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GENERALIZED RELATIVE TYPE (α, β) AND GENERALIZED RELATIVE WEAK TYPE (α, β) ORIENTED SOME GROWTH PROPERTIES OF COMPOSITE ENTIRE FUNCTION

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Abstract. The main aim of this paper is to prove some results related to the growth rates of composite entire functions on the basis of their generalized relative type (α, β) and generalized relative weak type (α, β) where α and β are continuous non-negative functions defined on $(-\infty, +\infty)$.

1. Introduction

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let f be an entire function defined on $\mathbb{C}$. The maximum modulus function $M_f(r)$ of $f = \sum_{n=0}^{\infty} a_n z^n$ on $|z| = r$ is defined as $M_f = \max_{|z|=r} |f(z)|$. Moreover, if f is non-constant entire then $M_f(r)$ is also strictly increasing and continuous function of r. Therefore its inverse $M_f^{-1} : (M_f(0), \infty) \to (0, \infty)$ exists and is such that $\lim_{s \to +\infty} M_f^{-1}(s) = \infty$. We use the standard notations and definitions of the theory of entire functions which are available in [8] and [9], and therefore we do not explain those in details.

Now let L be a class of continuous non-negative functions α defined on $(-\infty, +\infty)$ such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ with $\alpha(x) \uparrow +\infty$ as $x \to +\infty$ and $\alpha((1 + o(1))x) = (1 + o(1))\alpha(x)$ as $x \to +\infty$. We say that $\alpha \in L^0$, if $\alpha \in L$ and $\alpha(cx) = (1 + o(1))\alpha(x)$ as $x_0 \leq x \to +\infty$ for each $c \in (0, +\infty)$, i.e., α is slowly increasing function. Clearly $L^0 \subset L$.

Further we assume that throughout the present paper α, α_1, α_2, β, β_1 and β_2 always denote the functions belonging to L^0. The value

$$\rho_{(\alpha,\beta)}[f] = \limsup_{r \to +\infty} \frac{\alpha(\log M_f(r))}{\beta(\log r)} (\alpha \in L, \beta \in L)$$

is called [7] generalized order (α, β) of an entire function f.

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Now Biswas et al.[3] rewrote the definitions of the generalized order (α, β) and generalized lower order (α, β) of an entire function after giving a minor modification to the original definition of generalized order (α, β) of an entire function (see [7]).

Definition 1. [3] *The generalized order (α, β) denoted by $\rho_{(\alpha,\beta)}[f]$ and generalized lower order (α, β) denoted by $\lambda_{(\alpha,\beta)}[f]$ of an entire function f are defined as:*

$$\rho_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty} \frac{\alpha(M_f(r))}{\beta(r)} \text{ and } \lambda_{(\alpha,\beta)}[f] = \liminf_{r\to+\infty} \frac{\alpha(M_f(r))}{\beta(r)}.$$

In order to refine the growth scale namely the generalized order (α, β) of an entire function, Biswas et al.[4] have introduced the definitions of another growth indicators, called generalized type (α, β) and generalized lower type (α, β) respectively of an entire function which are as follows:

Definition 2. [4] *The generalized type (α, β) denoted by $\sigma_{(\alpha,\beta)}[f]$ and generalized lower type (α, β) denoted by $\overline{\sigma}_{(\alpha,\beta)}[f]$ of an entire function f having finite positive generalized order (α, β) $(0 < \rho_{(\alpha,\beta)}[f] < \infty)$ are defined as :*

$$\sigma_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty} \frac{\exp(\alpha(M_f(r)))}{(\exp(\beta(r))^{\rho_{(\alpha,\beta)}[f]}} \text{ and } \overline{\sigma}_{(\alpha,\beta)}[f] = \liminf_{r\to+\infty} \frac{\exp(\alpha(M_f(r)))}{(\exp(\beta(r))^{\rho_{(\alpha,\beta)}[f]}}.$$

It is obvious that $0 \leq \overline{\sigma}_{(\alpha,\beta)}[f] \leq \sigma_{(\alpha,\beta)}[f] \leq \infty$.

Analogously, to determine the relative growth of two entire functions having same non zero finite generalized lower order (α, β), one can introduce the definitions of generalized weak type (α, β) and generalized upper weak type (α, β) of an entire function f of finite positive generalized lower order (α, β), $\lambda_{(\alpha,\beta)}[f]$ in the following way:

Definition 3. [4] *The generalized upper weak type (α, β) denoted by $\tau_{(\alpha,\beta)}[f]$ and generalized weak type (α, β) denoted by $\overline{\tau}_{(\alpha,\beta)}[f]$ of an entire function f having finite positive generalized lower order (α, β) $(0 < \lambda_{(\alpha,\beta)}[f] < \infty)$ are defined as :*

$$\tau_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty} \frac{\exp(\alpha(M_f(r)))}{(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[f]}} \text{ and } \overline{\tau}_{(\alpha,\beta)}[f] = \liminf_{r\to+\infty} \frac{\exp(\alpha(M_f(r)))}{(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[f]}}.$$

It is obvious that $0 \leq \overline{\tau}_{(\alpha,\beta)}[f] \leq \tau_{(\alpha,\beta)}[f] \leq \infty$.

Mainly the growth investigation of entire functions has usually been done through their maximum moduli function in comparison with those of exponential function. But if one is paying attention to evaluate the growth rates of any entire function with respect to a new entire function, the notions of relative growth indicators (see [1, 2]) will come. Now in order to make some progresses in the study of relative order, Biswas et al.[5] have introduced the definitions of generalized relative order (α, β) and generalized relative lower order (α, β) of an entire function with respect to another entire function in the following way:

Definition 4. [5] *The generalized relative order (α, β), denoted by $\rho_{(\alpha,\beta)}[f]_g$ and generalized relative lower order (α, β), denoted by $\lambda_{(\alpha,\beta)}[f]_g$ of an entire function f with respect to an entire function g are defined as:*

$$\rho_{(\alpha,\beta)}[f]_g = \limsup_{r\to+\infty} \frac{\alpha(M_g^{-1}(M_f(r)))}{\beta(r)} \text{ and } \lambda_{(\alpha,\beta)}[f]_g = \liminf_{r\to+\infty} \frac{\alpha(M_g^{-1}(M_f(r)))}{\beta(r)}.$$

Now in order to refine the above growth scale, Biswas et al.[5] have introduced the definitions of other growth indicators, such as generalized relative type (α, β) and generalized relative lower type (α, β) of entire function with respect to an entire function which are as follows:

