi^2+\eta)\phi^2(\xi,\,t)d\xi - \mathfrak{C}\int_{\mathbb{R}}(\xi^2+\eta)\psi^2(\xi,\,t)d\xi \le 0. \tag{30}$$

In the next section, we use the semigroup theory of linear operators to obtain the existence, uniqueness, and smoothness theorem for the system (28).

3. Well-posedness of the problem

We define

$$\mathbb{H}^1(0,\,L) = \{z \in H^1(0,\,L):\ z(0)=0\},$$
$$\mathbb{H}^2(0,\,L) = \{z \in H^2_0(0,\,L) \cap H^3_0(0,\,L):\ z_{xxx}(0)=0\}.$$

Then

$$\mathcal{H} = [\mathbb{H}^1(0,\,L)\times L^2(0,\,L)\times L^2(\mathbb{R})]^2 \times [\mathbb{H}^2(0,\,L)\times L^2(0,\,L)\times L^2(\mathbb{R})], \tag{31}$$

equipped with the inner product given by

$$\begin{aligned} \langle \mathcal{U}, \widetilde{\mathcal{U}}\rangle_{\mathcal{H}} \;=\;& \varrho_1 h_1\int_0^L U\overline{\widetilde{U}}dx + \varrho_3 h_3\int_0^L V\overline{\widetilde{V}}dx + \varrho h\int_0^L W\overline{\widetilde{W}}dx \\ &+E_1h_1\int_0^L u_x\overline{\widetilde{u}}_x dx + E_3h_3\int_0^L v_x\overline{\widetilde{v}}_x dx + EI\int_0^L w_{xx}\overline{\widetilde{w}}_{xx}dx \\ &+k\int_0^L(-u+v+\gamma\, w_x)(-\overline{\widetilde{u}}+\overline{\widetilde{v}}+\gamma\overline{\widetilde{w}_x})dx \\ &+\mathfrak{C}\int_{\mathbb{R}}\varphi\overline{\widetilde{\varphi}}d\xi + \mathfrak{C}\int_{\mathbb{R}}\phi\overline{\widetilde{\phi}}d\xi + \mathfrak{C}\int_{\mathbb{R}}\psi\overline{\widetilde{\psi}}d\xi, \end{aligned} \tag{32}$$

where $\mathcal{U} = (u, U, \varphi, v, V, \phi, w, W, \psi)^T$ and $\widetilde{\mathcal{U}} = (\widetilde{u}, \widetilde{U},\ \widetilde{\varphi}\,,\ \widetilde{v}, \widetilde{V}, \widetilde{\phi}, \widetilde{w}, \widetilde{W}, \widetilde{\psi})^T$. We now wish to transform the initial boundary value problem (28) to an abstract problem in the Hilbert space $\mathcal{H}$. We introduce the functions $u_t = U$, $v_t = V$, $w_t = W$ and rewrite the system (28) as the following initial value problem

$$\begin{cases} \dfrac{d}{dt}\mathcal{U}(t) = \mathcal{A}\mathcal{U}(t), \\ \mathcal{U}(0) = \mathcal{U}_0, \quad \forall\, t>0, \end{cases} \tag{33}$$

where $\mathcal{U}=(u,U,\varphi,v,V,\phi,w,W,\psi)^T$, $\mathcal{U}_0=(u_0,\ u_1,\ \varphi_0,v_0,v_1,\phi_0,w_0,w_1,\psi_0)^T$, and the operator $\mathcal{A}:\mathcal{D}(\mathcal{A})\subset\mathcal{H}\rightarrow\mathcal{H}$ is given by

$$\mathcal{A}\begin{pmatrix} u\\ U\\ \varphi\\ v\\ V\\ \phi\\ w\\ W\\ \psi \end{pmatrix}=\begin{pmatrix} U\\ \frac{1}{\varrho_1 h_1}\left[E_1h_1u_{xx}+k\left(-u+v+\gamma w_x\right)\right]\\ -(\xi^2+\eta)\varphi+U(L)\mu(\xi)\\ V\\ \frac{1}{\varrho_3 h_3}\left[E_3h_3v_{xx}-k(-u+v+\gamma w_x)\right]\\ -(\xi^2+\eta)\phi+V(L)\mu(\xi)\\ W\\ \frac{1}{\varrho h}\left[-EIw_{xxxx}+k\gamma(-u+v+\gamma w_x)_x\right]\\ -(\xi^2+\eta)\psi+W(L)\mu(\xi) \end{pmatrix} \tag{34}$$

with the domain

$$\mathcal{D}(\mathcal{A})=\left\{\mathcal{U}\in\mathcal{H}\;\middle|\;\begin{array}{l} u,\ v\in H^2(0,\ L),\ w\in H^4(0,\ L),\\ U,\ V\in\mathbb{H}^1(0,\ L),\ W\in\mathbb{H}^2(0,\ L),\\ -(\xi^2+\eta)\varphi+U(L)\mu(\xi)\in L^2(\mathbb{R}),\\ -(\xi^2+\eta)\phi+V(L)\mu(\xi)\in L^2(\mathbb{R}),\\ -(\xi^2+\eta)\psi+W(L)\mu(\xi)\in L^2(\mathbb{R}),\\ E_1h_1u_x(L)+\mathfrak{C}\int_{\mathbb{R}}\mu(\xi)\varphi(\xi)d\xi=0,\\ E_3h_3v_x(L)+\mathfrak{C}\int_{\mathbb{R}}\mu(\xi)\phi(\xi)d\xi=0,\\ EIw_{xxx}(L)-\mathfrak{C}\int_{\mathbb{R}}\mu(\xi)\psi(\xi)d\xi=0,\\ |\xi|\varphi,\ \ |\xi|\phi,\ \ |\xi|\psi\in L^2(\mathbb{R}) \end{array}\right\}.$$

Note that $\mathcal{D}(\mathcal{A})$ is independent of time $t>0$ and clearly, $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{H}$. Now, we are ready to prove the following well-posedness result.

Theorem 3.1. *Let $\mathcal{U}_0\in\mathcal{H}$, then there exists a unique weak solution $\mathcal{U}\in C(\mathbb{R}^+,\mathcal{H})$ of problem (33). Moreover, if $\mathcal{U}_0\in D(\mathcal{A})$, then $\mathcal{U}\in C(\mathbb{R}^+,D(\mathcal{A}))\cap C^1(\mathbb{R}^+,\mathcal{H})$. In this case, it is called a strong solution.*

Proof. First, we prove that the operator $\mathcal{A}$ is dissipative.
For $\mathcal{U}=(u,U,\varphi,v,V,\phi,w,W,\psi)^T\in\mathcal{D}(\mathcal{A})$, we want to show that

$$\begin{aligned} Re\langle\mathcal{A}\mathcal{U},\,\mathcal{U}\rangle_{\mathcal{H}}=&-\mathfrak{C}\int_{\mathbb{R}}(\xi^2+\eta)\varphi^2(\xi,\,t)d\xi-\mathfrak{C}\int_{\mathbb{R}}(\xi^2+\eta)\phi^2(\xi,\,t)d\xi\\ &-\mathfrak{C}\int_{\mathbb{R}}(\xi^2+\eta)\psi^2(\xi,\,t)d\xi\leq 0. \end{aligned} \tag{35}$$

Direct computation, using (32), gives

$$
\begin{aligned}
\langle \mathcal{A}\mathcal{U},\ \mathcal{U}\rangle_{\mathcal{H}} =& E_1h_1\int_0^L u_{xx}\overline{U}dx + E_3h_3\int_0^L v_{xx}\overline{V}dx - EI\int_0^L w_{xxxx}\overline{W}dx\\
&+ k\int_0^L(-u+v+\gamma\, w_x)\overline{U}dx - k\int_0^L(-u+v+\gamma w_x)\overline{V}dx\\
&+ k\gamma\int_0^L(-u+v+\gamma w_x)_x\overline{W}dx\\
&+ E_1h_1\int_0^L U_x\overline{u}_xdx + E_3h_3\int_0^L V_x\overline{v}_xdx + EI\int_0^L W_{xx}\overline{w}_{xx}dx\\
&+ k\int_0^L(-U+V+\gamma W_x)(-\overline{u}+\overline{v}+\gamma\overline{w}_x)dx\\
&+ U(L)\mathfrak{C}\int_{\mathbb{R}}\mu(\xi)\overline{\varphi}d\xi + V(L)\mathfrak{C}\int_{\mathbb{R}}\mu(\xi)\overline{\phi}d\xi + W(L)\mathfrak{C}\int_{\mathbb{R}}\mu(\xi)\overline{\psi}d\xi\\
&- \mathfrak{C}\int_{\mathbb{R}}(\xi^2+\eta)\varphi^2d\xi - \mathfrak{C}\int_{\mathbb{R}}(\xi^2+\eta)\phi^2d\xi - \mathfrak{C}\int_{\mathbb{R}}(\xi^2+\eta)\psi^2d\xi.
\end{aligned}
$$

Integrating by parts and using $(28)_{BC}$ it follows that

$$
\begin{aligned}
\langle \mathcal{A}\mathcal{U},\ \mathcal{U}\rangle_{\mathcal{H}} =& -2iE_1h_1Im\int_0^L u_x\overline{U}_xdx - 2iE_3h_3Im\int_0^L v_x\overline{V}_xdx\\
&- 2iEIIm\int_0^L w_{xx}\overline{W}_{xx}dx\\
&- 2ikIm\int_0^L(-u+v+\gamma w_x)(-\overline{U}+\overline{V}+\gamma\overline{W}_x)dx\\
&- 2iIm\left[\overline{U}(L)\mathfrak{C}\int_{\mathbb{R}}\mu(\xi)\varphi dx\right] - 2iIm\left[\overline{V}(L)\mathfrak{C}\int_{\mathbb{R}}\mu(\xi)\phi dx\right]\\
&- 2iIm\left[\overline{W}(L)\mathfrak{C}\int_{\mathbb{R}}\mu(\xi)\psi dx\right]\\
&- \mathfrak{C}\int_{\mathbb{R}}(\xi^2+\eta)\varphi^2d\xi - \mathfrak{C}\int_{\mathbb{R}}(\xi^2+\eta)\phi^2d\xi - \mathfrak{C}\int_{\mathbb{R}}(\xi^2+\eta)\psi^2d\xi.
\end{aligned}
$$

