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HOMOCLINIC SOLUTIONS FOR THE NONPERIODIC FRACTIONAL HAMILTONIAN SYSTEMS

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ABSTRACT. A new result for existence of homoclinic solutions is obtained for the nonperiodic fractional Hamiltonian systems

$$-{}_tD^{\alpha}_{\infty}({}_{-\infty}D^{\alpha}_t x(t)) - L(t)x(t) + \nabla\left[W_1(t,x(t)) - W_2(t,x(t))\right] = 0,$$

where $\alpha \in (1/2, 1]$, $x \in H^{\alpha}(\mathbb{R}, \mathbb{R}^N)$, $W_1,\ W_2 \in C^1(\mathbb{R}\times\mathbb{R}^N, \mathbb{R})$ are not necessary periodic in t. This result generalizes and improves some existing results in the literatures.

1. Introduction

In this paper, we are concerned with the existence of homoclinic solutions for a class of fractional Hamiltonian systems of the following form

$$\begin{cases} -{}_tD^{\alpha}_{\infty}({}_{-\infty}D^{\alpha}_t u(t)) - L(t)u(t) + \nabla W(t,u(t)) = 0, \\ u \in H^{\alpha}(\mathbb{R}, \mathbb{R}^N), \end{cases} \tag{1}$$

where $\alpha \in (\frac{1}{2}, 1)$, $t \in \mathbb{R}$, $u \in \mathbb{R}^N$, $\nabla W(t,u)$ is the gradient of W at u, ${}_{-\infty}D^{\alpha}_t$ and ${}_tD^{\alpha}_{\infty}$ are left and right Liouville-Weyl fractional derivatives of order α on the whole axis $\mathbb{R}$ respectively, $L(t) \in C(\mathbb{R}, \mathbb{R}^{N^2})$ is symmetric and positive definite matrix for all $t \in \mathbb{R}$ and $W \in C^1(\mathbb{R}\times\mathbb{R}^N, \mathbb{R})$.

The study of fractional calculus (differentiation and integration) has emerged as an important and popular field in research. It is mainly due to the extensive application of fractional differential equations in many engineering and scientific disciplines such as physics, mechanics, control theory, viscoelasticity, electro chemistry, bioengineering, economics and others [1, 5, 10, 11, 13, 16]. An important characteristic of fractional-order differential operator that distinguishes it from the integer-order differential operator is its non local behavior, that is, the future state of a dynamical system or process involving fractional derivative depends on its current state as well as its past states. In other words, differential equations of arbitrary order describe memory and hereditary properties of various materials and process. This is one of the futures that has contributed to the popularity of the subject and has motivated the researchers to focus on fractional order models, which are more realistic and practical than the classical integer-order models. Recently, also

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equations including both left and right fractional derivatives were investigated and many results were obtained dealing with the existence and multiplicity of solutions of nonlinear fractional differential equations by using techniques of Nonlinear Analysis, such as fixed point theory [3, 27], topological degree theory [8], comparison methods [28], you can see also [14, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31]. In 2012, Jiao and Zhou [9] showed that critical point theory is an effective approach to tackle the existence of solutions for the fractional boundary-value problem

$$\begin{cases} {}_tD_T^{\alpha}({}_0D_t^{\alpha}u(t)) = \nabla W(t,u(t)), & \text{a.e. } t\in[0,T], \\ u(0)=u(T), \end{cases} \tag{2}$$

where $\alpha\in(1/2,1)$, $u\in\mathbb{R}^N$, $W\in C^1([0,T]\times\mathbb{R}^N,\mathbb{R}), \nabla W(t,u)$ is the gradient of W at u, and obtained the existence of at least one nontrivial solution. Inspired by this paper, Torres [20] studied the fractional Hamiltonian system (1) and he showed that (1) possesses at least one nontrivial solution via Mountain Pass Theorem, by assuming that L and W satisfy the following hypotheses:

- (L_1) $L(t)$ is symmetric and positive definite matrix for all $t\in\mathbb{R}$ and there exists an $l\in C(\mathbb{R},(0,\infty))$ such that $l(t)\to\infty$ as $|t|\to\infty$ and

$$(L(t)x,x)\geq l(t)\,|x|^2\,,\ \forall t\in\mathbb{R},\ x\in\mathbb{R}^N.$$

- (W_1) $|\nabla W(t,x)|=o(|x|)$ as $|x|\to 0$ uniformly in $t\in\mathbb{R}$;
- (W_2) there is $\overline{W}\in C(\mathbb{R}^N,\mathbb{R})$ such that

$$|W(t,x)|+|\nabla W(t,x)|\leq|\overline{W}(x)|,\quad \forall(t,x)\in\mathbb{R}\times\mathbb{R}^N.$$

- (W_3) there exists a constant $\mu>2$ such that

$$0<\mu W(t,x)\leq(\nabla W(t,x),x),\quad \forall t\in\mathbb{R},\ x\in\mathbb{R}^N\backslash\{0\}.$$

When $\alpha=1$, (1) reduces to the standard second-order Hamiltonian systems

$$\ddot{u}(t)-L(t)u(t)+\nabla W(t,u(t))=0. \tag{3}$$

Assuming that $L(t)$ and $W(t,u)$ are independent of t or periodic in t, many authors have studied the existence of homoclinic solutions for the Hamiltonian system (3) (see [2, 4, 18] and the references therein), and some more general Hamiltonian systems are considered in the recent papers [6, 7]. In this case, the existence of homoclinic solutions can be obtained by going to the limit of periodic solutions of approximating problems. If $L(t)$ and $W(t,u)$ are neither autonomous nor periodic in t, the existence of homoclinic solutions of (3) is quite different from the periodic systems, because of the lack of compactness of the Sobolev embedding (see for instance [4, 15, 19] and the references therein).