Definition 5. [5] *The generalized relative type (α, β), denoted by $\sigma_{(\alpha,\beta)}[f]_g$ and generalized relative lower type (α, β), denoted by $\overline{\sigma}_{(\alpha,\beta)}[f]_g$ of an entire function f with respect to an entire function g having non-zero finite generalized relative order (α, β) are defined as:*

$$\sigma_{(\alpha,\beta)}[f]_g = \limsup_{r\to+\infty} \frac{\exp(\alpha(M_g^{-1}(M_f(r))))}{(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[f]_g}}$$

$$\text{and } \overline{\sigma}_{(\alpha,\beta)}[f]_g = \liminf_{r\to+\infty} \frac{\exp(\alpha(M_g^{-1}(M_f(r))))}{(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[f]_g}}.$$

Analogously, to determine the relative growth of an entire function f having same non-zero finite generalized relative lower order (α, β) with respect to an entire function g, Biswas et al.[5] have introduced the definitions of generalized relative upper weak type (α, β) denoted by $\tau_{(\alpha,\beta)}[f]_g$ and generalized relative weak type (α, β) denoted by $\overline{\tau}_{(\alpha,\beta)}[f]_g$ of f with respect to g of finite positive generalized relative lower order (α, β) in the following way:

Definition 6. [5] *The generalized relative upper weak type (α, β), denoted by $\tau_{(\alpha,\beta)}[f]_g$ and generalized relative weak type (α, β), denoted by $\overline{\tau}_{(\alpha,\beta)}[f]_g$, of an entire function f with respect to an entire function g having non-zero finite generalized relative lower order (α, β) are defined as:*

$$\tau_{(\alpha,\beta)}[f]_g = \limsup_{r\to+\infty} \frac{\exp(\alpha(M_g^{-1}(M_f(r))))}{(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[f]_g}}$$

$$\text{and } \overline{\tau}_{(\alpha,\beta)}[f]_g = \liminf_{r\to+\infty} \frac{\exp(\alpha(M_g^{-1}(M_f(r))))}{(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[f]_g}}.$$

In this paper we wish to prove some results related to the growth rates of composite entire functions on the basis of their generalized relative order (α, β), generalized relative type (α, β) and generalized relative weak type (α, β) where α and β are continuous non-negative functions defined on $(-\infty, +\infty)$.

2. Main results

In this section first we present two lemmas which will be needed in the sequel.

Lemma 1. [6] *Let f and g are any two entire functions with $g(0) = 0$. Also let b satisfy $0 < b < 1$ and $c(b) = \frac{(1-b)^2}{4b}$. Then for all sufficiently large values of r,*

$$M_f(c(b)M_g(br)) \le M_{f(g)}(r) \le M_f(M_g(r)).$$

In addition if $b = \frac{1}{2}$, then for all sufficiently large values of r,

$$M_{f(g)}(r) \ge M_f\left(\frac{1}{8}M_g\left(\frac{r}{2}\right)\right).$$

Lemma 2. [2] *Suppose f is an entire function and $a > 1$, $0 < b < a$. Then for all sufficiently large r,*

$$M_f(ar) \ge bM_f(r).$$

Now we present the main results of the paper.

Theorem 1. *Let f, g and h be any three entire functions such that* $0 < \lambda_{(\alpha_1,\beta_1)}[f]_h \leq \rho_{(\alpha_1,\beta_1)}[f]_h < \infty$ *and* $\sigma_{(\alpha_2,\beta_2)}[g] < \infty$ *where* $\beta_1(r) \leq \exp(\alpha_2(r))$. *Then*

$$\limsup_{r\to+\infty} \frac{\alpha_1(M_h^{-1}(M_{f(g)}(r)))}{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]})))} \leq \frac{\sigma_{(\alpha_2,\beta_2)}[g]\cdot\rho_{(\alpha_1,\beta_1)}[f]_h}{\lambda_{(\alpha_1,\beta_1)}[f]_h}.$$

Proof. In view of Lemma 1 it follows that, for all sufficiently large values of r,

$$\alpha_1(M_h^{-1}(M_{f(g)}(r))) \leq \alpha_1(M_h^{-1}(M_f(M_g(r))))$$

$$i.e.,\ \alpha_1(M_h^{-1}(M_{f(g)}(r))) \leq (\rho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)\beta_1(M_g(r)).$$

Since $\beta_1(r) \leq \exp(\alpha_2(r))$, we get from above that, for all sufficiently large values of r,

$$\alpha_1(M_h^{-1}(M_{f(g)}(r))) \leq (\rho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)\exp(\alpha_2(M_g(r)))$$

$$i.e.,\ \alpha_1(M_h^{-1}(M_{f(g)}(r))) \leq (\rho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)(\sigma_{(\alpha_2,\beta_2)}[g] + \varepsilon)(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]}. \tag{1}$$

Now from the definition of $\lambda_{(\alpha_1,\beta_1)}[f]_h$, we obtain for all sufficiently large values of r that

$$\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]}))) \geq (\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]}. \tag{2}$$

Therefore from (1) and (2), it follows for all sufficiently large values of r that

$$\frac{\alpha_1(M_h^{-1}(M_{f(g)}(r)))}{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]})))} \leq \frac{(\rho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)(\sigma_{(\alpha_2,\beta_2)}[g] + \varepsilon)(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]}}{(\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]}}$$

$$i.e., \limsup_{r\to+\infty} \frac{\alpha_1(M_h^{-1}(M_{f(g)}(r)))}{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]})))} \leq \frac{\sigma_{(\alpha_2,\beta_2)}[g]\cdot\rho_{(\alpha_1,\beta_1)}[f]_h}{\lambda_{(\alpha_1,\beta_1)}[f]_h}.$$

Thus the theorem is established. □

Remark 1. *In Theorem 1, if we replace "$\sigma_{(\alpha_2,\beta_2)}[g]$" by "$\overline{\sigma}_{(\alpha_2,\beta_2)}[g]$", then Theorem 1 remains valid with "limit inferior" in place of "limit superior".*

Remark 2. *In Theorem 1, if we replace the condition "$0 < \lambda_{(\alpha_1,\beta_1)}[f]_h \leq \rho_{(\alpha_1,\beta_1)}[f]_h < \infty$" by "$\rho_{(\alpha_1,\beta_1)}[f]_h < \infty$, $\lambda_{(\alpha_2,\beta_2)}[g]_k > 0$" where k is an entire function and other conditions remain same, then Theorem 1 remains valid with "$\alpha_2(M_k^{-1}(M_g(\beta_2^{-1}(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]})))$" and "$\lambda_{(\alpha_2,\beta_2)}[g]_k$" in place of "$\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]})))$" and "$\lambda_{(\alpha_1,\beta_1)}[f]_h$" respectively.*