Taking the real part yields (35). Next, we will prove that the operator $\lambda I - \mathcal{A}$ is surjective for $\lambda > 0$. For this purpose, let $\mathcal{F}=(f_1,f_2,f_3,f_4,f_5,f_6,f_7,f_8,f_9)^T \in \mathcal{H}$, we seek $\mathcal{U}=(u,U,\varphi,v,V,\phi,w,W,\psi)^T \in \mathcal{D}(\mathcal{A})$ such that $(\lambda I - \mathcal{A})\mathcal{U} = \mathcal{F}$, that is,

$$
\begin{cases}
\lambda u - U = f_1 \quad \text{in} \quad \mathbb{H}^1(0,\,L),\\
\lambda\varrho_1h_1U - E_1h_1u_{xx} - k(-u+v+\gamma w_x) = \varrho_1h_1f_2 \quad \text{in} \quad L^2(0,\,L),\\
\lambda\varphi + (\xi^2+\eta)\varphi - U(L)\mu(\xi) = f_3 \quad \text{in} \quad L^2(\mathbb{R}),\\
\lambda v - V = f_4 \quad \text{in} \quad \mathbb{H}^1(0,\,L),\\
\lambda\varrho_3h_3V - E_3h_3v_{xx} + k(-u+v+\gamma w_x) = \varrho_3h_3f_5 \quad \text{in} \quad L^2(0,\,L),\\
\lambda\phi + (\xi^2+\eta)\phi - V(L)\mu(\xi) = f_6 \quad \text{in} \quad L^2(\mathbb{R}),\\
\lambda w - W = f_7 \quad \text{in} \quad \mathbb{H}^2(0,\,L),\\
\lambda\varrho hW + EIw_{xxxx} - k\gamma(-u+v+\gamma w_x)_x = \varrho hf_8 \quad \text{in} \quad L^2(0,\,L),\\
\lambda\psi + (\xi^2+\eta)\psi - W(L)\mu(\xi) = f_9 \quad \text{in} \quad L^2(\mathbb{R}).
\end{cases}
\tag{36}
$$

From $(36)_{3,\,6,\,9}$ we have

$$\begin{aligned} \varphi(\xi) &= \frac{f_3(\xi) + U(L)\mu(\xi)}{\xi^2 + \eta + \lambda}, \\ \phi(\xi) &= \frac{f_6(\xi) + V(L)\mu(\xi)}{\xi^2 + \eta + \lambda}, \\ \psi(\xi) &= \frac{f_9(\xi) + W(L)\mu(\xi)}{\xi^2 + \eta + \lambda} \end{aligned} \tag{37}$$

and, from $(36)_{1,\,4,\,7}$ it follows that

$$\begin{aligned} U &= \lambda u - f_1 \in \mathbb{H}^1(0,\,L), \\ V &= \lambda v - f_4 \in \mathbb{H}^1(0,\,L), \\ W &= \lambda w - f_7 \in \mathbb{H}^2(0,\,L). \end{aligned} \tag{38}$$

On the other hand, replacing $(36)_{1,\,4,\,7}$ into $(36)_{2,\,5,\,8}$ respectively we obtain

$$\begin{aligned} \lambda^2 \varrho_1 h_1 U - E_1 h_1 u_{xx} - k(-u + v + \gamma w_x) &= \varrho_1 h_1 f_2 + \lambda \varrho_1 h_1 f_1, \\ \lambda^2 \varrho_3 h_3 V - E_3 h_3 v_{xx} + k(-u + v + \gamma w_x) &= \varrho_3 h_3 f_5 + \lambda \varrho_3 h_3 f_4, \\ \lambda^2 \varrho h W + EI w_{xxxx} - k\gamma(-u + v + \gamma w_x)_x &= \varrho h f_8 + \lambda \varrho h f_7. \end{aligned} \tag{39}$$

To solve the system (39) is equivalent to finding $u,\, v \in H^2(0,\,L) \cap \mathbb{H}^1(0,\,L)$ and $w \in H^4(0,\,L) \cap \mathbb{H}^2(0,\,L)$ such that

$$\int_0^L \left[\lambda^2 \varrho_1 h_1 u - E_1 h_1 u_{xx} - k(-u + v + \gamma w_x)\right] \tilde{u} dx = \int_0^L \varrho_1 h_1 (f_2 + \lambda f_1) \tilde{u} dx, \tag{40}$$

$$\int_0^L \left[\lambda^2 \varrho_3 h_3 v - E_3 h_3 v_{xx} + k(-u + v + \gamma w_x)\right] \tilde{v} dx = \int_0^L \varrho_3 h_3 (f_5 + \lambda f_4) \tilde{v} dx, \tag{41}$$

$$\int_0^L \left[\lambda^2 \varrho_3 h_3 w + EI w_{xxxx} - k\gamma(-u + v + \gamma w_x)_x\right] \tilde{w} dx = \int_0^L \varrho h (f_8 + \lambda f_7) \tilde{w} dx, \tag{42}$$

for all $(\tilde{u},\, \tilde{v}) \in \mathbb{H}^1(0,\,L) \times \mathbb{H}^1(0,\,L)$ and $\tilde{w} \in \mathbb{H}^2(0,\,L)$. Firstly, we estimate (40), then

$$\begin{aligned} \int_0^L \lambda^2 \varrho_1 h_1 u \tilde{u} dx - E_1 h_1 \int_0^L u_{xx} \tilde{u} dx - k \int_0^L (-u + v + \gamma w_x) \tilde{u} dx \\ = \int_0^L \varrho_1 h_1 (f_2 + \lambda f_1) \tilde{u} dx. \end{aligned}$$

Integrating by parts, using $(28)_{BC_4}$ and $(37)_1$ we have

$$\begin{aligned} &\int_0^L (\lambda^2 \varrho_1 h_1 u \tilde{u} + E_1 h_1 u_x \tilde{u}_x) dx + \left[\mathfrak{C} \int_{\mathbb{R}} \frac{\mu^2(\xi)}{\xi^2 + \eta + \lambda} d\xi\right] U(L) \tilde{u}(L) \\ &- k \int_0^L (-u + v + \gamma w_x) \tilde{u} dx = \int_0^L \varrho_1 h_1 (f_2 + \lambda\, f_1) \tilde{u} dx - \left[\mathfrak{C} \int_{\mathbb{R}} \frac{\mu(\xi) f_3(\xi)}{\xi^2 + \eta + \lambda} d\xi\right] \tilde{u}(L). \end{aligned} \tag{43}$$

Replacing $(38)_1$ into (43) we obtain

$$\begin{aligned}&\int_0^L(\lambda^2\varrho_1h_1u\tilde{u}+E_1h_1u_x\tilde{u}_x)dx+\left[\mathfrak{C}\int_{\mathbb{R}}\frac{\mu^2(\xi)}{\xi^2+\eta+\lambda}d\xi\right]\lambda u(L)\tilde{u}(L)\\&-k\int_0^L(-u+v+\gamma w_x)\tilde{u}dx=\int_0^L\varrho_1h_1(f_2+\lambda\, f_1)\tilde{u}dx\\&-\left[\mathfrak{C}\int_{\mathbb{R}}\frac{\mu(\xi)f_3(\xi)}{\xi^2+\eta+\lambda}d\xi\right]\tilde{u}(L)+\left[\mathfrak{C}\int_{\mathbb{R}}\frac{\mu^2(\xi)}{\xi^2+\eta+\lambda}d\xi\right]f_1(L)\tilde{u}(L).\end{aligned}\tag{44}$$

In a similar way we estimate (41) and (42), that is,

$$\begin{aligned}&\int_0^L(\lambda^2\varrho_3h_3v\tilde{v}+E_3h_3v_x\tilde{v}_x)dx+\left[\mathfrak{C}\int_{\mathbb{R}}\frac{\mu^2(\xi)}{\xi^2+\eta+\lambda}d\xi\right]\lambda v(L)\tilde{v}(L)\\&+k\int_0^L(-u+v+\gamma w_x)\tilde{v}dx=\int_0^L\varrho_3h_3(f_5+\lambda\, f_4)\tilde{v}dx\\&-\left[\mathfrak{C}\int_{\mathbb{R}}\frac{\mu(\xi)f_6(\xi)}{\xi^2+\eta+\lambda}d\xi\right]\tilde{v}(L)+\left[\mathfrak{C}\int_{\mathbb{R}}\frac{\mu^2(\xi)}{\xi^2+\eta+\lambda}d\xi\right]f_4(L)\tilde{v}(L),\end{aligned}\tag{45}$$

$$\begin{aligned}&\int_0^L(\lambda^2\varrho_3h_3w\tilde{w}+EIw_{xx}\tilde{w}_{xx})dx+\left[\mathfrak{C}\int_{\mathbb{R}}\frac{\mu^2(\xi)}{\xi^2+\eta+\lambda}d\xi\right]\lambda w(L)\tilde{w}(L)\\&+k\gamma\int_0^L(-u+v+\gamma w_x)\tilde{w}_xdx=\int_0^L\varrho h(f_8+\lambda\, f_7)\tilde{w}dx\\&-\left[\mathfrak{C}\int_{\mathbb{R}}\frac{\mu(\xi)f_9(\xi)}{\xi^2+\eta+\lambda}d\xi\right]\tilde{w}(L)+\left[\mathfrak{C}\int_{\mathbb{R}}\frac{\mu^2(\xi)}{\xi^2+\eta+\lambda}d\xi\right]f_7(L)\tilde{w}(L).\end{aligned}\tag{46}$$

The equations (44), (45) and (46) are equivalents to the problem

$$\mathfrak{a}\left((u,\,v,\,w),\,(\tilde{u},\,\tilde{v},\,\tilde{w})\right)=\mathcal{L}(\tilde{u},\,\tilde{v},\,\tilde{w}),\tag{47}$$

where the bilinear form $\mathfrak{a}:\left[\mathbb{H}^1(0,\,L)\times\mathbb{H}^1(0,\,L)\times\mathbb{H}^2(0,\,L)\right]^2\to\mathbb{R}$ and the linear form $\mathcal{L}:\mathbb{H}^1(0,\,L)\times\mathbb{H}^1(0,\,L)\times\mathbb{H}^2(0,\,L)\to\mathbb{R}$ are defined by

$$\begin{aligned}&\mathfrak{a}\left((u,\,v,\,w),\,(\tilde{u},\,\tilde{v},\,\tilde{w})\right)=\\&\int_0^L(\lambda^2\varrho_1h_1u\tilde{u}+E_1h_1u_x\tilde{u}_x)dx+\left[\mathfrak{C}\int_{\mathbb{R}}\frac{\mu^2(\xi)}{\xi^2+\eta+\lambda}d\xi\right]\lambda u(L)\tilde{u}(L)\\&\quad+\int_0^L(\lambda^2\varrho_3h_3v\tilde{v}+E_3h_3v_x\tilde{v}_x)dx+\left[\mathfrak{C}\int_{\mathbb{R}}\frac{\mu^2(\xi)}{\xi^2+\eta+\lambda}d\xi\right]\lambda v(L)\tilde{v}(L)\\&\quad+\int_0^L(\lambda^2\varrho_3h_3w\tilde{w}+EIw_{xx}\tilde{w}_{xx})dx+\left[\mathfrak{C}\int_{\mathbb{R}}\frac{\mu^2(\xi)}{\xi^2+\eta+\lambda}d\xi\right]\lambda w(L)\tilde{w}(L)\\&\quad+k\int_0^L(-u+v+\gamma w_x)(-\tilde{u}+\tilde{v}+\gamma\tilde{w}_x)dx\end{aligned}\tag{48}$$

and

$$
\begin{aligned}
&\mathcal{L}(\tilde{u},\, \tilde{v},\, \tilde{w}) = \\
&\int_0^L \varrho_1 h_1 (f_2 + \lambda\, f_1)\tilde{u} dx + \int_0^L \varrho_3 h_3 (f_5 + \lambda\, f_4)\tilde{v} dx + \int_0^L \varrho h (f_8 + \lambda\, f_7)\tilde{w} dx \\
&- \left[\mathfrak{C}\int_{\mathbb{R}} \frac{\mu(\xi) f_3(\xi)}{\xi^2+\eta+\lambda} d\xi\right] \tilde{u}(L) + \left[\mathfrak{C}\int_{\mathbb{R}} \frac{\mu^2(\xi)}{\xi^2+\eta+\lambda} d\xi\right] f_1(L)\tilde{u}(L) \\
&- \left[\mathfrak{C}\int_{\mathbb{R}} \frac{\mu(\xi) f_6(\xi)}{\xi^2+\eta+\lambda} d\xi\right] \tilde{v}(L) + \left[\mathfrak{C}\int_{\mathbb{R}} \frac{\mu^2(\xi)}{\xi^2+\eta+\lambda} d\xi\right] f_4(L)\tilde{v}(L) \\
&- \left[\mathfrak{C}\int_{\mathbb{R}} \frac{\mu(\xi) f_9(\xi)}{\xi^2+\eta+\lambda} d\xi\right] \tilde{w}(L) + \left[\mathfrak{C}\int_{\mathbb{R}} \frac{\mu^2(\xi)}{\xi^2+\eta+\lambda} d\xi\right] f_7(L)\tilde{w}(L). \qquad (49)
\end{aligned}
$$