Motivated by the above results, we will improve the result in[20] along another direction. For the statement of our main result, the potential $W(t,x)$ is supposed to satisfay the following conditions:

- (H_1) $W(t,x)=W_1(t,x)-W_2(t,x),\ W_1,\ W_2\in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$, and there is an $R>0$ such that

$$\frac{1}{l(t)}|\nabla W(t,x)|=o(|x|)\ as\ |x|\to 0,$$

uniformly in $t\in(-\infty,-R]\cup[R,+\infty)$,

(H_2) there are two constants $\mu > 2$ and $\nu \in [0, \frac{\mu}{2} - 1)$ such that

$$\frac{\mu\nu}{\mu - 2}(L(t)x, x) < \mu W_1(t, x) \leq (\nabla W_1(t, x), x) + \nu(L(t)x, x),$$

$\forall t \in \mathbb{R},\ x \in \mathbb{R}^N \backslash \{0\}$,

(H_3) $W_2(t, 0) = 0$ and there is a constant $\sigma \in [2, \mu)$ such that

$$W_2(t, x) \geq 0, \quad (\nabla W_2(t, x), x) \leq \sigma W_2(t, x), \quad \forall t \in \mathbb{R},\ x \in \mathbb{R}^N \backslash \{0\}.$$

In this paper, we will prove the following theorem.

Theorem 1.1. *Suppose that (L_1), $(H_1) - (H_3)$ hold. Then (1) possesses at least one nontrivial homoclinic solution.*

Remark 1.1. *If $W_2(t, x) = 0$, then our Theorem 1.1 improves Theorem 1.1 in* [20] *by relaxing conditions (W_1) and (W_2) (see (H_1) and (H_2) and removing condition (W_3)).*

The rest of the paper is organized as follows: in section 2, subsection 2.1, we describe the Liouville-Weyl fractional calculus; in subsection 2.2 we introduce the fractional space that we use in our work and some lemmas are proven which will aid in our analysis. In section 3, we will prove Theorem 1.1.

2. Preliminaries

2.1. **Liouville-Weyl Fractional Calculus.**

Definition 2.1. The left and right Liouville-Weyl fractional integrals of order $0 < \alpha < 1$ on the whole axis $\mathbb{R}$ are defined by

$$_{-\infty}I_x^\alpha u(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} (x - \xi)^{\alpha - 1} u(\xi) d\xi,$$
$$_xI_\infty^\alpha u(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} (\xi - x)^{\alpha - 1} u(\xi) d\xi\,,$$

respectively, where $x \in \mathbb{R}$.

Definition 2.2. The left and right Liouville-Weyl fractional derivatives of order $0 < \alpha < 1$ on the whole axis $\mathbb{R}$ are defined by

$$_{-\infty}D_x^\alpha u(x) = \frac{d}{dx}\, {}_{-\infty}I_x^{1-\alpha} u(x), \tag{4}$$
$$_xD_\infty^\alpha u(x) = -\frac{d}{dx}\, {}_xI_\infty^{1-\alpha} u(x)\,, \tag{5}$$

respectively, where $x \in \mathbb{R}$.

Remark 2.1. Definitions (4) and (5) may be written in the alternative forms:

$$_{-\infty}D_x^\alpha u(x) = \frac{\alpha}{\Gamma(1 - \alpha)} \int_0^\infty \frac{u(x) - u(x - \xi)}{\xi^{\alpha + 1}} d\xi,$$
$$_xD_\infty^\alpha u(x) = \frac{\alpha}{\Gamma(1 - \alpha)} \int_0^\infty \frac{u(x) - u(x + \xi)}{\xi^{\alpha + 1}} d\xi\,.$$

Recall that the Fourier transform $\widehat{u}(w)$ of $u(x)$ is defined by

$$\widehat{u}(w) = \int_{-\infty}^{\infty} e^{-ix.w} u(x)dx.$$

We establish the Fourier transform properties of the fractional integral and fractional operators as follows:

$$\widehat{{}_{-\infty}I_x^\alpha u(x)}(w) = (iw)^{-\alpha}\widehat{u}(w),$$
$$\widehat{{}_xI_\infty^\alpha u(x)}(w) = (-iw)^{-\alpha}\widehat{u}(w),$$
$$\widehat{{}_{-\infty}D_x^\alpha u}(x)(w) = (iw)^{\alpha}\widehat{u}(w),$$
$$\widehat{{}_xD_\infty^\alpha u(x)}(w) = (-iw)^{\alpha}\widehat{u}(w).$$