Remark 3. *In Remark 2, if we replace "$\sigma_{(\alpha_2,\beta_2)}[g]$" by "$\overline{\sigma}_{(\alpha_2,\beta_2)}[g]$", then Remark 2 remains valid with "limit inferior" in place of "limit superior".*

Remark 4. *We remark that in Remark 2, if we replace the condition "$\rho_{(\alpha_1,\beta_1)}[f]_h < \infty$" by "$\lambda_{(\alpha_1,\beta_1)}[f]_h < \infty$", then*

$$\liminf_{r\to+\infty} \frac{\alpha_1(M_h^{-1}(M_{f(g)}(r)))}{\alpha_2(M_k^{-1}(M_g(\beta_2^{-1}(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]})))} \leq \frac{\sigma_{(\alpha_2,\beta_2)}[g]\cdot\lambda_{(\alpha_1,\beta_1)}[f]_h}{\lambda_{(\alpha_2,\beta_2)}[g]_k}. \tag{3}$$

Remark 5. *In Remark 4, if we replace the conditions "* $\lambda_{(\alpha_2,\beta_2)}[g]_k > 0$ *and* $\lambda_{(\alpha_1,\beta_1)}[f]_h < \infty$*" by "*$\rho_{(\alpha_2,\beta_2)}[g]_k > 0$ *and* $\rho_{(\alpha_1,\beta_1)}[f]_h < \infty$*" respectively, then the same replacement is need to go in right part of* (3).

Remark 6. *In Theorem 1, if we replace the condition "*$\sigma_{(\alpha_2,\beta_2)}[g] < \infty$*" by "*$\tau_{(\alpha_2,\beta_2)}[g] < \infty$*"and other conditions remain same, then Theorem 1 remains valid with "*$\lambda_{(\alpha_2,\beta_2)}[g]$*" and "*$\tau_{(\alpha_2,\beta_2)}[g]$*" in place of "*$\rho_{(\alpha_2,\beta_2)}[g]$*" and "*$\sigma_{(\alpha_2,\beta_2)}[g]$*" respectively.*

Remark 7. *We remark that in Remark 6, if we replace the condition "* $0 < \lambda_{(\alpha_1,\beta_1)}[f]_h \leq \rho_{(\alpha_1,\beta_1)}[f]_h < \infty$ *and* $\tau_{(\alpha_2,\beta_2)}[g] < \infty$*" by "* $0 < \lambda_{(\alpha_1,\beta_1)}[f]_h < \infty$ *or* $0 < \rho_{(\alpha_1,\beta_1)}[f]_h < \infty$ *and* $\sigma_{(\alpha_2,\beta_2)}[g] < \infty$*", then*

$$\liminf_{r\to+\infty} \frac{\alpha_1(M_h^{-1}(M_{f(g)}(r)))}{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]})))} \leq \sigma_{(\alpha_2,\beta_2)}[g].$$

Remark 8. *In Remark 7, if we replace the condition "*$\sigma_{(\alpha_2,\beta_2)}[g] < \infty$*" by "*$\tau_{(\alpha_2,\beta_2)}[g] < \infty$*"and other conditions remain same, then Remark 7 remains valid with "*$\lambda_{(\alpha_2,\beta_2)}[g]$*" and "*$\tau_{(\alpha_2,\beta_2)}[g]$*" in place of "*$\rho_{(\alpha_2,\beta_2)}[g]$*" and "*$\sigma_{(\alpha_2,\beta_2)}[g]$*".*

Remark 9. *In Remark 2, if we replace the condition "*$\sigma_{(\alpha_2,\beta_2)}[g] < \infty$*" by "*$\tau_{(\alpha_2,\beta_2)}[g] < \infty$*"and other conditions remain same, then Remark 2 remains valid with "*$\alpha_2(M_k^{-1}(M_g(\beta_2^{-1}(\exp(\beta_2(r)))^{\lambda_{(\alpha_2,\beta_2)}[g]})))$*" and "*$\tau_{(\alpha_2,\beta_2)}[g]$*" in place of "*$\alpha_2(M_k^{-1}(M_g(\beta_2^{-1}(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]})))$*" and "*$\sigma_{(\alpha_2,\beta_2)}[g]$*" respectively where k is an entire function.*

Remark 10. *In Theorem 1, if we replace the condition "*$\sigma_{(\alpha_2,\beta_2)}[g] < \infty$*" by "*$\overline{\tau}_{(\alpha_2,\beta_2)}[g] < \infty$*" and other conditions remain same, then Theorem 1 remains valid with "limit inferior", "*$\lambda_{(\alpha_2,\beta_2)}[g]$*", and "*$\overline{\tau}_{(\alpha_2,\beta_2)}[g]$*" in place of "limit superior", "*$\rho_{(\alpha_2,\beta_2)}[g]$*" and "*$\sigma_{(\alpha_2,\beta_2)}[g]$*" respectively.*

Now we state the following theorem without its proof as it can easily be carried out in the line of Theorem 1.

Theorem 2. *Let f, g, h and k be any four entire functions such that* $\rho_{(\alpha_1,\beta_1)}[f]_h < \infty$, $\lambda_{(\alpha_2,\beta_2)}[g]_k > 0$ *and* $\overline{\tau}_{(\alpha_2,\beta_2)}[g] < \infty$ *where* $\beta_1(r) \leq \exp(\alpha_2(r))$*. Then*

$$\liminf_{r\to+\infty} \frac{\alpha_1(M_h^{-1}(M_{f(g)}(r)))}{\alpha_2(M_k^{-1}(M_g(\beta_2^{-1}(\exp(\beta_2(r)))^{\lambda_{(\alpha_2,\beta_2)}[g]})))} \leq \frac{\overline{\tau}_{(\alpha_2,\beta_2)}[g] \cdot \rho_{(\alpha_1,\beta_1)}[f]_h}{\lambda_{(\alpha_2,\beta_2)}[g]_k}.$$