It is easy to verify to $\mathfrak{a}$ is continuous and coercive, and $\mathcal{L}$ is continuous. So applying the Lax-Milgram Theorem, we deduce for all

$$(\tilde{u},\, \tilde{v},\, \tilde{w}) \in \mathbb{H}^1(0,\, L) \times \mathbb{H}^1(0,\, L) \times \mathbb{H}^2(0,\, L)$$

the problem (47) admits a unique solution

$$(u,\, v,\, w) \in \mathbb{H}^1(0,\, L) \times \mathbb{H}^1(0,\, L) \times \mathbb{H}^2(0,\, L).$$

Using elliptic regularity, it follows from (44)-(46) that

$$(u,\, v,\, w) \in H^2(0,\, L) \times H^2(0,\, L) \times H^4(0,\, L).$$

Therefore, the operator $\lambda I - \mathcal{A}$ is surjective for any $\lambda > 0$. As consequence of the Hille-Yosida Theorem [22, Theorem 1.2.2, page 3], we have that $\mathcal{A}$ generates a C_0-semigroup of contractions $S(t) = e^{t\mathcal{A}}$ on $\mathcal{H}$. From semigroup theory, $\mathcal{U}(t) = e^{t\mathcal{A}}\mathcal{U}_0$ is the unique solution of (33) satisfying the conditions of theorem and the proof is complete. □

4. Strong stability

This section deals with strong stability, in the following approach:

Theorem 4.1. *The C_0-semigroup $e^{t\mathcal{A}}$ is strongly stable in $\mathcal{H}$, that is, for all $U_0 \in \mathcal{H}$, the solution of* (73) *satisfies*

$$\lim_{t\to+\infty} \|e^{t\mathcal{A}} U_0\|_{\mathcal{H}} = 0.$$

According to ideas from the works of the first author [23, 31], we will apply a general criteria due to Arendt-Batty and Lyubich-Vũ to prove the strong stability of the C_0-semigroup $e^{t\mathcal{A}}$, associated to the system (33) in the absence of the compactness of the resolvent of $\mathcal{A}$.

Theorem 4.2 (Arendt-Batty and Lyubich-Vũ, [2, 19])**.** *Let B be a reflexive Banach space and $\{\mathcal{S}(t)\}_{t\geq 0}$ be a C_0-semigroup generated by $\mathcal{A}$ on B. Assume that $\{\mathcal{S}(t)\}_{t\geq 0}$ is bounded and that no eigenvalues of $\mathcal{A}$ lie on the imaginary axis. If $\sigma(\mathcal{A}) \cap i\,\mathbb{R}$ is countable, then $S(t)$ is strongly stable.*

Adapting the theorem 4.2 to Hilbert space, in the context of this work, the strong stability result is given by:

Theorem 4.3. *Let $\mathcal{A}$ be the infinitesimal generator of a uniformly bounded C_0-semigroup $\{\mathcal{S}(t)\}_{t\geq 0}$ on a Hilbert space $\mathcal{H}$. If*

(i) *If* $\sigma(\mathcal{A}) \cap i\,\mathbb{R}$ *is at most a countable set, where* $\sigma(\mathcal{A})$ *denotes the spectrum of* $\mathcal{A}$*;*

(ii) *If* $\sigma_r(\mathcal{A}) \cap i\,\mathbb{R} = \emptyset$*, where* $\sigma_r(\mathcal{A})$ *denotes the set of residual spectrum of* $\mathcal{A}$*.*

Then the semigroup $\{\mathcal{S}(t)\}_{t\geq 0}$ *is asymptotically stable, that is,*

$$\|\mathcal{S}(t)y\|_{\mathcal{H}} \to 0 \quad as \quad t \to \infty, \ \ for\ any\ y \in \mathcal{H}.$$

The prove of Theorem 4.3 will be done by some lemmas.

Lemma 4.4. *We have*

$$\sigma(\mathcal{A}) \cap \{i\,\lambda,\ \lambda \in \mathbb{R},\ \lambda \neq 0\} = \emptyset.$$

Proof. The proof is by contradiction. We suppose that there $\lambda \in \mathbb{R}$, $\lambda \neq 0$ and $\mathcal{U} \neq 0$, such that $\mathcal{A}\mathcal{U} = i\,\lambda\mathcal{U}$, that is, $(i\,\lambda - \mathcal{A})\mathcal{U} = 0$. Then

$$\begin{cases} i\lambda u - U = 0, \\ i\lambda\varrho_1 h_1 U - E_1 h_1 u_{xx} - k(-u + v + \gamma w_x) = 0, \\ i\lambda\varphi + (\xi^2 + \eta)\varphi - U(L)\mu(\xi) = 0, \\ i\lambda v - V = 0, \\ i\lambda\varrho_3 h_3 V - E_3 h_3 v_{xx} + k(-u + v + \gamma w_x) = 0, \\ i\lambda\phi + (\xi^2 + \eta)\phi - V(L)\mu(\xi) = 0, \\ i\lambda w - W = 0, \\ i\lambda\varrho h W + EI w_{xxxx} - k\gamma(-u + v + \gamma w_x)_x = 0, \\ i\lambda\psi + (\xi^2 + \eta)\psi - W(L)\mu(\xi) = 0. \end{cases} \tag{50}$$

Note that by $(50)_{1,\,4,\,7}$ we have

$$\begin{cases} U(L) = i\,\lambda\,u(L), \\ V(L) = i\,\lambda\,v(L), \\ W(L) = i\,\lambda\,w(L). \end{cases} \tag{51}$$

Now, from (35) we have $\varphi(\xi) = \phi(\xi) = \psi(\xi) = 0$. Hence from $(50)_{3,\,6}$ we have

$$U(L,\,t) = V(L,\,t) = V(L,\,t) = 0. \tag{52}$$

Moreover, from the systems $(50)_{1,\,4,\,7}$ and $(28)_{BC_{4,\,5,\,6}}$ we get

$$u(L) = v(L) = w(L) = 0, \qquad u_x(L) = w_x(L) = w_{xxx}(L) = 0. \tag{53}$$

On the other hand, replacing $(50)_{1,\,4,\,7}$ into $(50)_{2,\,5,\,8}$ respectively we obtain

$$\begin{cases} -\lambda^2\varrho_1 h_1 u - E_1 h_1 u_{xx} - k(-u + v + \gamma w_x) = 0, \\ -\lambda^2\varrho_3 h_3 v - E_3 h_3 v_{xx} + k(-u + v + \gamma w_x) = 0, \\ -\lambda^2\varrho h w + EI w_{xxxx} - k\gamma(-u + v + \gamma w_x)_x = 0. \end{cases} \tag{54}$$

Let's consider $X = (u, u_x, v, v_x, w, w_x, w_{xx}, w_{xxx})$. Then we can rewrite (52)-(54) as the initial value problem

$$\begin{cases} \dfrac{d}{dx}X = \mathbb{A}X, \\ \\ X(L) = 0, \end{cases} \tag{55}$$

where

$$\mathbb{A} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{(-\lambda^2 \varrho_3 h_3 + k)}{E_1 h_1} & \frac{-k}{E_1 h_1} & 0 & 0 & 0 & \frac{-k\gamma}{E_1 h_1} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{-k}{E_3 h_3} & \frac{(-\lambda^2 \varrho_3 h_3 + k)}{E_3 h_3} & 0 & 0 & 0 & \frac{k}{E_3 h_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{k\gamma}{EI} & \frac{k\gamma}{EI} & \frac{\lambda^2 \varrho h}{EI} & 0 & \frac{k\gamma^2}{EI} & 0 \end{pmatrix}. \quad (56)$$

Using the Picard theorem (ordinary differential equations), (55) has a unique solution $X = 0$. Thus, $u = 0$, $v = 0$ and $w = 0$. It follows from $(50)_{1,4,7}$ that $U = 0$, $V = 0$, and $W = 0$. Therefore, $\mathcal{U} = 0$. Then, $i\mathbb{R} \subset \rho(\mathcal{A}) = \mathbb{C}\,\sigma(\mathcal{A})$ and consequently, $\mathcal{A}$ does not have purely imaginary eigenvalues. □

By Theorem 4.3, the condition (i) holds if we show that any point $\sigma(\mathbb{A}) \cap \{i\,\mathbb{R}\}$ is at most a countable set. It's will proved in the following two lemmas.