2.2. **Fractional derivative spaces.** Let us recall for any $\alpha > 0$, the semi-norm

$$|u|_{I_{-\infty}^\alpha} = \|{}_{-\infty}D_x^\alpha u\|_{L^2},$$

and the norm

$$\|u\|_{I_{-\infty}^\alpha} = \left(\|u\|_{L^2}^2 + |u|_{I_{-\infty}^\alpha}^2\right)^{1/2}.$$

Let the space $I_{-\infty}^\alpha(\mathbb{R})$ denote the completion of $C_0^\infty(\mathbb{R})$ with respect to the norm $\|\cdot\|_{I_{-\infty}^\alpha}$, i.e.,

$$I_{-\infty}^\alpha(\mathbb{R}) = \overline{C_0^\infty(\mathbb{R})}^{\|\cdot\|_{I_{-\infty}^\alpha}}.$$

Next, we define the fractional Sobolev space $H^\alpha(\mathbb{R})$ in terms of the Fourier transform. For $0 < \alpha < 1$, define the semi-norm

$$|u|_\alpha = \||w|^\alpha \widehat{u}\|_{L^2},$$

and the norm

$$\|u\|_\alpha = (\|u\|_{L^2}^2 + |u|_\alpha^2)^{1/2},$$

and let

$$H^\alpha(\mathbb{R}) = \overline{C_0^\infty(\mathbb{R})}^{\|\cdot\|_\alpha}.$$

We note that a function $u \in L^2(\mathbb{R})$ belongs to $I_{-\infty}^\alpha(\mathbb{R})$ if and only if

$$|w|^\alpha \widehat{u} \in L^2(\mathbb{R}).$$

In particular, $|u|_{I_{-\infty}^\alpha} = \||w|^\alpha \widehat{u}\|_{L^2(\mathbb{R})}$. Therefore $H^\alpha(\mathbb{R})$ and $I_{-\infty}^\alpha(\mathbb{R})$ are equivalent, with equivalent semi-norm and norm (see [20]). Analogous to $I_{-\infty}^\alpha(\mathbb{R})$, we introduce $I_\infty^\alpha(\mathbb{R})$. Let us define the semi-norm

$$|u|_{I_\infty^\alpha} = \|{}_xD_\infty^\alpha\|_{L^2(\mathbb{R})},$$

and norm

$$\|u\|_{I_\infty^\alpha} = (\|u\|_{L^2}^2 + |u|_{I_\infty^\alpha}^2)^{1/2},$$

and let

$$I_{-\infty}^\alpha(\mathbb{R}) = \overline{C_0^\infty(\mathbb{R})}^{\|\cdot\|_{I_{-\infty}^\alpha}}.$$

Moreover $I_\infty^\alpha(\mathbb{R})$ and $I_{-\infty}^\alpha(\mathbb{R})$ are equivalent, with equivalent semi-norm and norm.

Lemma 2.1 ([20]). *If $\alpha > 1/2$, then $H^\alpha(\mathbb{R}) \subset C(\mathbb{R})$ and there is a constant $C = C_\alpha$ such that*

$$\|u\|_{L^\infty} = \sup_{u\in\mathbb{R}} |u(x)| \le C\|u\|_\alpha \tag{6}$$

where $C(\mathbb{R})$ denote the space of continuous functions on $\mathbb{R}$.

Remark 2.2. *If $u \in H^\alpha(\mathbb{R})$, then $u \in L^q(\mathbb{R})$ for all $q \in [2,\infty]$, since*

$$\int_{\mathbb{R}} |u(x)|^q dx \leq \|u\|_{L^\infty}^{q-2} \|u\|_{L^2}^2 .$$

In what follows, we introduce the fractional space in which we will construct the variational framework of (1). Let

$$X^\alpha = \left\{ u \in H^\alpha(\mathbb{R},\mathbb{R}^n) : \int_{\mathbb{R}} |{}_{-\infty}D_t^\alpha u(t)|^2 + (L(t)u(t),u(t))dt < \infty \right\}.$$

The space X^α is a reflexive and separable Hilbert space with the inner product

$$(u,v)_{X^\alpha} = \int_{\mathbb{R}} ({}_{-\infty}D_t^\alpha u(t).{}_{-\infty}D_t^\alpha v(t)) + (L(t)u(t),v(t))dt,$$

and the corresponding norm is

$$\|u\|^2 = (u,u)_{X^\alpha}.$$

Lemma 2.2. *Suppose L satisfies (L_1). Then, X^α is continuously embedded in $H^\alpha(\mathbb{R},\mathbb{R}^n)$.*

Proof. Since $l \in C(\mathbb{R},(0,\infty))$ and l is coercive, then $l_* = \min_{t\in\mathbb{R}} l(t)$ exists, so we have

$$(L(t)u(t),u(t)) \geq l(t)\left|t\right|^2 \geq l_* \left|t\right|^2, \forall t \in \mathbb{R}.$$