Remark 11. *We remark that in Theorem 2, if we replace the condition "*$\rho_{(\alpha_1,\beta_1)}[f]_h < \infty$ *and* $\overline{\tau}_{(\alpha_2,\beta_2)}[g] < \infty$*" by "*$\lambda_{(\alpha_1,\beta_1)}[f]_h < \infty$ *and* $\tau_{(\alpha_2,\beta_2)}[g] < \infty$*", then Theorem 2 remains valid with "*$\tau_{(\alpha_2,\beta_2)}[g]$*" and "*$\lambda_{(\alpha_1,\beta_1)}[f]_h$*" in place of "*$\overline{\tau}_{(\alpha_2,\beta_2)}[g]$*", and "*$\rho_{(\alpha_1,\beta_1)}[f]_h$*" respectively.*

Remark 12. *In Remark 11, if we replace the conditions "*$\lambda_{(\alpha_2,\beta_2)}[g]_k > 0$ *and* $\lambda_{(\alpha_1,\beta_1)}[f]_h < \infty$*" by "*$\rho_{(\alpha_2,\beta_2)}[g]_k > 0$ *and* $\rho_{(\alpha_1,\beta_1)}[f]_h < \infty$*" respectively, then it is needed to go the same replacement in right part of the inequality in Remark 11.*

Theorem 3. *Let f, g and h be any three entire functions such that* $0 < \lambda_{(\alpha_1,\beta_1)}[f]_h \leq \rho_{(\alpha_1,\beta_1)}[f]_h < \infty$ *and* $\overline{\sigma}_{(\alpha_2,\beta_2)}[g] < \infty$ *where* $\beta_1(r) \geq \exp(\alpha_2(r))$*. Then for any* $\eta > 16$

$$\liminf_{r\to+\infty} \frac{\alpha_1(M_h^{-1}(M_{f(g)}(\eta r)))}{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]})))} \geq \frac{\overline{\sigma}_{(\alpha_2,\beta_2)}[g] \cdot \lambda_{(\alpha_1,\beta_1)}[f]_h}{\rho_{(\alpha_1,\beta_1)}[f]_h}.$$

Proof. In view of Lemma 1 and Lemma 2, we get that, for any $\eta > 16$ and all sufficiently large values of r,

$$\alpha_1(M_h^{-1}(M_{f(g)}(\eta r))) \geq \alpha_1(M_h^{-1}(M_f(M_g(r))))$$
$$i.e.,\ \alpha_1(M_h^{-1}(M_{f(g)}(\eta r))) \geq (\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)\beta_1(M_g(r)).$$

Since $\beta_1(r) \geq \exp(\alpha_2(r))$, we get that, from above for all sufficiently large values of r,

$$\alpha_1(M_h^{-1}(M_{f(g)}(\eta r))) \geq (\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)\exp(\alpha_2(M_g(r)))$$
$$i.e.,\ \alpha_1(M_h^{-1}(M_{f(g)}(\eta r))) \geq (\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)(\overline{\sigma}_{(\alpha_2,\beta_2)}[g] - \varepsilon)(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]}. \quad (4)$$

Now from the definition of $\rho_{(\alpha_1,\beta_1)}[f]_h$, we obtain that, for all sufficiently large values of r,

$$\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]}))) \leq (\rho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]}. \quad (5)$$

Therefore from (4) and (5), it follows that, for all sufficiently large values of r,

$$\frac{\alpha_1(M_h^{-1}(M_{f(g)}(\eta r)))}{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]})))} \geq \frac{(\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)(\overline{\sigma}_{(\alpha_2,\beta_2)}[g] - \varepsilon)(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]}}{(\rho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]}}$$

$$i.e., \liminf_{r\to+\infty}\frac{\alpha_1(M_h^{-1}(M_{f(g)}(\eta r)))}{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]})))} \geq \frac{\overline{\sigma}_{(\alpha_2,\beta_2)}[g]\cdot\lambda_{(\alpha_1,\beta_1)}[f]_h}{\rho_{(\alpha_1,\beta_1)}[f]_h}.$$

Thus the theorem is established. □

Remark 13. *In Theorem 3, if we replace "$\overline{\sigma}_{(\alpha_2,\beta_2)}[g]$" by "$\sigma_{(\alpha_2,\beta_2)}[g]$", then Theorem 3 remains valid with "limit superior" instead of " limit inferior".*

Remark 14. *We remark that in Theorem 3, if we replace the condition " $0 < \lambda_{(\alpha_1,\beta_1)}[f]_h \leq \rho_{(\alpha_1,\beta_1)}[f]_h < \infty$" by " $0 < \lambda_{(\alpha_1,\beta_1)}[f]_h < \infty$ or $0 < \rho_{(\alpha_1,\beta_1)}[f]_h < \infty$", then Theorem 3 remains valid with "limit superior" instead of "limit inferior" as well as right hand side is replaced by "$\overline{\sigma}_{(\alpha_2,\beta_2)}[g]$".*

Now we state the following theorem without its proof as it can easily be carried out in the line of Theorem 3.

Theorem 4. *Let f, g, h and k be any four entire functions such that $\lambda_{(\alpha_1,\beta_1)}[f]_h > 0$, $\rho_{(\alpha_2,\beta_2)}[g]_k < \infty$ and $\overline{\sigma}_{(\alpha_2,\beta_2)}[g] < \infty$ where $\beta_1(r) \geq \exp(\alpha_2(r))$. Then for any $\eta > 16$*

$$\liminf_{r\to+\infty}\frac{\alpha_1(M_h^{-1}(M_{f(g)}(\eta r)))}{\alpha_2(M_k^{-1}(M_g(\beta_2^{-1}(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]})))} \geq \frac{\overline{\sigma}_{(\alpha_2,\beta_2)}[g]\cdot\lambda_{(\alpha_1,\beta_1)}[f]_h}{\rho_{(\alpha_2,\beta_2)}[g]_k}.$$

Remark 15. *In Theorem 4, if we replace "$\overline{\sigma}_{(\alpha_2,\beta_2)}[g]$" by "$\sigma_{(\alpha_2,\beta_2)}[g]$", then Theorem 4 remains valid with " limit inferior" replaced by "limit superior".*

Remark 16. *We remark that in Theorem 4, if we replace the condition " $\rho_{(\alpha_2,\beta_2)}[g]_k < \infty$" by "$\lambda_{(\alpha_2,\beta_2)}[g]_k < \infty$", then*

$$\limsup_{r\to+\infty}\frac{\alpha_1(M_h^{-1}(M_{f(g)}(\eta r)))}{\alpha_2(M_k^{-1}(M_g(\beta_2^{-1}(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]})))} \geq \frac{\overline{\sigma}_{(\alpha_2,\beta_2)}[g]\cdot\lambda_{(\alpha_1,\beta_1)}[f]_h}{\lambda_{(\alpha_2,\beta_2)}[g]_k}. \quad (6)$$