Lemma 4.5. *The operator $i\,\lambda\, I - \mathcal{A}$ is surjective for $\lambda \neq 0$.*

Proof. In fact, we will prove that the operator $i\,\lambda\, I - \mathcal{A}$ is surjective for $\lambda \neq 0$. For this purpose, let $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9)^T \in \mathbb{H}$, we seek $\mathcal{U} = (u, U, \varphi, v, V, \phi, w, W, \psi)^T \in D(\mathcal{A})$ such that $(i\,\lambda\, I - \mathcal{A})\mathcal{U} = F$, that lead to,

$$\begin{cases} i\lambda u - U = f_1 \quad \text{in} \quad \mathbb{H}^1(0,\, L), \\ i\lambda \varrho_1 h_1 U - E_1 h_1 u_{xx} - k(-u + v + \gamma w_x) = \varrho_1 h_1 f_2 \quad \text{in} \quad L^2(0,\, L), \\ i\lambda\varphi + (\xi^2 + \eta)\varphi - U(L)\mu(\xi) = f_3 \quad \text{in} \quad L^2(\mathbb{R}), \\ i\lambda v - V = f_4 \quad \text{in} \quad \mathbb{H}^1(0,\, L), \\ i\lambda \varrho_3 h_3 V - E_3 h_3 v_{xx} + k(-u + v + \gamma w_x) = \varrho_3 h_3 f_5 \quad \text{in} \quad L^2(0,\, L), \\ i\lambda\phi + (\xi^2 + \eta)\phi - V(L)\mu(\xi) = f_6 \quad \text{in} \quad L^2(\mathbb{R}), \\ i\lambda w - W = f_7 \quad \text{in} \quad \mathbb{H}^2(0,\, L), \\ i\lambda \varrho h W + EI w_{xxxx} - k\gamma(-u + v + \gamma w_x)_x = \varrho h f_8 \quad \text{in} \quad L^2(0,\, L), \\ i\lambda\psi + (\xi^2 + \eta)\psi - W(L)\mu(\xi) = f_9, \quad \text{in} \quad L^2(\mathbb{R}) \end{cases} \quad (57)$$

with the following conditions

$$\begin{cases} E_1 h_1 u_x(L) = -\mathfrak{C} \int_{\mathbb{R}} \mu(\xi)\varphi(\xi) d\xi, \\ E_3 h_3 v_x(L) = -\mathfrak{C} \int_{\mathbb{R}} \mu(\xi)\phi(\xi) d\xi, \\ EI w_{xxx}(L) = \mathfrak{C} \int_{\mathbb{R}} \mu(\xi)\psi(\xi) d\xi. \end{cases} \quad (58)$$

Suppose that we have found u, v and w with the appropriated regularity. Therefore, from $(57)_{1,4,7}$ we have

$$\begin{cases} U = i\,\lambda\, u - f_1, \\ V = i\,\lambda\, v - f_4, \\ W = i\,\lambda\, w - f_7. \end{cases} \quad (59)$$

and

$$\begin{cases} U(L) = i\lambda u(L) - f_1(L), \\ V(L) = i\lambda v(L) - f_4(L), \\ W(L) = i\lambda w(L) - f_7(L). \end{cases} \quad (60)$$

It is clear that $u,\, v \in \mathbb{H}^1(0,\, L)$ and $w \in \mathbb{H}^2(0,\, L)$. Then, replacing $(59)_{1,\,2,\,3}$ into $(57)_{2,\,5,\,8}$ it follows that

$$\left\{ \begin{array}{l} -\lambda^2 \varrho_1 h_1 u - E_1 h_1 u_{xx} - k(-u + v + \gamma w_x) = \varrho_1 h_1 f_2 + i\lambda \varrho_1 h_1 f_1, \\ -\lambda^2 \varrho_3 h_3 v - E_3 h_3 v_{xx} + k(-u + v + \gamma w_x) = \varrho_3 h_3 f_5 + i\lambda \varrho_3 h_3 f_4, \\ -\lambda^2 \varrho h w + EI w_{xxxx} - k\gamma(-u + v + \gamma w_x)_x = \varrho h f_8 + i\lambda \varrho h f_7. \end{array} \right. \tag{61}$$

Solving system (61) is equivalent to finding $(u,\, v) \in \left[H^2(0,\, L) \cap \mathbb{H}^1(0,\, L)\right]^2$ and $w \in H^4(0,\, L) \cap \mathbb{H}^2(0,\, L)$ such that

$$\begin{aligned} \int_0^L \left[-\lambda^2 \varrho_1 h_1 u - E_1 h_1 u_{xx} - k(-u + v + \gamma w_x)\right] \tilde{u} dx &= \int_0^L \left[\varrho_1 h_1 f_2 + i\lambda \varrho_1 h_1 f_1\right] \tilde{u} dx, \\ \int_0^L \left[-\lambda^2 \varrho_3 h_3 v - E_3 h_3 v_{xx} + k(-u + v + \gamma w_x)\right] \tilde{v} dx &= \int_0^L \left[\varrho_3 h_3 f_5 + i\lambda \varrho_3 h_3 f_4\right] \tilde{v} dx, \\ \int_0^L \left[-\lambda^2 \varrho h w + EI w_{xxxx} - k\gamma(-u + v + \gamma w_x)_x\right] \tilde{w} dx &= \int_0^L \left[\varrho h f_8 + i\lambda \varrho h f_7\right] \tilde{w} dx, \end{aligned} \tag{62}$$

for all $\tilde{u},\, \tilde{v} \in \mathbb{H}^1(0,\, L)$ and $\tilde{w} \in \mathbb{H}^2(0,\, L)$. Performing similar estimates as (40), (41) and (42) we obtain

$$\begin{aligned} &\int_0^L (-\lambda^2 \varrho_1 h_1 u \tilde{u} + E_1 h_1 u_x \tilde{u}_x) dx + i\lambda \left[\mathfrak{C} \int_{\mathbb{R}} \frac{\mu^2(\xi)}{\xi^2 + \eta + i\lambda} d\xi\right] u(L) \tilde{u}(L) \\ &- k \int_0^L (-u + v + \gamma w_x) \tilde{u} dx = \int_0^L \varrho_1 h_1 (f_2 + i\lambda f_1) \tilde{u} dx \\ &- \left[\mathfrak{C} \int_{\mathbb{R}} \frac{\mu(\xi) f_3(\xi)}{\xi^2 + \eta + i\lambda} d\xi\right] \tilde{u}(L) + \left[\mathfrak{C} \int_{\mathbb{R}} \frac{\mu^2(\xi)}{\xi^2 + \eta + i\lambda} d\xi\right] f_1(L) \tilde{u}(L), \end{aligned} \tag{63}$$

$$\begin{aligned} &\int_0^L (-\lambda^2 \varrho_3 h_3 v \tilde{v} + E_3 h_3 v_x \tilde{v}_x) dx + i\lambda \left[\mathfrak{C} \int_{\mathbb{R}} \frac{\mu^2(\xi)}{\xi^2 + \eta + i\lambda} d\xi\right] v(L) \tilde{v}(L) \\ &+ k \int_0^L (-u + v + \gamma w_x) \tilde{v} dx = \int_0^L \varrho_3 h_3 (f_5 + i\lambda f_4) \tilde{v} dx \\ &- \left[\mathfrak{C} \int_{\mathbb{R}} \frac{\mu(\xi) f_6(\xi)}{\xi^2 + \eta + i\lambda} d\xi\right] \tilde{v}(L) + \left[\mathfrak{C} \int_{\mathbb{R}} \frac{\mu^2(\xi) dx}{\xi^2 + \eta + i\lambda}\right] f_4(L) \tilde{v}(L), \end{aligned} \tag{64}$$

$$\begin{aligned} &\int_0^L (-\lambda^2 \varrho_3 h_3 w \tilde{w} + EI w_{xx} \tilde{w}_{xx}) dx + i\lambda \left[\mathfrak{C} \int_{\mathbb{R}} \frac{\mu^2(\xi)}{\xi^2 + \eta + i\lambda} d\xi\right] w(L) \tilde{w}(L) \\ &+ k \int_0^L (-u + v + \gamma w_x) \gamma \tilde{w}_x dx = \int_0^L \varrho h (f_8 + i\lambda f_7) \tilde{w} dx \\ &- \left[\mathfrak{C} \int_{\mathbb{R}} \frac{\mu(\xi) f_9(\xi)}{\xi^2 + \eta + i\lambda} d\xi\right] \tilde{w}(L) + \left[\mathfrak{C} \int_{\mathbb{R}} \frac{\mu^2(\xi) dx}{\xi^2 + \eta + i\lambda}\right] f_7(L) \tilde{w}(L). \end{aligned} \tag{65}$$

The system (63)-(65) is equivalent to the problem

$$-\langle \mathbb{L}_\lambda \mathcal{U},\, \mathcal{V} \rangle_{[\mathbb{H}^1(0,\, L) \times \mathbb{H}^1(0,\, L) \times \mathbb{H}^2(0,\, 1)]^2} + \langle \mathcal{U},\, \mathcal{V} \rangle_{[\mathbb{H}^1(0,\, L) \times \mathbb{H}^1(0,\, L) \times \mathbb{H}^2(0,\, 1)]^2} = \Phi(\mathcal{V}), \tag{66}$$

where

$$\begin{aligned}\langle \mathbb{L}_\lambda \mathcal{U},\, \mathcal{V}\rangle =&\lambda^2 \int_0^L [\varrho_1 h_1 u\tilde{u} + \varrho_3 h_3 v\tilde{v} + \varrho h w\tilde{w}]\, dx - i\lambda \left[\mathfrak{C}\int_{\mathbb{R}} \frac{\mu^2(\xi)}{\xi^2+\eta+i\lambda} d\xi\right] u(L)\tilde{u}(L)\\ &- i\lambda \left[\mathfrak{C}\int_{\mathbb{R}} \frac{\mu^2(\xi)}{\xi^2+\eta+i\lambda} d\xi\right] v(L)\tilde{v}(L) - i\lambda \left[\mathfrak{C}\int_{\mathbb{R}} \frac{\mu^2(\xi)}{\xi^2+\eta+i\lambda} d\xi\right] w(L)\tilde{w}(L)\end{aligned}$$

and

$$\begin{aligned}\langle \mathcal{U},\, \mathcal{V}\rangle =&E_1 h_1 \int_0^L u_x \tilde{u}_x dx + E_3 h_3 \int_0^L v_x \tilde{v}_x dx + EI \int_0^L w_x \tilde{w}_x dx\\ &- k\int_0^L (-u+v+\gamma w_x)(-\tilde{u}+\tilde{v}+\gamma\tilde{w}_x)dx.\end{aligned}$$

Using that

$$\begin{aligned}L^2(0,\, L) \overset{c}{\hookrightarrow} H^{-1}(0,\, L), \text{ that is, } (L^2(0,\, L) \overset{c}{\hookrightarrow}, H^{-2}(0,\, L)),\\ \mathbb{H}^1(0,\, L) \overset{c}{\hookrightarrow} L^2(0,\, L), \text{ that is, } (\mathbb{H}^2(0,\, L) \overset{c}{\hookrightarrow} L^2(0,\, L)),\end{aligned}$$

it follows that the operator $\mathbb{L}_\lambda$ is compact from $[L^2(0,\, L)]^3$ into $[L^2(0,\, L)]^3$. This way, by Fredholm alternative, proving the existence of $\mathcal{U}$ solution of (66) reduces to show that 1 is not a eigenvalue of $\mathbb{L}_\lambda$. In fact, if 1 is an eigenvalue, then there exists $\mathbb{U} \neq 0$, such that

$$\langle \mathbb{L}_\lambda \mathbb{U},\, \mathbb{V}\rangle_{\mathbb{M}^2} = \langle \mathbb{U},\, \mathbb{V}\rangle_{\mathbb{M}^2}, \tag{67}$$

for all $\mathbb{V} = (\tilde{u},\, \tilde{v},\, \tilde{w}) \in \mathbb{M}$, where $\mathbb{M} = \left\{\tilde{u},\, \tilde{v} \in \mathbb{H}^1(0,\, L) \quad \text{and} \quad \tilde{w} \in \mathbb{H}^2(0,\, L)\right\}$. In particular for $\mathbb{U} = \mathbb{V}$, we have

$$\begin{aligned}&\lambda^2 \left[\varrho_1 h_1 \|u\|^2_{L^2(0,\, L)} + \varrho_3 h_3 \|v\|^2_{L^2(0,\, L)} + \varrho h \|w\|^2_{L^2(0,\, L)}\right]\\ &- i\lambda \left[\mathfrak{C}\int_{\mathbb{R}} \frac{\mu^2(\xi)}{\xi^2+\eta+i\lambda} d\xi\right] |u(L)|^2 - i\lambda \left[\mathfrak{C}\int_{\mathbb{R}} \frac{\mu^2(\xi)}{\xi^2+\eta+i\lambda} d\xi\right] |v(L)|^2\\ &- i\lambda \left[\mathfrak{C}\int_{\mathbb{R}} \frac{\mu^2(\xi)}{\xi^2+\eta+i\lambda} d\xi\right] |w(L)|^2\\ &= E_1 h_1 \|u_x\|^2_{L^2(0,\, L)} + E_3 h_3 \|v_x\|^2_{L^2(0,\, L)} + EI \|w_x\|^2_{L^2(0,\, L)} + k\|-u+v+\gamma w_x\|^2_{L^2(0,\, L)}.\end{aligned}$$