Then

$$\begin{aligned} \|u\|_\alpha^2 &= \int_{\mathbb{R}} (|{}_{-\infty}D_t^\alpha u(t)|^2 + (L(t)u(t),u(t)))dt \\ &\leq \int_{\mathbb{R}} |{}_{-\infty}D_t^\alpha u(t)|^2 dt + \frac{1}{l_*}\int_{\mathbb{R}} (L(t)u(t),u(t))dt \end{aligned}$$

So

$$\|u\|_\alpha^2 \leq K\|u\|^2 \tag{7}$$

where $K = \max(1, \frac{1}{l_*})$. □

Lemma 2.3. *Suppose L satisfies (L_1). Then the embedding of X^α in $L^2(\mathbb{R})$ is compact.*

Proof. We note first that by lemma 2.2 and Remark 2.2 we have

$$X^\alpha \hookrightarrow L^2(\mathbb{R}) \textit{ is continuous.}$$

Now, let $(u_k) \in X^\alpha$ be a sequence such that $u_k \rightharpoonup u$ *in* X^α. We will show that $u_k \to u$ *in* $L^2(\mathbb{R})$. The Banach Steinhauss theorem implies

$$A = \sup_{k\in\mathbb{N}} \|u_k - u\| < \infty.$$

Let $\epsilon > 0$, since $\lim_{|t|\to\infty} l(t) = \infty$, then there is $T_0 > 0$ such that $\frac{1}{l(t)} \leq \epsilon, \forall\, |t| \geq T_0$. So

$$\begin{aligned} \int_{|t|\geq T_0} |u_k(t)-u(t)|^2\, dt &\leq \epsilon \int_{|t|\geq T_0} l(t)\, |u_k(t)-u(t)|^2\, dt \\ &\leq \epsilon\, \|u_k - u\|^2 \\ &\leq \epsilon A^2. \end{aligned} \tag{8}$$

Besides, Sobolev's Theorem (see [12]) implies that $u_k \to u$ uniformly on $[-T_0, T_0]$, so there is a $k_0 \in \mathbb{N}$ such that

$$\int_{|t|\leq T_0} |u_k(t) - u(t)|^2 \, dt \quad \leq \quad \epsilon, \forall k \geq k_0. \tag{9}$$

Combining (8) and (9) we obtain $u_k \to u \; in \; L^2(\mathbb{R})$. □

Lemma 2.4. *Assume that (H_2) and (H_3) Hold . Then for every $t \in \mathbb{R}$ we have*

$$\begin{aligned} W_1(t,x) &\leq \left[W_1\left(t, \frac{x}{|x|}\right) - \frac{\nu}{\mu-2}\left(L(t)\frac{x}{|x|}, \frac{x}{|x|}\right)\right]|x|^\mu \\ &+ \frac{\nu}{\mu-2}(L(t)x,x), \; if \; 0 < |x| < 1, \end{aligned} \tag{10}$$

$$\begin{aligned} W_1(t,x) &\geq \left[W_1\left(t, \frac{x}{|x|}\right) - \frac{\nu}{\mu-2}\left(L(t)\frac{x}{|x|}, \frac{x}{|x|}\right)\right]|x|^\mu \\ &+ \frac{\nu}{\mu-2}(L(t)x,x), \; if \; |x| \geq 1, \end{aligned} \tag{11}$$

and

$$W_2(t,x) \leq W_2\left(t, \frac{x}{|x|}\right)|x|^\sigma, \; if \; |x| \geq 1. \tag{12}$$

Proof. Set $\phi(s) = s^{-\mu}W_1(t, sx)$. Then by (H_2), we have

$$\begin{aligned} \phi' &= -\mu s^{-\mu-1}W_1(t,sx) + s^{-\mu}(\nabla W_1(t,sx), x) \\ &= s^{-\mu-1}\left[-\mu W_1(t,sx) + (\nabla W_1(t,sx), sx)\right] \\ &\geq -\nu s^{1-\mu}(L(t)x,x), \; s > 0. \end{aligned} \tag{13}$$

If $s \geq 1$, then it follows that

$$\phi(1) \leq \phi(s) + \frac{\nu}{\mu-2}(1 - s^{2-\mu})(L(t)x,x),$$

which implies that (10) holds. If $0 < s \leq 1$, then it follows from (13) that

$$\phi(1) \geq \phi(s) + \frac{\nu}{\mu-2}(1 - s^{2-\mu})(L(t)x,x),$$

which implies that (11) holds. By a similar fashion, we can prove that (12) holds. The proof is complete. □

Lemma 2.5. *Under the conditions of Theorem 1.1 , φ' is compact, i.e., $\varphi'(u_k) \to \varphi'(u)$ if $u_k \rightharpoonup u$ in X^α, where $\varphi : X^\alpha \to \mathbb{R}$ is defined by*

$$\varphi(u) = \int_{\mathbb{R}} W(t,u)dt. \tag{14}$$

Proof. Assume that $u_k \rightharpoonup u \; in \; X^\alpha$. Then there exists a constant $M > 0$ such that

$$\|u_k\| \leq M \;\; and \;\; \|u\| \leq M$$

for $k \in \mathbb{N}$. In addition, from (H_1), for any $\epsilon > 0$, we can choose $R > 0$ and $\delta > 0$ such that

$$|\nabla W(t,u)| \leq \epsilon l(t)\,|u|\,, \; \forall\,|t| \geq R, \forall\,|u| \leq \delta.$$