Remark 17. *In Remark 16, if we replace the conditions " $0 < \lambda_{(\alpha_1,\beta_1)}[f]_h$ and $\lambda_{(\alpha_2,\beta_2)}[g]_k < \infty$" by "$0 < \rho_{(\alpha_1,\beta_1)}[f]_h$ and $\rho_{(\alpha_2,\beta_2)}[g]_k < \infty$" respectively, then the inequality* (6) *is true if we replace "$\lambda_{(\alpha_1,\beta_1)}[f]_h$" and "$\lambda_{(\alpha_2,\beta_2)}[g]_k$" in right part of* (6) *by "$\rho_{(\alpha_1,\beta_1)}[f]_h$" and "$\rho_{(\alpha_2,\beta_2)}[g]_k$" respectively.*

Using the concept of generalized weak type (α, β) of an entire function, we may state the subsequent two theorems without their proofs since those can be carried out in the line of Theorem 3 and Theorem 4 respectively.

Theorem 5. *Let f, g and h be any three entire functions such that $0 < \lambda_{(\alpha_1,\beta_1)}[f]_h \leq \rho_{(\alpha_1,\beta_1)}[f]_h < \infty$ and $\overline{\tau}_{(\alpha_2,\beta_2)}[g] < \infty$ where $\beta_1(r) \geq \exp(\alpha_2(r))$. Then for any $\eta > 16$*

$$\liminf_{r\to+\infty} \frac{\alpha_1(M_h^{-1}(M_{f(g)}(\eta r)))}{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\lambda_{(\alpha_2,\beta_2)}[g]})))} \geq \frac{\overline{\tau}_{(\alpha_2,\beta_2)}[g] \cdot \lambda_{(\alpha_1,\beta_1)}[f]_h}{\rho_{(\alpha_1,\beta_1)}[f]_h}.$$

Remark 18. *In Theorem 5, if we replace "$\overline{\tau}_{(\alpha_2,\beta_2)}[g]$" by "$\tau_{(\alpha_2,\beta_2)}[g]$", then Theorem 5 remains valid with "limit superior" instead of "limit inferior".*

Remark 19. *We remark that in Theorem 5, if we replace the condition " $0 < \lambda_{(\alpha_1,\beta_1)}[f]_h \leq \rho_{(\alpha_1,\beta_1)}[f]_h < \infty$" by "$0 < \lambda_{(\alpha_1,\beta_1)}[f]_h < \infty$ or $0 < \rho_{(\alpha_1,\beta_1)}[f]_h < \infty$", then*

$$\limsup_{r\to+\infty} \frac{\alpha_1(M_h^{-1}(M_{f(g)}(\eta r)))}{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\lambda_{(\alpha_2,\beta_2)}[g]})))} \geq \overline{\tau}_{(\alpha_2,\beta_2)}[g].$$

Theorem 6. *Let f, g, h and k be any four entire functions such that $\lambda_{(\alpha_1,\beta_1)}[f]_h > 0$, $\rho_{(\alpha_2,\beta_2)}[g]_k < \infty$ and $\overline{\tau}_{(\alpha_2,\beta_2)}[g] < \infty$ where $\beta_1(r) \geq \exp(\alpha_2(r))$. Then for any $\eta > 16$*

$$\liminf_{r\to+\infty} \frac{\alpha_1(M_h^{-1}(M_{f(g)}(\eta r)))}{\alpha_2(M_k^{-1}(M_g(\beta_2^{-1}(\exp(\beta_2(r)))^{\lambda_{(\alpha_2,\beta_2)}[g]})))} \geq \frac{\overline{\tau}_{(\alpha_2,\beta_2)}[g] \cdot \lambda_{(\alpha_1,\beta_1)}[f]_h}{\rho_{(\alpha_2,\beta_2)}[g]_k}.$$

Remark 20. *In Theorem 6, if we replace "$\overline{\tau}_{(\alpha_2,\beta_2)}[g]$" by "$\tau_{(\alpha_2,\beta_2)}[g]$", then Theorem 6 remains valid with "limit superior" instead of "limit inferior".*

Remark 21. *We remark that in Theorem 6, if we replace the condition " $0 < \lambda_{(\alpha_1,\beta_1)}[f]_h$" by " $0 < \rho_{(\alpha_1,\beta_1)}[f]_h$", then*

$$\limsup_{r\to+\infty} \frac{\alpha_1(M_h^{-1}(M_{f(g)}(\eta r)))}{\alpha_2(M_k^{-1}(M_g(\beta_2^{-1}(\exp(\beta_2(r)))^{\lambda_{(\alpha_2,\beta_2)}[g]})))} \geq \frac{\overline{\tau}_{(\alpha_2,\beta_2)}[g] \cdot \rho_{(\alpha_1,\beta_1)}[f]_h}{\rho_{(\alpha_2,\beta_2)}[g]_k}. \quad (7)$$

Remark 22. *In Remark 21, if we replace the conditions " $0 < \rho_{(\alpha_1,\beta_1)}[f]_h$ and $0 < \rho_{(\alpha_2,\beta_2)}[g]_k$" by "$0 < \lambda_{(\alpha_1,\beta_1)}[f]_h$ and $0 < \lambda_{(\alpha_2,\beta_2)}[g]_k$", then is need to go the same replacement in right part of* (7).

Theorem 7. *Let f, g and h be any three entire functions such that $0 < \rho_{(\alpha_1,\beta_1)}[f]_h < \infty$, $\rho_{(\alpha_1,\beta_1)}[f]_h = \rho_{(\alpha_2,\beta_2)}[g]$, $\sigma_{(\alpha_2,\beta_2)}[g] < \infty$ and $0 < \sigma_{(\alpha_1,\beta_1)}[f]_h < \infty$ where $\beta_1(r) \leq \exp(\alpha_2(r))$. Then*

$$\liminf_{r\to+\infty} \frac{\alpha_1(M_h^{-1}(M_{f(g)}(r)))}{\exp(\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\beta_2(r))))))} \leq \frac{\rho_{(\alpha_1,\beta_1)}[f]_h \cdot \sigma_{(\alpha_2,\beta_2)}[g]}{\sigma_{(\alpha_1,\beta_1)}[f]_h}. \quad (8)$$