Thus, by the above equation the imaginary term are equal to zero, then we have

$$u(L) = v(L) = w(L) = 0. \tag{68}$$

From (67) we obtain

$$u_x(L) = v_x(L) = w_x(L) = 0 \tag{69}$$

and

$$\begin{aligned}-\lambda^2 \varrho_1 h_1 u - E_1 h_1 u_{xx} - k(-u+v+\gamma w_x) = 0,\\ -\lambda^2 \varrho_3 h_3 v - E_3 h_3 v_{xx} + k(-u+v+\gamma w_x) = 0,\\ -\lambda^2 \varrho h w + EI w_{xxxx} - k\gamma(-u+v+\gamma w_x)_x = 0.\end{aligned} \tag{70}$$

Similar to what was done in (54) us consider $X = (u, v, u_x, v_x, w, w_x, w_{xx}, w_{xxx})$. Then we can rewrite (68)-(70) as the initial value problem

$$\begin{cases} \dfrac{d}{dx}X = \mathbb{A}X, \\ \\ X(L) = 0, \end{cases} \tag{71}$$

Using the Picard theorem (ordinary differential equations), (71) has a unique solution $X = 0$. Thus, $u = 0$, $v = 0$, and $w = 0$. It follows from (57) that $U = 0$, $V = 0$, and $W = 0$. Therefore, $\mathbb{U} = 0$. □

Lemma 4.6. *If* $\lambda \neq 0$, *we have that* $0 \in \varrho(\mathcal{A})$.

Proof. We have that $\mathcal{U} = (u, U, \varphi, v, V, \phi, w, W, \psi)^T \in \ker(\mathcal{A})$ if and only if $\mathcal{A}\mathcal{U} = 0$. From (34), we have

$$\begin{cases} U = 0, \\ E_1h_1u_{xx} + k(-u + v + \gamma w_x) = 0, \\ (\xi^2 + \eta)\varphi - U(L)\mu(\xi) = 0, \\ V = 0, \\ E_3h_3v_{xx} - k(-u + v + \gamma w_x) = 0, \\ (\xi^2 + \eta)\phi - V(L)\mu(\xi) = 0, \\ W = 0, \\ EIw_{xxxx} - k\gamma(-u + v + \gamma w_x)_x = 0, \\ (\xi^2 + \eta)\psi - W(L)\mu(\xi) = 0. \end{cases} \tag{72}$$

Replacing $(72)_{1,4,7}$ into $(72)_{3,6,9}$ implies $U = V = W = \varphi = \phi = \psi = 0$. Multiplying $(72)_{2,5,8}$ by u, v, and w respectively, integrating each equation over $(0, L)$, and using the definition of $\mathbb{H}^1(0, L)$ and $\mathbb{H}^2(0, L)$ we obtain

$$\begin{aligned} -E_1h_1\int_0^L u_x^2dx + k\int_0^L(-u + v + \gamma w_x)udx + u(L)E_1h_1u_x(L) = 0, \\ -E_3h_3\int_0^L v_x^2dx - k\int_0^L(-u + v + \gamma w_x)vdx + v(L)E_3h_3v_x(L) = 0, \\ -EI\int_0^L w_{xx}^2dx - k\int_0^L(-u + v + \gamma w_x)\gamma w_x - w_x(L)EIw_{xxx}(L)dx = 0. \end{aligned} \tag{73}$$

Using $(28)_{BC_{4,5,6}}$ and performing straightforward calculations we get

$$\begin{aligned} &- E_1h_1\int_0^L u_x^2dx - E_3h_3\int_0^L v_x^2dx - EI\int_0^L w_{xx}^2dx - k\int_0^L |-u + v + \gamma w_x|^2dx \\ &- u(L)\mathfrak{C}\int_{\mathbb{R}} \mu(\xi)\varphi(\xi)d\xi - v(L)\mathfrak{C}\int_{\mathbb{R}} \mu(\xi)\phi(\xi)d\xi - w_x(L)\mathfrak{C}\int_{\mathbb{R}} \mu(\xi)\psi(\xi)d\xi = 0. \end{aligned}$$

Now, using that $\varphi = \phi = 0$ it follows that

$$E_1h_1\|u_x\|^2_{L^2(0,L)} + E_3h_3\|v_x\|^2_{L^2(0,L)} + EI\|w_{xx}\|^2_{L^2(0,L)} + k\|-u + v + \gamma w_x\|^2_{L^2(0,L)} = 0. \tag{74}$$

From (74) we have that u and v are constant functions and the last term in (74) implies that w is a constant function. Thereby, $\mathcal{U} = (u, U, \varphi, v, V, \phi, w, W, \psi)^T = 0$ and $\mathcal{A}$ is injective.

Now, given $\mathcal{F} = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9) \in \mathcal{H}$, we must show that there exists a unique $\mathcal{U} = (u, U, \varphi, v, V, \phi, w, W, \psi)^T$ in $\mathcal{D}(\mathcal{A})$, such that $-\mathcal{A}\mathcal{U} = \mathcal{F}$, namely

$$\begin{cases} -U = f_1, \\ -E_1 h_1 u_{xx} - k(-u + v + \gamma w_x) = \varrho_1 h_1 f_2, \\ (\xi^2 + \eta)\varphi - U(L)\mu(\xi) = f_3, \\ -V = f_4, \\ -E_3 h_3 v_{xx} + k(-u + v + \gamma w_x) = \varrho_3 h_3 f_5, \\ (\xi^2 + \eta)\phi - V(L)\mu(\xi) = f_6, \\ -W = f_7, \\ EI w_{xxxx} - k\gamma(-u + v + \gamma w_x)_x = \varrho h f_8, \\ (\xi^2 + \eta)\psi - W(L)\mu(\xi) = f_9, \end{cases} \tag{75}$$

with the following boundary conditions

$$\begin{aligned} E_1 h_1 u_x(L) &= -\mathfrak{C} \int_{\mathbb{R}} \mu(\xi)\varphi(\xi) d\xi, \\ E_3 h_3 v_x(L) &= -\mathfrak{C} \int_{\mathbb{R}} \mu(\xi)\phi(\xi) d\xi, \\ EI w_{xxx}(L) &= \mathfrak{C} \int_{\mathbb{R}} \mu(\xi)\psi(\xi) d\xi, \end{aligned} \tag{76}$$

where $\mathfrak{C} = \pi^{-1}\sin(\alpha\pi)$. Using the same idea to the system (63)-(65) for $\lambda = 0$ it follows that

$$\begin{aligned} &\int_0^L E_1 h_1 u_x \tilde{u}_x dx - k \int_0^L (-u + v + \gamma w_x)\tilde{u} dx = \int_0^L \varrho_1 h_1 f_2 \tilde{u} dx \\ &- \left[\mathfrak{C} \int_{\mathbb{R}} \frac{\mu(\xi) f_3(\xi)}{\xi^2 + \eta} d\xi\right] \tilde{u}(L) + \left[\mathfrak{C} \int_{\mathbb{R}} \frac{\mu^2(\xi)}{\xi^2 + \eta} d\xi\right] f_1(L)\Gamma(L), \end{aligned} \tag{77}$$

$$\begin{aligned} &\int_0^L E_3 h_3 v_x \tilde{v}_x dx + k \int_0^L (-u + v + \gamma w_x)\tilde{v} dx = \int_0^L \varrho_3 h_3 f_5 \tilde{v} dx \\ &- \left[\mathfrak{C} \int_{\mathbb{R}} \frac{\mu(\xi) f_6(\xi)}{\xi^2 + \eta} d\xi\right] \tilde{v}(L) + \left[\mathfrak{C} \int_{\mathbb{R}} \frac{\mu^2(\xi)}{\xi^2 + \eta} d\xi\right] f_4(L)\tilde{v}(L), \end{aligned} \tag{78}$$

$$\begin{aligned} &\int_0^L EI w_{xx} \tilde{w}_{xx} dx + k \int_0^L (-u + v + \gamma w_x)\gamma \tilde{w}_x dx = \int_0^L \varrho h f_8 \tilde{w} dx \\ &- \left[\mathfrak{C} \int_{\mathbb{R}} \frac{\mu(\xi) f_9(\xi)}{\xi^2 + \eta} d\xi\right] \tilde{w}(L) + \left[\mathfrak{C} \int_{\mathbb{R}} \frac{\mu^2(\xi)}{\xi^2 + \eta} d\xi\right] f_7(L)\tilde{w}(L). \end{aligned} \tag{79}$$

The system (77)-(79) is equivalent to the problem

$$\mathfrak{a}_\eta\left((u,\, v,\, w),\ (\tilde{u},\, \tilde{v},\, \tilde{w})\right) = \mathcal{L}_\eta(\tilde{u},\, \tilde{v},\, \tilde{w}), \tag{80}$$

where the bilinear form continuous and coercive

$$\mathfrak{a}_\eta : \left[\mathbb{H}^1(0,\, L) \times \mathbb{H}^1(0,\, L) \times \mathbb{H}^2(0,\, L)\right]^2 \to \mathbb{R}$$

and the continuous linear form

$$\mathcal{L}_\eta : \mathbb{H}^1(0,\, L) \times \mathbb{H}^1(0,\, L) \times \mathbb{H}^2(0,\, L) \to \mathbb{R}$$

is defined by

$$\mathfrak{a}_\eta\left((u,\, v,\, w),\ (\tilde{u},\, \tilde{v},\, \tilde{w})\right) = E_1h_1\int_0^L u_x\tilde{u}_x dx + E_3h_3\int_0^L v_x\tilde{v}_x dx + EI\int_0^L w_{xx}\,\tilde{w}_{xx}\ dx + k\int_0^L(-u+v+\gamma w_x)(-\tilde{u}+\tilde{v}+\gamma\tilde{w}_x)dx \tag{81}$$

and

$$\begin{aligned}\mathcal{L}(\tilde{u},\, \tilde{v},\, \tilde{w}) =& \varrho_1h_1\int_0^L f_2\tilde{u}dx + \varrho_3h_3\int_0^L f_5\tilde{v}dx + \varrho h\int_0^L f_8\tilde{w}dx\\ &-\left[\mathfrak{C}\int_{\mathbb{R}}\frac{\mu(\xi)f_3(\xi)}{\xi^2+\eta}d\xi\right]\tilde{u}(L) + \left[\mathfrak{C}\int_{\mathbb{R}}\frac{\mu^2(\xi)}{\xi^2+\eta}d\xi\right]f_1(L)\tilde{u}(L)\\ &-\left[\mathfrak{C}\int_{\mathbb{R}}\frac{\mu(\xi)f_6(\xi)}{\xi^2+\eta}d\xi\right]\tilde{v}(L) + \left[\mathfrak{C}\int_{\mathbb{R}}\frac{\mu^2(\xi)}{\xi^2+\eta}d\xi\right]f_4(L)\tilde{v}(L)\\ &-\left[\mathfrak{C}\int_{\mathbb{R}}\frac{\mu(\xi)f_9(\xi)}{\xi^2+\eta}d\xi\right]\tilde{w}(L) + \left[\mathfrak{C}\int_{\mathbb{R}}\frac{\mu^2(\xi)}{\xi^2+\eta}d\xi\right]f_7(L)\tilde{w}(L).\end{aligned} \tag{82}$$