Since $X^\alpha \subset C^0(\mathbb{R}, \mathbb{R}^N)$ the space of continuous functions u on $\mathbb{R}$ such that $u(t) \to 0$ as $|t| \to \infty$.Then, there exists $\bar{R} > 0$ such that

$$|u_k(t)| \leq \delta, \; \forall\,|t| \geq \bar{R}, \; k \in \mathbb{N}. \tag{15}$$

Noting that $u_k \rightharpoonup u$ in X^α, it is easy to verify that $u_k(t)$ converge to $u(t)$ pointwise for all $t \in \mathbb{R}$. Hence, we have by (15)

$$|u(t)| \leq \delta,\ \forall |t| \geq T,\ k \in \mathbb{N},\ where\ T = \max(R, \bar{R}). \tag{16}$$

It follows from (15), (16) and Hölder inequality that

$$\begin{aligned}
& \int_{|t|\geq T} |\nabla W(t, u_k(t)) - \nabla W(t, u(t))|\, |v(t)|\, dt \\
\leq\ & \int_{|t|\geq T} (|\nabla W(t, u_k(t))| + |\nabla W(t, u(t))|)\, (|v(t)|)\, dt \\
\leq\ & \epsilon \int_{|t|\geq T} l(t)\, (|u_k(t)| + |u(t)|)\, (|v(t)|)\, dt \\
\leq\ & \epsilon \left(\int_{|t|\geq T} l(t)\, |u_k|^2\, dt \right)^{\frac{1}{2}} \left(\int_{|t|\geq T} l(t)\, |v(t)|^2\, dt \right)^{\frac{1}{2}} \\
+\ & \epsilon \left(\int_{|t|\geq T} l(t)\, |u|^2\, dt \right)^{\frac{1}{2}} \left(\int_{|t|\geq T} l(t)\, |v(t)|^2\, dt \right)^{\frac{1}{2}} \\
\leq\ & \epsilon\, (\|u_k\| + \|u\|)\, \|v\| \\
\leq\ & 2\epsilon M\, \|v\|\,,\ k \in \mathbb{N}.
\end{aligned} \tag{17}$$

On the other hand, there is a $k_0 \in \mathbb{N}$ such that

$$\int_{|t|\leq T} |\nabla W(t, u_k(t)) - \nabla W(t, u(t))|\, |v(t)|\, dt < \epsilon\, \|v\|_\infty\,,\ for\ k \geq k_0. \tag{18}$$

Combining (17) and (18) we get

$$|\varphi'(u_k) - \varphi'(u)| \leq \epsilon\, (C_\infty + 2M)\, \|v\|\,,\ for\ k \geq k_0. \tag{19}$$

Hence we get

$$\begin{aligned}
\|\varphi'(u_k) - \varphi'(u)\| &= \sup_{\|v\|=1} \left| \int_{\mathbb{R}} (\nabla W(t, u_k(t)) - \nabla W(t, u(t)), v(t))\, dt \right| \\
&\leq \epsilon\, (C_\infty + 2M)\,,
\end{aligned} \tag{20}$$

which yields $\varphi'(u_k) \to \varphi'(u)$ as $u_k \rightharpoonup u$, that is, φ' is compact. □

Let E be a real Banach space. Recall that $I \in C^1(E, \mathbb{R})$ is said to satisfy the Palais-Smale condition (PS) if any sequence $(u_n) \subset E$, for which $(I(u_n))$ is bounded and $I'(u_n) \to 0$ as $n \to \infty$, possesses a convergent subsequence in E. We obtain the existence of solutions to (1) by using the following well-known Mountain Pass Theorem.

Lemma 2.6 ([17]). *Let E be a real Banach space and $I \in C^1(E, \mathbb{R})$ satisfying the Palais-Smale condition. If I satisfies the following conditions:*

(i) $I(0) = 0$,
(ii) *there exist constants $\rho, \beta > 0$ such that $I_{/\partial B_\rho(0)} \geq \beta$,*
(iii) *there exist $e \in E \backslash \overline{B}_\rho(0)$ such that $I(e) \leq 0$.*

Then I possesses a critical value $c \geq \gamma$ given by

$$c = \inf_{g\in\Gamma} \max_{s\in[0,1]} I(g(s)),$$

where

$$\Gamma = \{g \in C([0,1], E) : g(0) = 0, g(1) = e\}.$$

3. Proof of Theorem

Now we establish the corresponding variational framework to obtain the existence of solutions for (1). Define the functional $I : X^\alpha \to \mathbb{R}$ by

$$\begin{aligned} I(u) &= \int_{\mathbb{R}} \Big[\frac{1}{2}|{}_{-\infty}D_t^\alpha u(t)|^2 + \frac{1}{2}(L(t)u(t), u(t)) - W(t, u(t)\Big] dt \\ &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} W(t, u(t))dt. \end{aligned} \tag{21}$$