Proof. In view of the condition $\rho_{(\alpha_1,\beta_1)}[f]_h = \rho_{(\alpha_2,\beta_2)}[g]$, we obtain that, from (1) for all sufficiently large values of r,

$$\alpha_1(M_h^{-1}(M_{f(g)}(r))) \leq (\rho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)(\sigma_{(\alpha_2,\beta_2)}[g] + \varepsilon)(\exp(\beta_2(r)))^{\rho_{(\alpha_1,\beta_1)}[f]_h}. \quad (9)$$

Now using the definition of $\sigma_{(\alpha_1,\beta_1)}[f]_h$, we get from above that, for a sequence of values of r tending to infinity,

$$\exp(\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\beta_2(r)))))) \geq (\sigma_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)(\exp(\beta_2(r)))^{\rho_{(\alpha_1,\beta_1)}[f]_h}. \quad (10)$$

Now from (9) and (10), it follows that, for a sequence of values of r tending to infinity,

$$\begin{aligned} &\frac{\alpha_1(M_h^{-1}(M_{f(g)}(r)))}{\exp(\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\beta_2(r))))))} \\ \leq \ & \frac{(\rho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)(\sigma_{(\alpha_2,\beta_2)}[g] + \varepsilon)(\exp(\beta_2(r)))^{\rho_{(\alpha_1,\beta_1)}[f]_h}}{(\sigma_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)(\exp(\beta_2(r)))^{\rho_{(\alpha_1,\beta_1)}[f]_h}}. \end{aligned}$$

Since $\varepsilon(> 0)$ is arbitrary, it follows from above that

$$\liminf_{r\to+\infty} \frac{\alpha_1(M_h^{-1}(M_{f(g)}(r)))}{\exp(\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\beta_2(r))))))} \leq \frac{\rho_{(\alpha_1,\beta_1)}[f]_h \cdot \sigma_{(\alpha_2,\beta_2)}[g]}{\sigma_{(\alpha_1,\beta_1)}[f]_h}.$$

□

Remark 23. *In Theorem 7, if we replace the conditions " $\sigma_{(\alpha_2,\beta_2)}[g] < \infty$" and "$0 < \sigma_{(\alpha_1,\beta_1)}[f]_h < \infty$" by "$\overline{\sigma}_{(\alpha_2,\beta_2)}[g] < \infty$" and "$0 < \overline{\sigma}_{(\alpha_1,\beta_1)}[f]_h < \infty$", then is need to go the same replacement in right part of* (8).

Remark 24. *If we replace the conditions $0 < \rho_{(\alpha_1,\beta_1)}[f]_h < \infty$ and $0 < \sigma_{(\alpha_1,\beta_1)}[f]_h < \infty$ of Theorem 7 by $0 < \lambda_{(\alpha_1,\beta_1)}[f]_h \leq \rho_{(\alpha_1,\beta_1)}[f]_h < \infty$ and $0 < \overline{\sigma}_{(\alpha_1,\beta_1)}[f]_h < \infty$ respectively, then Theorem 7 remains valid with "$\lambda_{(\alpha_1,\beta_1)}[f]_h$", and "$\overline{\sigma}_{(\alpha_1,\beta_1)}[f]_h$" in place of "$\rho_{(\alpha_1,\beta_1)}[f]_h$" and "$\sigma_{(\alpha_1,\beta_1)}[f]_h$" respectively.*

Remark 25. *If we replace the condition $0 < \sigma_{(\alpha_1,\beta_1)}[f]_h < \infty$ of Theorem 7 by $0 < \overline{\sigma}_{(\alpha_1,\beta_1)}[f]_h < \infty$, then Theorem 7 remains valid with "limit superior" and "$\overline{\sigma}_{(\alpha_1,\beta_1)}[f]_h$" instead of "limit inferior" and "$\sigma_{(\alpha_1,\beta_1)}[f]_h$".*

Remark 26. *In Theorem 7, if we replace the condition "$\rho_{(\alpha_1,\beta_1)}[f]_h = \rho_{(\alpha_2,\beta_2)}[g]$, $\sigma_{(\alpha_2,\beta_2)}[g] < \infty$" and "$0 < \sigma_{(\alpha_1,\beta_1)}[f]_h < \infty$" by "$\lambda_{(\alpha_1,\beta_1)}[f]_h = \lambda_{(\alpha_2,\beta_2)}[g]$, $\tau_{(\alpha_2,\beta_2)}[g] < \infty$" and "$0 < \tau_{(\alpha_1,\beta_1)}[f]_h < \infty$" and other conditions remain same, then Theorem 7 remains valid with "$\tau_{(\alpha_2,\beta_2)}[g]$" and "$\tau_{(\alpha_1,\beta_1)}[f]_h$" in place of "$\sigma_{(\alpha_2,\beta_2)}[g]$" and "$\sigma_{(\alpha_1,\beta_1)}[f]_h$" respectively.*

Remark 27. *In Theorem 7, if we replace the condition "$\rho_{(\alpha_1,\beta_1)}[f]_h = \rho_{(\alpha_2,\beta_2)}[g]$, $\sigma_{(\alpha_2,\beta_2)}[g] < \infty$" and "$0 < \sigma_{(\alpha_1,\beta_1)}[f]_h < \infty$" by "$\lambda_{(\alpha_1,\beta_1)}[f]_h = \lambda_{(\alpha_2,\beta_2)}[g]$, $\overline{\tau}_{(\alpha_2,\beta_2)}[g] < \infty$" and "$0 < \overline{\tau}_{(\alpha_1,\beta_1)}[f]_h < \infty$" and other conditions remain same, then Theorem 7 remains valid with "$\overline{\tau}_{(\alpha_2,\beta_2)}[g]$" and "$\overline{\tau}_{(\alpha_1,\beta_1)}[f]_h$" in place of "$\sigma_{(\alpha_2,\beta_2)}[g]$" and "$\sigma_{(\alpha_1,\beta_1)}[f]_h$" respectively.*

Remark 28. *If we replace the conditions $0 < \lambda_{(\alpha_1,\beta_1)}[f]_h \leq \rho_{(\alpha_1,\beta_1)}[f]_h < \infty$ and $0 < \tau_{(\alpha_1,\beta_1)}[f]_h < \infty$ of Remark 26 by $0 < \lambda_{(\alpha_1,\beta_1)}[f]_h < \infty$ and $0 < \overline{\tau}_{(\alpha_1,\beta_1)}[f]_h < \infty$ respectively, then Remark 26 remains valid with "$\lambda_{(\alpha_1,\beta_1)}[f]_h$", "$\tau_{(\alpha_2,\beta_2)}[g]$" and "$\overline{\tau}_{(\alpha_1,\beta_1)}[f]_h$" in place of "$\rho_{(\alpha_1,\beta_1)}[f]_h$", "$\sigma_{(\alpha_2,\beta_2)}[g]$" and "$\sigma_{(\alpha_1,\beta_1)}[f]_h$" respectively.*