Applying the Lax-Milgran theorem, we have that for all

$$(\tilde{u},\, \tilde{v},\, \tilde{w}) \in \mathbb{H}^1(0,\, L)\times\mathbb{H}^1(0,\, L)\times\mathbb{H}^2(0,\, L)$$

the problem (80) admits a unique solution

$$(u,\, v,\, w) \in \mathbb{H}^1(0,\, L)\times\mathbb{H}^1(0,\, L)\times\mathbb{H}^2(0,\, L).$$

Using elliptic regularity, it follows from (77)-(79) that

$$(u,\, v,\, w) \in H^2(0,\, L)\times H^2(0,\, L)\times H^4(0,\, L).$$

Therefore, the operator $\mathcal{A}$ is surjective. □

Now we introduce the operator $\mathcal{A}^*$.

Lemma 4.7. *Let* $\mathcal{A}$ *be defined by* (34)*, then*

$$\mathcal{A}^*\begin{pmatrix} u\\ U\\ \varphi\\ v\\ V\\ \phi\\ w\\ W\\ \psi\end{pmatrix} = \begin{pmatrix} -U\\ \frac{1}{\varrho_1h_1}\left[E_1h_1u_{xx}+k\left(-u+v+\gamma w_x\right)\right]\\ -(\xi^2+\eta)\varphi - U(L)\mu(\xi)\\ -V\\ \frac{1}{\varrho_3h_3}\left[E_3h_3v_{xx}-k(-u+v+\gamma w_x)\right]\\ -(\xi^2+\eta)\phi - V(L)\mu(\xi)\\ -W\\ \frac{1}{\varrho h}\left[-EIw_{xxxx}+k\gamma(-u+v+\gamma w_x)_x\right]\\ -(\xi^2+\eta)\psi - W(L)\mu(\xi)\end{pmatrix}, \tag{83}$$

with the domain

$$\mathcal{D}(\mathcal{A}^*) = \left\{\mathcal{U}\in\mathcal{H}\ \middle|\ \begin{array}{l} u,\, v\in H^2(0,\, L),\ w\in H^4(0,\, L),\\ U,\, V\in\mathbb{H}^1(0,\, L),\ W\in\mathbb{H}^2(0,\, L),\\ -(\xi^2+\eta)\,\varphi - U(L)\,\mu(\xi)\in L^2(\mathbb{R}),\\ -(\xi^2+\eta)\,\phi - V(L)\,\mu(\xi)\in L^2(\mathbb{R}),\\ -(\xi^2+\eta)\,\psi - W(L)\mu(\xi)\in L^2(\mathbb{R}),\\ E_1h_1\,u_x(L)+\mathfrak{C}\int_{\mathbb{R}}\mu(\xi)\,\varphi(\xi)d\xi = 0,\\ E_3h_3\,v_x(L)+\mathfrak{C}\int_{\mathbb{R}}\mu(\xi)\,\phi(\xi)d\xi = 0,\\ EI\,w_{xxx}(L)-\mathfrak{C}\int_{\mathbb{R}}\mu(\xi)\,\psi(\xi)d\xi = 0,\\ |\xi|\varphi,\ |\xi|\phi,\ |\xi|\psi\in L^2(\mathbb{R}) \end{array}\right\}.$$

Proof. It is not difficult to show that $\langle \mathcal{A}\mathcal{U}, \mathcal{U}\rangle = \langle \mathcal{U}, \mathcal{A}^*\mathcal{U}\rangle$. □

The prove that no eigenvalues of $\mathcal{A}$ lie on the imaginary axis is given by the next lemma.

Lemma 4.8. *$\sigma_r(\mathcal{A}) = \emptyset$, where $\sigma_r(\mathcal{A})$ denotes the set of residual spectrum of $\mathcal{A}$.*

Proof. Since $\lambda \in \sigma_r(\mathcal{A})$, $\overline{\lambda} \in \sigma_p(\mathcal{A}^*)$ the proof will be successful if we can show that $\sigma_r(\mathcal{A}) = \sigma_p(\mathcal{A}^*)$. This is because we have considering that the eigenvalues of $\mathcal{A}$ are symmetric on the real axis. In fact, we will consider the eigenvalue problem $\mathcal{A}^*U = \lambda U$ for $\lambda \in \mathbb{C}$ and $0 \neq \mathcal{U} = (u, U, \varphi, v, V, \phi, w, W, \psi)$ in $\mathcal{D}(\mathcal{A}^*)$, that is, from (83)

$$\begin{cases} \lambda u + U = 0, \\ \lambda \varrho_1 h_1 U - E_1 h_1 u_{xx} + k(-u + v - \gamma w_x) = 0, \\ \lambda \varphi + (\xi^2 + \eta)\varphi + U(L)\mu(\xi) = 0, \\ \lambda v + V = 0, \\ \lambda \varrho_3 h_3 V - E_3 h_3 v_{xx} + k(-u + v + \gamma w_x) = 0, \\ \lambda \phi + (\xi^2 + \eta)\phi + V(L)\mu(\xi) = 0, \\ \lambda w + W = 0, \\ \lambda \varrho h W + EI w_{xxxx} - k(-u + v + \gamma w_x)_x = 0, \\ \lambda \psi + (\xi^2 + \eta)\psi + W(L)\mu(\xi) = 0. \end{cases} \tag{84}$$

Replacing $(84)_{1,4,7}$ into $(84)_{2,5,8}$ respectively we obtain

$$\begin{aligned} &-\lambda^2 \varrho_1 h_1 U - E_1 h_1 u_{xx} + k(-u + v - \gamma w_x) = 0, \\ &-\lambda^2 \varrho_3 h_3 V - E_3 h_3 v_{xx} + k(-u + v + \gamma w_x) = 0, \\ &-\lambda^2 \varrho h W + EI w_{xxxx} - k(-u + v + \gamma w_x)_x = 0, \end{aligned} \tag{85}$$

with the following boundary conditions

$$\begin{aligned} u(0, t) = v(0, t) &= 0, \\ E_1 h_1 u_x(L) &= -\lambda(\lambda + \eta)^{\alpha - 1} u(L), \\ E_3 h_3 v_x(L) &= -\lambda(\lambda + \eta)^{\alpha - 1} v(L). \end{aligned} \tag{86}$$

On the other hand, $EI w_{xxx}(L) = \mathfrak{C} \int_{\mathbb{R}} \mu(\xi)\psi(\xi)$. Then from $(84)_{7,9}$ and Lemma 2.3 we obtain

$$\begin{aligned} EI w_{xxx}(L) &= \mathfrak{C} \int_{\mathbb{R}} \mu(\xi)\psi(\xi) dx = -W(L)\mathfrak{C} \int_{\mathbb{R}} \frac{\mu^2(\xi)}{\xi^2 + \eta + \lambda} d\xi \\ &= \lambda(\lambda + \eta)^{\alpha - 1} w(L) \end{aligned} \tag{87}$$

with the following conditions

$$w(0) = 0, \quad w_x(0) = 0, \quad w_{xx}(L) = 0. \tag{88}$$

Thereby, the system (86)-(88) is exactly the eigenvalue problem of $\mathcal{A}$. Thus, $\mathcal{A}^*$ has the same eigenvalues with $\mathcal{A}$. □

5. Polynomial stability

In this section, we show that the C_0-semigroup $e^{t\mathcal{A}}$ is polynomially stable by using Borichev-Tomilov Theorem 5.1.

Theorem 5.1 (Borichev-Tomilov, [4]). *Let $\mathcal{S}(t) = e^{\mathcal{A}\,t}$ be a C_0-semigroup of contractions on Hilbert space $\mathcal{H}$. If*

$$i\,\mathbb{R} \subseteq \varrho(\mathcal{A}) \quad \text{and} \quad \sup_{|\beta|\geq 1} \frac{1}{\beta^\ell}\,\|(i\,\beta\,I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < M,$$

for some ℓ, then there exist c such that

$$\|e^{\mathcal{A}\,t}U_0\|^2 \leq \frac{c}{t^{2/\ell}}\,\|U_0\|^2_{\mathcal{D}(\mathcal{A})}.$$

The main theorem of this section is presented as follows.

Theorem 5.2. *The semigroup $\mathcal{S}_{\mathcal{A}}(t)_{t\geq 0}$ is polynomially stable and*

$$\|\mathcal{S}_{\mathcal{A}}(t)\mathcal{U}_0\|_{\mathcal{H}} \leq \frac{1}{t^{1/2\,(1-\alpha)}}\,\|U_0\|_{\mathcal{D}(\mathcal{A})}. \tag{89}$$

Proof. We will study the resolvent equation $(i\,\lambda\,I - \mathcal{A})\mathcal{U} = \mathcal{F}$, $\lambda \in \mathbb{R}$. That is,

$$\begin{cases} i\lambda u - U = f_1, \\ i\lambda\varrho_1 h_1 U - E_1 h_1 u_{xx} - k(-u+v+\gamma w_x) = \varrho_1 h_1\, f_2, \\ i\lambda\varphi + (\xi^2+\eta)\varphi - U(L)\,\mu(\xi) = f_3, \\ i\lambda v - V = f_4, \\ i\lambda\varrho_3 h_3 V - E_3 h_3 v_{xx} + k(-u+v+\gamma w_x) = \varrho_3 h_3 f_5, \\ i\lambda\phi + (\xi^2+\eta)\phi - V(L)\,\mu(\xi) = f_6, \\ i\lambda w - W = f_7, \\ i\lambda\varrho h W + EI w_{xxxx} - k\gamma(-u+v+\gamma w_x)_x = \varrho h f_8, \\ i\lambda\psi + (\xi^2+\eta)\psi - W(L)\,\mu(\xi) = f_9, \end{cases} \tag{90}$$

where $\mathcal{F} = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9)^T$. Taking the inner product in $\mathcal{H}$ with U and using (35) we have

$$|Re\langle \mathcal{A}\mathcal{U},\, \mathcal{U}\rangle_{\mathcal{H}}| \leq \|\mathcal{U}\|_{\mathcal{H}}\;\|\mathcal{F}\|_{\mathcal{H}},$$

that is,

$$\begin{cases} \mathfrak{C}\int_{\mathbb{R}} \left(\xi^2+\eta\right)|\varphi|^2 d\xi \leq \|\mathcal{U}\|_{\mathcal{H}}\;\|\mathcal{F}\|_{\mathcal{H}}, \\ \mathfrak{C}\int_{\mathbb{R}} \left(\xi^2+\eta\right)|\phi|^2 d\xi \leq \|\mathcal{U}\|_{\mathcal{H}}\;\|\mathcal{F}\|_{\mathcal{H}}, \\ \mathfrak{C}\int_{\mathbb{R}} \left(\xi^2+\eta\right)|\psi|^2 d\xi \leq \|\mathcal{U}\|_{\mathcal{H}}\;\|\mathcal{F}\|_{\mathcal{H}}. \end{cases} \tag{91}$$