Under the conditions of Theorem 1.1, we see that I is a continuously Fréchet-differentiable functional defined on X^α, i.e., $I \in C^1(X^\alpha, \mathbb{R})$. Moreover, we have

$$I'(u)v = \int_{\mathbb{R}} [({}_{-\infty}D_t^\alpha u(t), {}_{-\infty} D_t^\alpha v(t)) + (L(t)u(t), v(t)) - (\nabla W(t, u(t)), v(t))]dt, \tag{22}$$

for all $u, v \in X^\alpha$, which yields

$$I'(u)u = \|u\|^2 - \int_{\mathbb{R}} (\nabla W(t, u(t)), u(t))dt. \tag{23}$$

Lemma 3.1. *Under the conditions of Theorem 1.1 , I satisfies the (PS) condition.*

Proof. Assume that $(u_k)_{k\in\mathbb{N}} \in X^\alpha$ is a sequence such that $(I(u_k))$ is bounded and $I'(u_k) \to 0$ as $k \to \infty$. Then there exists a constant $C_1 > 0$ such that

$$|I(u_k)| \leq C_1 \;\; and \;\; \|I'(u_k)\| \leq C_1 \tag{24}$$

for every $k \in \mathbb{N}$. We first prove that (u_k) is bounded in X^α. By (21), (22), (24), (H_2) and(H_3) , we obtain

$$\begin{aligned} C_1 + \frac{C_1}{\mu}\|u_k\| &\geq I(u_k) - \frac{1}{\mu}\|I'(u_k)\| u_k \\ &= (\frac{1}{2} - \frac{1}{\mu})\|u_k\|^2 + \int_{\mathbb{R}} \Big[W_2(t, u_k(t)) - \frac{1}{\mu}(\nabla W_2(t, u_k(t)), u_k(t))\Big] dt \\ &- \int_{\mathbb{R}} \Big[W_1(t, u_k(t)) - \frac{1}{\mu}(\nabla W_1(t, u_k(t)), u_k(t))\Big] dt \\ &\geq (\frac{1}{2} - \frac{1}{\mu})\|u_k\|^2, \; k \in \mathbb{N}. \end{aligned} \tag{25}$$

Since $\mu > 2$, the inequality (25) shows that (u_k) is bounded in X^α. So passing to a subsequence if necessary, it can be assumed that $u_k \rightharpoonup u$ in X^α. Since

$$(I'(u_k) - I'(u))(u_k - u) = \|u_k - u\|^2 - \int_{\mathbb{R}} (\nabla W(t, u_k(t)) - \nabla W(t, u(t)), u_k - u(t))\, dt.$$

Then by lemma 2.5, we deduce that $\|u_k - u\|^2 \to 0 \; as \; k \to \infty$. □

Now we are in the position to give the proof of Theorem 1.1. We divide the proof into several steps.

3.1. **Proof of Theorem. Step 1.** It is clear that $I(0) = 0$ and $I \in C^1(X^\alpha, \mathbb{R})$ satisfies the (PS) condition.

Step 2. Now we show that there exist constants ρ and $\beta > 0$ such that I satisfies the assumption (ii) of lemma 2.6. Choose $\delta \in (0, 1]$ such that

$$\frac{\mu-2-2\nu}{2(\mu-2)}\delta - M\delta^{\mu-1} = \max_{x\in[0,1]}\left(\frac{\mu-2-2\nu}{2(\mu-2)}x - Mx^{\mu-1}\right),$$

where $M = \sup\left\{\frac{W_1(t,x)}{l(t)} / \ t \in \mathbb{R},\ |x| = 1\right\}$.

Then

$$\frac{\mu-2-2\nu}{2(\mu-2)}\delta - M\delta^{\mu-1} = \begin{cases} \frac{\mu-2-2\nu}{2(\mu-2)}\left[\frac{\mu-2-2\nu}{2M(\mu-1)(\mu-2)}\right]^{\frac{1}{\mu-2}}, & \text{if } M > \frac{\mu-2-2\nu}{2(\mu-1)(\mu-2)}, \\ \frac{\mu-2-2\nu}{2(\mu-2)} - M, & \text{if } M \le \frac{\mu-2-2\nu}{2(\mu-1)(\mu-2)}. \end{cases} \tag{26}$$

By lemma 2.1 and (7), there is a constant K_α such that

$$\|u\|_\alpha \le K_\alpha \|u\|. \tag{27}$$

If $\|u\| = \frac{\delta}{K_\alpha} = \rho$, then it follows from (27) that $|u(t)| \le \delta \le 1$ for $t \in \mathbb{R}$. By (10) we have

$$\begin{aligned} \int_{\mathbb{R}} W_1(t,u(t))dt &= \int_{\{t\in\mathbb{R}/u(t)\neq 0\}} W_1(t,u(t))dt \\ &\le \int_{\{t\in\mathbb{R}/u(t)\neq 0\}} \left[W_1(t, \frac{u(t)}{|u(t)|})\,|u(t)|^\mu + \frac{\nu}{\mu-2}\,(L(t)u(t),u(t))\right]dt \\ &\le \int_{\mathbb{R}} \left[Ml(t)\,|u(t)|^\mu + \frac{\nu}{\mu-2}\,(L(t)u(t),u(t))\right]dt \\ &\le \int_{\mathbb{R}} \left[M\delta^{\mu-2}l(t)\,|u(t)|^2 + \frac{\nu}{\mu-2}\,(L(t)u(t),u(t))\right]dt \\ &\le \left(M\delta^{\mu-2} + \frac{\nu}{\mu-2}\right)\int_{\mathbb{R}} (L(t)u(t),u(t))\,dt. \end{aligned} \tag{28}$$