Remark 29. *If we replace the condition $0 < \tau_{(\alpha_1,\beta_1)}[f]_h < \infty$ of Remark 26 by $0 < \overline{\tau}_{(\alpha_1,\beta_1)}[f]_h < \infty$, then Remark 26 remains valid with "limit superior" instead of "limit inferior".*

Remark 30. *If we replace the conditions* $0 < \rho_{(\alpha_1,\beta_1)}[f]_h < \infty$, $\rho_{(\alpha_1,\beta_1)}[f]_h = \rho_{(\alpha_2,\beta_2)}[g]$ *and* $0 < \sigma_{(\alpha_1,\beta_1)}[f]_h < \infty$ *of Theorem 7 by* $0 < \lambda_{(\alpha_1,\beta_1)}[f]_h \leq \rho_{(\alpha_1,\beta_1)}[f]_h < \infty$, $\lambda_{(\alpha_1,\beta_1)}[f]_h = \rho_{(\alpha_2,\beta_2)}[g]$ *and* $0 < \tau_{(\alpha_1,\beta_1)}[f]_h < \infty$ *respectively, then Theorem 7 remains valid with "*$\tau_{(\alpha_1,\beta_1)}[f]_h$*" in place of "*$\sigma_{(\alpha_1,\beta_1)}[f]_h$*".*

Remark 31. *If we replace the condition* $0 < \tau_{(\alpha_1,\beta_1)}[f]_h < \infty$ *of Remark 30 by* $0 < \overline{\tau}_{(\alpha_1,\beta_1)}[f]_h < \infty$, *then Remark 30 remains valid with "limit superior" instead of "limit inferior".*

Remark 32. *In Remark 30, if we replace the conditions "* $\sigma_{(\alpha_2,\beta_2)}[g] < \infty$*" and "*$0 < \tau_{(\alpha_1,\beta_1)}[f]_h < \infty$*" by "*$\overline{\sigma}_{(\alpha_2,\beta_2)}[g] < \infty$*" and "*$0 < \overline{\tau}_{(\alpha_1,\beta_1)}[f]_h < \infty$*", then the conclusion of Theorem 7 remains valid with "*$\overline{\sigma}_{(\alpha_2,\beta_2)}[g]$*" and "*$\overline{\tau}_{(\alpha_1,\beta_1)}[f]_h$*" in place of "*$\sigma_{(\alpha_2,\beta_2)}[g]$*" and "*$\sigma_{(\alpha_1,\beta_1)}[f]_h$*" respectively.*

Remark 33. *If we replace the conditions* $0 < \lambda_{(\alpha_1,\beta_1)}[f]_h \leq \rho_{(\alpha_1,\beta_1)}[f]_h < \infty$ *and* $0 < \tau_{(\alpha_1,\beta_1)}[f]_h < \infty$ *of Remark 30 by* $0 < \lambda_{(\alpha_1,\beta_1)}[f]_h < \infty$ *and* $0 < \overline{\tau}_{(\alpha_1,\beta_1)}[f]_h < \infty$ *respectively, then the conclusion of Theorem 7 remains valid with "*$\lambda_{(\alpha_1,\beta_1)}[f]_h$*" and "*$\overline{\tau}_{(\alpha_1,\beta_1)}[f]_h$*" in place of "*$\rho_{(\alpha_1,\beta_1)}[f]_h$*" and "*$\sigma_{(\alpha_1,\beta_1)}[f]_h$*" respectively.*

Remark 34. *If we replace the conditions* $\rho_{(\alpha_1,\beta_1)}[f]_h = \rho_{(\alpha_2,\beta_2)}[g]$ *and* $\sigma_{(\alpha_2,\beta_2)}[g] < \infty$ *of Theorem 7 by* $\rho_{(\alpha_1,\beta_1)}[f]_h = \lambda_{(\alpha_2,\beta_2)}[g]$ *and* $\tau_{(\alpha_2,\beta_2)}[g] < \infty$ *respectively, then Theorem 7 remains valid with "*$\tau_{(\alpha_2,\beta_2)}[g]$*" in place of "*$\sigma_{(\alpha_2,\beta_2)}[g]$*".*

Remark 35. *If we replace the condition* $0 < \sigma_{(\alpha_1,\beta_1)}[f]_h < \infty$ *of Remark 34 by* $0 < \overline{\sigma}_{(\alpha_1,\beta_1)}[f]_h < \infty$, *then the conclusion of Theorem 7 remains valid with "limit superior", "*$\overline{\sigma}_{(\alpha_1,\beta_1)}[f]_h$*" and "*$\tau_{(\alpha_2,\beta_2)}[g]$*" in place of "limit inferior", "*$\sigma_{(\alpha_1,\beta_1)}[f]_h$*" and "*$\sigma_{(\alpha_2,\beta_2)}[g]$*" respectively.*

Remark 36. *If we replace the conditions* $0 < \rho_{(\alpha_1,\beta_1)}[f]_h < \infty$ *and* $0 < \sigma_{(\alpha_1,\beta_1)}[f]_h < \infty$ *of Remark 34 by* $0 < \lambda_{(\alpha_1,\beta_1)}[f]_h \leq \rho_{(\alpha_1,\beta_1)}[f]_h < \infty$ *and* $0 < \overline{\sigma}_{(\alpha_1,\beta_1)}[f]_h < \infty$ *respectively, then the conclusion of Theorem 7 remains valid with "*$\lambda_{(\alpha_1,\beta_1)}[f]_h$*", "*$\tau_{(\alpha_2,\beta_2)}[g] < \infty$*" and "*$\overline{\sigma}_{(\alpha_1,\beta_1)}[f]_h$*" in place of "*$\rho_{(\alpha_1,\beta_1)}[f]_h$*", "*$\sigma_{(\alpha_2,\beta_2)}[g]$*" and "*$\sigma_{(\alpha_1,\beta_1)}[f]_h$*".*