Moreover, from $(90)_{1,\,4,\,7}$ we have

$$\begin{aligned} ||\lambda||u(L,\,t)| - |f_1(L)|| &\leq |i\lambda u(L) - f_1(L)| = |U(L)|\,, \\ ||\lambda||v(L)| - |f_4(L)|| &\leq |i\lambda v(L) - f_4(L)| = |V(L)|\,, \\ ||\lambda||w(L)| - |f_7(L)|| &\leq |i\lambda w(L) - f_7(L)| = |W(L)|\,, \end{aligned}$$

then

$$\begin{aligned} |\lambda|^2|u(L)|^2 &\leq C|f_1(L)|^2 + C|U(L)|^2, \\ |\lambda|^2|v(L)|^2 &\leq C|f_4(L)|^2 + C|V(L)|^2, \\ |\lambda|^2|w(L)|^2 &\leq C|f_7(L)|^2 + C|W(L)|^2. \end{aligned} \tag{92}$$

On the other hand, from $(28)_{BC_{4,\,5,\,6}}$ and using the Cauchy-Schwartz inequality we have

$$\begin{cases} E_1 h_1 |u_x(L)|^2 \leq C\|\mathcal{U}\|_{\mathcal{H}}\|\mathcal{F}\|_{\mathcal{H}}, \\ E_3 h_3 |v_x(L)|^2 \leq C\|\mathcal{U}\|_{\mathcal{H}}\|\mathcal{F}\|_{\mathcal{H}}, \\ EI|w_{xxx}(L)|^2 \leq C\|\mathcal{U}\|_{\mathcal{H}}\|\mathcal{F}\|_{\mathcal{H}}. \end{cases} \tag{93}$$

From $(90)_{3,6,9}$ we obtain

$$\begin{aligned} U(L)\mu(\xi) &= (\xi^2+\eta+i\,\lambda)\varphi - f_3(\xi),\\ V(L)\mu(\xi) &= (\xi^2+\eta+i\,\lambda)\phi - f_6(\xi),\\ W(L)\mu(\xi) &= (\xi^2+\eta+i\,\lambda)\psi - f_9(\xi). \end{aligned} \tag{94}$$

Now, multiplying $(94)_1$ by $(\xi^2+\eta+i\,\lambda)^{-1}\mu(\xi)$, applying absolute values and integrating over $\xi\in\mathbb{R}$ we get

$$\begin{aligned} \left(\int_{\mathbb{R}} \frac{|\mu(\xi)|^2}{|\xi^2+\eta+i\,\lambda|}\,d\xi\right)|U(L)| &\le \int_{\mathbb{R}} |\mu(\xi)|\,|\varphi|\,d\xi + \int_{\mathbb{R}} \frac{|\mu(\xi)|\,|f_3(\xi)|}{|(\xi^2+\eta+i\,\lambda)|}\,d\xi\\ &\le \int_{\mathbb{R}} (\xi^2+\eta)^{-1/2}\,|\mu(\xi)|\,(\xi^2+\eta)^{1/2}\,|\varphi|\,d\xi\\ &\quad + \int_{\mathbb{R}} \frac{|\mu(\xi)|\,|f_3(\xi)|}{|(\xi^2+\eta+i\,\lambda)|}\,d\xi. \end{aligned}$$

Using the Cauchy-Schwartz inequality and straightforward estimates it follows that

$$\begin{aligned} \left(\int_{\mathbb{R}} \frac{|\mu(\xi)|^2}{\xi^2+\eta+|\lambda|}\,d\xi\right)|U(L)| &\le \left(\int_{\mathbb{R}} \frac{|\mu(\xi)|^2}{\xi^2+\eta}\,d\xi\right)^{1/2}\left(\int_{\mathbb{R}} (\xi^2+\eta)\,|\varphi|^2\,d\xi\right)^{1/2}\\ &\quad + \left(\int_{\mathbb{R}} \frac{|\mu(\xi)|^2}{(\xi^2+\eta+|\lambda|)^2}\,d\xi\right)^{1/2}\left(\int_{\mathbb{R}} |f_3|^2\,d\xi\right)^{1/2}. \end{aligned}$$

Applying power squared on both sides of the inequality and using $2\,a\,b\le a^2+b^2$ we obtain

$$\begin{aligned} \left(\int_{\mathbb{R}} \frac{|\mu(\xi)|^2\,d\xi}{\xi^2+\eta+|\lambda|}\right)^2|U(L)|^2 &\le 2\left(\int_{\mathbb{R}} \frac{|\mu(\xi)|^2}{\xi^2+\eta}\,d\xi\right)\left(\int_{\mathbb{R}} (\xi^2+\eta)\,|\varphi|^2\,d\xi\right)\\ &\quad + 2\left(\int_{\mathbb{R}} \frac{|\mu(\xi)|^2}{(\xi^2+\eta+|\lambda|)^2}\,d\xi\right)\left(\int_{\mathbb{R}} |f_3(\xi)|^2\,d\xi\right). \end{aligned} \tag{95}$$

Hence, using $(91)_1$ we have

$$\begin{aligned} \left(\int_{\mathbb{R}} \frac{|\mu(\xi)|^2}{|\lambda|+\xi^2+\eta}\,d\xi\right)^2|U(L)|^2 &\le C\left(\int_{\mathbb{R}} \frac{|\mu(\xi)|^2}{(\xi^2+\eta)}\,d\xi\right)\|\mathcal{U}\|_{\mathcal{H}}\,\|\mathcal{F}\|_{\mathcal{H}}\\ &\quad + C\left(\int_{\mathbb{R}} \frac{|\mu(\xi)|^2}{(|\lambda|+\xi^2+\eta)^2}\,d\xi\right)\|\mathcal{F}\|^2_{\mathcal{H}}. \end{aligned} \tag{96}$$

Now, from Lemma 2.3, it follows that

$$|U(L)|^2 \le C\,|\lambda|^{2-2\,\alpha}\,\|\mathcal{U}\|_{\mathcal{H}}\,\|\mathcal{F}\|_{\mathcal{H}} + C\,\|\mathcal{F}\|^2_{\mathcal{H}}. \tag{97}$$

Similarly, we estimate $(94)_{2,3}$, that is,

$$\begin{cases} |V(L)|^2 \le C\,|\lambda|^{2-2\,\alpha}\,\|\mathcal{U}\|_{\mathcal{H}}\,\|\mathcal{F}\|_{\mathcal{H}} + C\,\|\mathcal{F}\|^2_{\mathcal{H}},\\ |W(L)|^2 \le C\,|\lambda|^{2-2\,\alpha}\,\|\mathcal{U}\|_{\mathcal{H}}\,\|\mathcal{F}\|_{\mathcal{H}} + C\,\|\mathcal{F}\|^2_{\mathcal{H}}. \end{cases} \tag{98}$$

Therefore

$$|U(L)|^2+|V(L)|^2+|W(L)|^2 \le C|\lambda|^{2-2\alpha}\|\mathcal{U}\|_{\mathcal{H}}\|\mathcal{F}\|_{\mathcal{H}} + C\|\mathcal{F}\|^2_{\mathcal{H}}. \tag{99}$$

Consequently,

$$|U_x(L)|^2 + |V_x(L)|^2 + |W_x(L)|^2 \le C\left(\frac{1}{|\lambda|^{2\,\alpha}} + 1\right)\|\mathcal{U}\|_{\mathcal{H}}\|\mathcal{F}\|_{\mathcal{H}} + \frac{C}{|\lambda|^2}\|\mathcal{F}\|^2_{\mathcal{H}}. \quad (100)$$

To conclude the proof of the theorem we need the following lemma:

Lemma 5.3. *Let $q(x) \in H^2(0, L)$. Then we have*

$$\int_0^L \left[2\varrho_1 h_1 q|U|^2 + 2E_1h_1q|u_x|^2 - E_1h_1q_{xx}|u|^2\right]dx = 2kRe\int_0^L q\overline{u}(-u+v+\gamma w_x)dx + 2E_1h_1q\overline{u}u_x\Big|_0^L - q_x|u|^2\Big|_0^L + R_1 \quad (101)$$

where

$$R_1 = 2\varrho_1 h_1\int_0^L qRe(\overline{u}f_2)dx - 2\varrho_1h_1\int_0^L qRe(U\overline{f}_1)dx. \quad (102)$$

$$\int_0^L\left[2\varrho_3h_3q|V|^2 + 2E_3h_3q|v_x|^2 - E_3h_3q_{xx}|v|^2\right]dx = -2kRe\int_0^L q\,\overline{v}(-u+v+\gamma w_x)dx + 2E_3h_3q\overline{v}v_x\Big|_0^L - q_x|v|^2\Big|_0^L + R_2 \quad (103)$$

where

$$R_2 = 2\varrho_3h_3\int_0^L qRe(\overline{v}f_5)dx - 2\varrho_3h_3\int_0^L qRe(V\overline{f}_4)dx. \quad (104)$$

In the third case we consider $p(x) \in H^4(0, L)$ then we have

$$\int_0^L p_x\left[\varrho h|W|^2 + 3EI|w_{xx}|^2\right]dx - 2EI\int_0^L p_{xx}\frac{d}{dx}|w_x|^2dx - 2k\gamma Re\int_0^L p\overline{w}_x(-u+v+\gamma w_x)_xdx = -2EIpRe(\overline{w}_xw_{xxx})\Big|_0^L + EIp|W|^2\Big|_0^L + EIp|w_{xx}|^2\Big|_0^L + R_3 \quad (105)$$

where

$$R_3 = 2\varrho hRe\int_0^L p\overline{w}_xf_8dx + 2\varrho hRe\int_0^L p\overline{W}_x\overline{f}_{7x}dx. \quad (106)$$

Remark. For each R_i, $(i = 1, 2, 3)$ we have

$$R_i \le C\|\mathcal{U}\|_{\mathcal{H}}\,\|\mathcal{F}\|_{\mathcal{H}}, \quad i = 1, 2. \quad (107)$$