Set

$$\beta = \left[\frac{\mu-2-2\nu}{2(\mu-2)}\delta - M\delta^{\mu-1}\right]\frac{\delta}{K_\alpha^2}. \tag{29}$$

It follows from (26) that $\beta > 0$. Hence from (26), (28) and (29) we have

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2 + \int_{\mathbb{R}} [W_2(t,u(t)) - W_1(t,u(t))]\,dt \\ &\ge \frac{1}{2}\int_{\mathbb{R}} |{}_{-\infty}D_t^\alpha u(t)|^2 dt + (\frac{1}{2} - \frac{\nu}{\mu-2} - M\delta^{\mu-2})\int_{\mathbb{R}} (L(t)u(t),u(t))dt \\ &\ge (\frac{1}{2} - \frac{\nu}{\mu-2} - M\delta^{\mu-2})\int_{\mathbb{R}} \left[|{}_{-\infty}D_t^\alpha u(t)|^2 + (L(t)u(t),u(t))\right]dt \\ &\ge (\frac{1}{2} - \frac{\nu}{\mu-2} - M\delta^{\mu-2})\,\|u\|^2 = \beta. \end{aligned} \tag{30}$$

Hence (30), shows that $\|u\| = \rho$ implies that $I(u) \ge \beta$.

Step 3. It remains to prove that there exists $e \in X^\alpha$ such that $\|e\| > \rho$ and

$I(e) \leq 0$, where ρ is defined in Step 2. Set

$$a_1 = \max\left\{W_2(t,x)/t \in [-2,2],\ x \in \mathbb{R},\ |x| = 1\right\}$$

and

$$a_1 = \max\left\{W_2(t,x)/t \in [-2,2],\ x \in \mathbb{R},\ |x| \leq 1\right\}.$$

Then by (H_3) and (12), $0 \leq a_1 \leq a_2 < \infty$ and

$$W_2(t,x) \leq a_1 |x|^{\sigma} + a_2, \quad for\ (t,x) \in [-2,2] \times \mathbb{R}^N. \tag{31}$$

Take $w \in X^{\alpha}$ such that

$$|w(t)| = \begin{cases} 1, & \text{if } |t| \leq 1, \\ 0, & \text{if } |t| \geq 2, \end{cases} \tag{32}$$

and $|\omega(t)| \leq 1$ for $|t| \in (1,2]$. For $\xi > 1$, by (11) and (32), we have

$$\begin{aligned} \int_{-1}^{1} W(t,\xi\omega(t))dt &\geq |\xi|^{\mu} \int_{-1}^{1} \left[W_1(t,\omega(t)) - \frac{\nu}{\mu-2}(L(t)\omega(t),\omega(t))\right] dt \\ &\geq 2m\,|\xi|^{\mu}, \end{aligned} \tag{33}$$

where

$$m = \min_{-1\leq t\leq 1, |x|=1} \left[W_1(t,x) - \frac{\nu}{\mu-2}(L(t)x,x)\right] > 0,$$

see (H_2). By (21), (31), (32), (33), (H_2) and (H_3), we have for $\xi > 0$,

$$\begin{aligned} I(\xi\omega) &= \frac{1}{2}\|\xi\omega\|^2 + \int_{\mathbb{R}} \left[W_2(t,\xi\omega(t)) - W_1(t,\xi\omega(t))\right] dt \\ &\leq \frac{|\xi|^2}{2}\|\omega\|^2 + \int_{-2}^{2} W_2(t,\xi\omega(t))dt - \int_{-1}^{1} W_1(t,\xi\omega(t))dt \\ &\leq \frac{1}{2}\|\xi\omega\|^2 + a_1|\xi|^{\sigma}\int_{-2}^{2}|\omega(t)|^{\sigma}\,dt + 4a_2 - 2m\,|\xi|^{\mu}. \end{aligned} \tag{34}$$

Since $\mu > \sigma$ and $m > 0$, (34) implies that there exists $\xi > 1$ such that $\|\xi\omega\| > \rho$ and $I(\xi\omega) < 0$. Set $e(t) = \xi\omega(t)$. Then $e \in X^{\alpha}$, $\|e\| = \|\xi\omega\| > \rho$ and $I(e) = I(\xi\omega) < 0$. By Lemma 2.6, I possesses a critical value $d \geq \beta$ given by

$$d = \inf_{g\in\Gamma}\max_{s\in[0,1]} I(g(s)),$$

where

$$\Gamma = \{g \in C([0,1],E) : g(0) = 0, g(1) = e\}.$$

Hence, there exists $u^* \in X^{\alpha}$ such that

$$I(u^*) = d,\ and\ I'(u^*) = 0.$$

Then function u^* is a desired classical solution of (1). Since $d > 0$, u^* is a nontrivial homoclinic solution.

References

[1] Agrawal O; Tenreieo Machado J, Sabatier J, *Fractional derivatives and Their Application,* Nonlinear Dynamics. Springer-Verlag: Berlin, 2004.

[2] A. Ambrosetti, V. C. Zelati; *Multiple homoclinic orbits for a class of conservative systems,* Rend. Semin. Mat. Univ. Padova, (1993), 177–194.

[3] Z. B. Bai, H. S. Lü; *Positive solutions for boundary value problem of nonlinear fractional differential equation*, J. Math. Anal. Appl., (2005), 495–505.

[4] Y. H. Ding; *Existence and multiplicity results for homoclinic solutions to a class of Hamiltonian systems*, Nonlinear Anal., (1995), 1095–1113.

[5] R. Hilfer; *Applications of fractional calculus in physics*, World Science, Singapore, 2000.

[6] M. Izydorek, J. Janczewska; *Homoclinic solutions for a class of the second order Hamiltonian systems*, J. Differential Equations, (2005), 375–389.

[7] M. Izydorek, J. Janczewska; *Homoclinic solutions for nonautonomous second order Hamiltonian systems with a coercive potential*, Journal of Mathematical Analysis and Applications 2007:1119–1127.

[8] W. H. Jiang; *The existence of solutions for boundary value problems of fractional differential equatios at resonance*, Nonlinear Anal., (2011), 1987–1994.

[9] F. Jiao, Y. Zhou; *Existence results for fractional boundary value problem via critical point theory*, Inter. Journal Bif. Chaos, (2012), 1–17.

[10] A. Kilbas, B. Bonilla, J. J. Trujillo; Existence and uniqueness theorems for nonlinear fractional differential equations. Demonstratio Mathematica, (2000), 583-602.

[11] A. Kilbas, H. Srivastava, J. Trujillo; Theory and applications of fractional differential equations, North-Holland Mathematics Studies, 204, Singapore, (2006).

[12] J. Mawhin, M. Willem; *Critical Point Theory and Hamiltonian Systems*, Springer, New York, (1989).

[13] K. Miller, B. Ross; *An introduction to the fractional calculus and fractional differential equations*, Wiley and Sons, New York, (1993).

[14] N. Nyamoradi, Y. Zhou; *Bifurcation results for a class of fractional Hamiltonian systems with Liouville-Wely fractional derivatives*, J. Vib. Control, 2014, DOI:10.1177/1077546314535827.

[15] W. Omana, M. Willem; *Homoclinic orbits for a class of Hamiltonian systems*, Differential Integral Equations, (1992), 1115–1120.

[16] I. Podlubny; *Fractional differential equations*, Academic Press, New York, (1999).

[17] P. H. Rabinowitz; *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, in: CBMS Reg. Conf. Ser. in. Math., vol. 65, American Mathematical Society, Provodence, RI, (1986).

[18] P. H. Rabinowitz; *Homoclinic orbits Some for a class of Hamiltonian systems*, Proceeding of the Royal Society Of Edimburgh, Section A (1990), 114 (1-2):33-38.

[19] P. H. Rabinowitz, K. Tanaka; *Some results on connecting orbits for a class of Hamiltonian systems*, Math. Z., (1991), 473–499.

[20] C. Torres; *Existence of solution for a class of fractional Hamiltonian systems*, Electron. J. Differential Equations, (2013), 1–12.

[21] C. Torres; *Existence of solutions for perturbed fractional Hamiltonian systems* , JFCA, (2015), 62–70.

[22] C. Torres and W.Zugiaba *Liouville-Weyl Fractional Hamiltonian Systems: Existence Results* , Progr.Fract.Differ.Appl.5.No.3,1-9(2019).

[23] C. Torres *Existenceof solution for a general fractional advection-dispersion equation* ,Anal.Math.Phys.(2018).

[24] C. Torres *Tempered fractional differential equation: Variational approach* ,Math.Meth.Appl.Sci.(2017),40,4962-4973.

[25] C. Torres and Z.Zhang *Concentratio of Ground State Solutions for Fractional Hamiltonian Systems* , Topological Methods in Nonlinear Analysis ,Volume 50,No .2,2017,623-643.

[26] G. M. Zaslavsky; *Hamiltonian Chaos and Fractional Dynamics*, Oxford University Press, (2005).

[27] S. Q. Zhang; *Existence of a solution for the fractional differential equation with nonlinear boundary conditions*, Comput. Math. Appl., (2011), 1202–1208.

[28] X. Wu, Z. Zhang; *Solutions for perturbed fractional Hamiltonian systems without coercive conditions*, BVP, (2015), 1–12

[29] J. Xu, D. O'Regan, K. Zhang; *Multiple solutions for a calss of fractional Hamiltonian systems*, Fract. Calc. Appl. Anal., (2015), 48–63.

[30] Z.Zhang,C. Torres,R.Yuan *Homoclinic Solutions for Fractional Hamiltonian Systems with indifinite Conditions*,Filomat32:7(2018),2403-2419.

[31] Z. Zhang, R. Yuan; *Existence of solutions to fractional Hamiltonian systems with combined nonlinearities*, Electron. J. Differential Equations, (2016), 1–13.

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