Proof. Multiplying $(90)_2$ by $q\overline{u}$ and integrating over $(0, L)$ we have

$$-\varrho_1h_1\int_0^L q(\overline{i\lambda\,u})Udx - E_1h_1\int_0^L q\overline{u}u_{xx}dx - k\int_0^L q\overline{u}(-u+v+\gamma w_x)dx = \varrho_1h_1\int_0^L q\,\overline{u}\,f_2dx.$$

From $(90)_1$ we obtain

$$\begin{aligned}&\varrho_1 h_1 \int_0^L q|U|^2 dx - E_1 h_1 \int_0^L q\overline{u} u_{xx} dx\\ &-k\int_0^L q\overline{u}(-u+v+\gamma w_x)dx = \varrho_1 h_1 \int_0^L q\overline{u} f_2 dx - \varrho_1 h_1 \int_0^L qU\overline{f}_1 dx.\end{aligned}$$

Then integrating by parts and using the fact $Re(\Phi\overline{\Phi}_x) = \frac{1}{2}\frac{d}{dx}|\Phi|^2$ we have

$$\begin{aligned}\int_0^L \left[2\varrho_1 h_1 q|U|^2 + 2E_1 h_1 q|u_x|^2 - E_1 h_1 q_{xx}|u|^2\right] dx =& 2kRe\int_0^L q\,\overline{u}(-u+v+\gamma w_x)dx\\ &+ 2E_1 h_1 q\overline{u} u_x \Big|_0^L\\ &- q_x|u|^2 \Big|_0^L + R_1, \qquad (108)\end{aligned}$$

where

$$R_1 = 2\varrho_1 h_1 \int_0^L qRe(\overline{u} f_2)dx - 2\varrho_1 h_1 \int_0^L qRe(U\overline{f}_1)dx. \qquad (109)$$

Performing similar calculation to (108) we obtain

$$\begin{aligned}\int_0^L \left[2\varrho_3 h_3 q|V|^2 + 2E_3 h_3 q|v_x|^2 - E_3 h_3 q_{xx}|v|^2\right] dx =& -2kRe\int_0^L q\,\overline{v}(-u+v+\gamma w_x)dx\\ &+ 2E_3 h_3 q\overline{v} v_x \Big|_0^L\\ &- q_x|v|^2 \Big|_0^L + R_2, \qquad (110)\end{aligned}$$

where

$$R_2 = 2\varrho_3 h_3 \int_0^L qRe(\overline{v} f_5)dx - 2\varrho_3 h_3 \int_0^L qRe(V\overline{f}_4)dx. \qquad (111)$$

On the other hand, multiplying $(90)_8$ by $p\overline{w}_x$, integrating over $x \in (0, L)$ we have

$$\begin{aligned}&-\varrho h \int_0^L p(\overline{i\lambda w_x})W dx + EI\int_0^L p\overline{w}_x w_{xxxx} dx\\ &-k\gamma \int_0^L p\overline{w}_x(-u+v+\gamma w_x)_x dx = \varrho h \int_0^L p\overline{w}_x f_8 dx.\end{aligned}$$

Performing similar calculations to what was done previously

$$\begin{aligned}&\int_0^L \left[\varrho h p_x|W|^2 + 3EIp_x|w_{xx}|^2\right] dx - 2EI\int_0^L p_{xx}\frac{d}{dx}|w_x|^2 dx\\ &-2k\gamma Re\int_0^L p\overline{w}_x(-u+v+\gamma w_x)_x dx = -2EIpRe(\overline{w}_x w_{xxx})\Big|_0^L\\ &+ EIp|W|^2\Big|_0^L + EIp|w_{xx}|^2\Big|_0^L + R_3, \qquad (112)\end{aligned}$$

where

$$R_3 = 2\varrho hRe\int_0^L q\overline{w}_x f_8 dx + 2\varrho hRe\int_0^L q\overline{W}_x\overline{f}_{7x} dx. \qquad (113)$$

The lemma follows. □

Returning to the proof of the Theorem 5.2 taking $q(x) = 1$ into (101) and (103) we have

$$\int_0^L \left[2\varrho_1 h_1|U|^2 + 2E_1h_1|u_x|^2\right] dx = 2kRe\int_0^L \overline{u}(-u+v+\gamma w_x)dx + 2E_1h_1\overline{u}u_x\Big|_0^L + R_1, \tag{114}$$

and

$$\int_0^L \left[2\varrho_3 h_3|V|^2 + 2E_3h_3|v_x|^2\right] dx = -2kRe\int_0^L \overline{v}(-u+v+\gamma w_x)dx + 2E_3h_3\overline{v}v_x\Big|_0^L + R_2. \tag{115}$$

Adding (114) with (115) we obtain

$$\begin{aligned}&\int_0^L \left[2\varrho_1 h_1|U|^2 + 2E_1h_1|u_x|^2\right] dx + \int_0^L \left[2\varrho_3 h_3|V|^2 + 2E_3h_3|v_x|^2\right] dx\\ &= 2kRe\int_0^L \overline{u}(-u+v+\gamma w_x)dx - 2kRe\int_0^L \overline{v}(-u+v+\gamma w_x)dx\\ &+ 2E_1h_1\overline{u}(L)u_x(L) + 2E_3h_3\overline{v}(L)v_x(L) + R_1 + R_2.\end{aligned}$$

Using the Young and Cauchy-Schwartz inequalities and $(93)_{1,2}$ we have

$$\begin{aligned}&2\int_0^L \left[\varrho_1 h_1|U|^2 + E_1h_1|u_x|^2\right] dx + 2\int_0^L \left[\varrho_3 h_3|V|^2 + E_3h_3|v_x|^2\right] dx\\ \le& k(3+\gamma)\int_0^L |u|^2dx + k(3+\gamma)\int_0^L |v|^2dx + 4k\gamma\int_0^L |w_x|^2dx\\ &+ E_1h_1|u(L)|^2 + E_1h_1|u_x(L)|^2 + E_3h_3|v(L)|^2 + E_3h_3|v_x(L)|^2 + C\|\mathcal{U}\|_{\mathcal{H}}\ \|\mathcal{F}\|_{\mathcal{H}}\\ \le& E_1h_1|u(L)|^2 + E_3h_3|v(L)|^2 + C\|\mathcal{U}\|_{\mathcal{H}}\ \|\mathcal{F}\|_{\mathcal{H}}\end{aligned}$$

for a positive constant C. Moreover, taking $p(x) = x$ into (105) we have

$$\begin{aligned}&\int_0^L [\varrho h|W|^2 + 3EI|w_{xx}|^2]dx = 2k\gamma Re\int_0^L x\overline{w}_x(-u+v+\gamma w_x)_x dx\\ &- 2EIxRe(\overline{w}_x w_{xxx})\Big|_0^L + EIx|W|^2\Big|_0^L + EIx|w_{xx}|^2\Big|_0^L + R_3.\end{aligned} \tag{116}$$

Performing similar estimate those given above together with $(93)_3$ we obtain

$$\int_0^L [\varrho h|W|^2 + 3EI|w_{xx}|^2]dx \le EI|w_{xx}(L)|^2 + C\|\mathcal{U}\|_{\mathcal{H}}\ \|\mathcal{F}\|_{\mathcal{H}}. \tag{117}$$

Now, using $(90)_{1,4,7}$ in the sense that

$$u = \frac{U+f_1}{i\,\lambda} \iff |u| \le \frac{|U|+|f_1|}{|\lambda|} \iff |u|^2 \le \frac{2|U|^2+2|f_1|^2}{|\lambda|^2}.$$

Then,

$$|u(L)|^2 \le \frac{2|U(L)|^2 + 2|f_1(L)|^2}{|\lambda|^2}. \tag{118}$$

Similarly,

$$|v(L)|^2 \le \frac{2|V(L)|^2 + 2|f_4(L)|^2}{|\lambda|^2} \tag{119}$$

and

$$|w(L)|^2 \leq \frac{2|W(L)|^2 + 2|f_7(L)|^2}{|\lambda|^2}. \tag{120}$$

Now, from (99) with straightforward estimates we obtain for $\lambda \neq 0$,

$$\begin{aligned}
&\int_0^L \left[\varrho_1 h_1 |U|^2 + E_1 h_1 |u_x|^2\right] dx \\
&+ \int_0^L \left[\varrho_3 h_3 |V|^2 + E_3 h_3 |w_x|^2\right] dx \\
&+ \int_0^L \left[\varrho h |W|^2 + EI |w_x|^2\right] dx \\
&\leq C\,|\lambda|^{2\,\alpha-2}\, \|\mathcal{U}\|_{\mathcal{H}}\, \|\mathcal{F}\|_{\mathcal{H}} + C\, \|\mathcal{U}\|_{\mathcal{H}}\, \|\mathcal{F}\|_{\mathcal{H}} \\
&+ C\, \|\mathcal{F}\|^2_{\mathcal{H}} + \frac{C}{|\lambda|^2}\, \|\mathcal{U}\|^2_{\mathcal{H}} \\
&+ \frac{C}{|\lambda|^2}\, \|\mathcal{F}\|^2_{\mathcal{H}} + \frac{C}{|\lambda|^2}\, \|\mathcal{U}\|_{\mathcal{H}}\, \|\mathcal{F}\|_{\mathcal{H}}.
\end{aligned}$$

Moreover, for we have that

$$\begin{aligned}
&\int_{\mathbb{R}} |\varphi(\xi)|^2 d\xi \leq C \int_{\mathbb{R}} (\xi^2+\eta)|\varphi(\xi)|^2 d\xi, \quad \text{for} \quad \lambda \neq 0, \\
&\int_{\mathbb{R}} |\phi(\xi)|^2 d\xi \leq C \int_{\mathbb{R}} (\xi^2+\eta)|\phi(\xi)|^2 d\xi, \quad \text{for} \quad \lambda \neq 0, \\
&\int_{\mathbb{R}} |\psi(\xi)|^2 d\xi \leq C \int_{\mathbb{R}} (\xi^2+\eta)\,|\psi(\xi)|^2 d\xi, \quad \text{for} \quad \lambda \neq 0.
\end{aligned}$$

If $|\lambda| > 1$ we get

$$\|\mathcal{U}\|^2_{\mathcal{H}} \leq |\lambda|^{4\,(1-\alpha)}\, \|\mathcal{F}\|^2_{\mathcal{H}} \iff \|\mathcal{U}\|_{\mathcal{H}} \leq |\lambda|^{2\,(1-\alpha)}\, \|\mathcal{F}\|_{\mathcal{H}}.$$

It follows that

$$\frac{1}{|\lambda|^{2\,(1-\alpha)}}\, \|(i\,\lambda\, I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq C, \quad \forall\, \lambda \in \mathbb{R},$$

for a positive constant C. The conclusion then follows by applying the Theorem 5.1. □

Conclusion and open problem: In this manuscript, we prove the existence, uniqueness, and smoothness theorem for Rao-Nakra sandwich beam with boundary dissipation of fractional derivative type. Furthermore, we establish strong stability and polynomial stability results. Since the approach for Rao-Nakra model with boundary dissipation of fractional derivative type is new, it is an interesting open problem to study the lack of exponential stability of the system, that the question: the system is not exponentially stable?

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