Remark 37. *If we replace the conditions* $\rho_{(\alpha_1,\beta_1)}[f]_h = \rho_{(\alpha_2,\beta_2)}[g]$, $\sigma_{(\alpha_2,\beta_2)}[g] < \infty$ *and* $0 < \sigma_{(\alpha_1,\beta_1)}[f]_h < \infty$ *of Theorem 7 by* $\rho_{(\alpha_1,\beta_1)}[f]_h = \lambda_{(\alpha_2,\beta_2)}[g]$, $\overline{\tau}_{(\alpha_2,\beta_2)}[g] < \infty$ *and* $0 < \overline{\sigma}_{(\alpha_1,\beta_1)}[f]_h < \infty$ *respectively, then Theorem 7 remains valid with "*$\overline{\tau}_{(\alpha_2,\beta_2)}[g] < \infty$*" and "*$\overline{\sigma}_{(\alpha_1,\beta_1)}[f]_h$*" in place of "*$\sigma_{(\alpha_2,\beta_2)}[g]$*" and "*$\sigma_{(\alpha_1,\beta_1)}[f]_h$*".*

Theorem 8. *Let f, g and h be any three entire functions such that* $0 < \lambda_{(\alpha_1,\beta_1)}[f]_h \leq \rho_{(\alpha_1,\beta_1)}[f]_h < \infty$, $\rho_{(\alpha_1,\beta_1)}[f]_h = \rho_{(\alpha_2,\beta_2)}[g]$, $\overline{\sigma}_{(\alpha_2,\beta_2)}[g] < \infty$ *and* $0 < \overline{\sigma}_{(\alpha_1,\beta_1)}[f]_h < \infty$ *where* $\beta_1(r) \geq \exp(\alpha_2(r))$. *Then for any* $\eta > 16$

$$\limsup_{r\to+\infty} \frac{\alpha_1(M_h^{-1}(M_{f(g)}(\eta r)))}{\exp(\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\beta_2(r))))))} \geq \frac{\lambda_{(\alpha_1,\beta_1)}[f]_h \cdot \overline{\sigma}_{(\alpha_2,\beta_2)}[g]}{\overline{\sigma}_{(\alpha_1,\beta_1)}[f]_h}. \tag{11}$$

Proof. In view of the condition $\rho_{(\alpha_1,\beta_1)}[f]_h = \rho_{(\alpha_2,\beta_2)}[g]$, we obtain from (4) that, for all sufficiently large values of r,

$$\alpha_1(M_h^{-1}(M_{f(g)}(\eta r))) \geq (\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)(\overline{\sigma}_{(\alpha_2,\beta_2)}[g] - \varepsilon)(\exp(\beta_2(r)))^{\rho_{(\alpha_1,\beta_1)}[f]_h}. \tag{12}$$

Further in view of definition of $\overline{\sigma}_{(\alpha_1,\beta_1)}[f]_h$, we get that, for a sequence of values of r tending to infinity,

$$\exp(\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\beta_2(r)))))) \geq (\overline{\sigma}_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)(\exp(\beta_2(r)))^{\rho_{(\alpha_1,\beta_1)}[f]_h}. \tag{13}$$

Now from (12) and (13), it follows that for a sequence of values of r tending to infinity,

$$\frac{\alpha_1(M_h^{-1}(M_{f(g)}(\eta r)))}{\exp(\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\beta_2(r))))))} \geq \frac{(\lambda_{(\alpha_1,\beta_1)}[f]_h-\varepsilon)(\overline{\sigma}_{(\alpha_2,\beta_2)}[g]-\varepsilon)(\exp(\beta_2(r)))^{\rho_{(\alpha_1,\beta_1)}[f]_h}}{(\overline{\sigma}_{(\alpha_1,\beta_1)}[f]_h+\varepsilon)(\exp(\beta_2(r)))^{\rho_{(\alpha_1,\beta_1)}[f]_h}}.$$

Since $\varepsilon(>0)$ is arbitrary, it follows from above that

$$\limsup_{r\to+\infty}\frac{\alpha_1(M_h^{-1}(M_{f(g)}(\eta r)))}{\exp(\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\beta_2(r))))))} \geq \frac{\lambda_{(\alpha_1,\beta_1)}[f]_h\cdot\overline{\sigma}_{(\alpha_2,\beta_2)}[g]}{\overline{\sigma}_{(\alpha_1,\beta_1)}[f]_h}.$$

□

Remark 38. *In Theorem 8, if we replace the conditions "* $\overline{\sigma}_{(\alpha_2,\beta_2)}[g]<\infty$*" and "*$0<\overline{\sigma}_{(\alpha_1,\beta_1)}[f]_h<\infty$*" by "*$\sigma_{(\alpha_2,\beta_2)}[g]<\infty$*" and "*$0<\sigma_{(\alpha_1,\beta_1)}[f]_h<\infty$*", then is need to go the same replacement in right part of* (11).

Remark 39. *If we replace the conditions* $0<\lambda_{(\alpha_1,\beta_1)}[f]_h\leq\rho_{(\alpha_1,\beta_1)}[f]_h<\infty$ *and* $0<\overline{\sigma}_{(\alpha_1,\beta_1)}[f]_h<\infty$ *of Theorem 8 by* $0<\rho_{(\alpha_1,\beta_1)}[f]_h<\infty$ *and* $0<\sigma_{(\alpha_1,\beta_1)}[f]_h<\infty$ *respectively, then Theorem 8 remains valid with "*$\rho_{(\alpha_1,\beta_1)}[f]_h$*" and "*$\sigma_{(\alpha_1,\beta_1)}[f]_h$*" in place of "*$\lambda_{(\alpha_1,\beta_1)}[f]_h$*" and "*$\overline{\sigma}_{(\alpha_1,\beta_1)}[f]_h$*". Further if we replace the condition* $0<\overline{\sigma}_{(\alpha_1,\beta_1)}[f]_h<\infty$ *of Theorem 8 by* $0<\sigma_{(\alpha_1,\beta_1)}[f]_h<\infty$*, then the conclusion of Theorem 8 remains valid with "limit inferior" replaced by "limit superior".*

Remark 40. *If we replace the conditions* $0<\lambda_{(\alpha_1,\beta_1)}[f]_h\leq\rho_{(\alpha_1,\beta_1)}[f]_h<\infty$, $\rho_{(\alpha_1,\beta_1)}[f]_h=\rho_{(\alpha_2,\beta_2)}[g]$, $\overline{\sigma}_{(\alpha_2,\beta_2)}[g]<\infty$ *and* $0<\overline{\sigma}_{(\alpha_1,\beta_1)}[f]_h<\infty$ *of Theorem 8 by* $0<\lambda_{(\alpha_1,\beta_1)}[f]_h<\infty$, $\lambda_{(\alpha_1,\beta_1)}[f]_h=\lambda_{(\alpha_2,\beta_2)}[g]$, $\overline{\tau}_{(\alpha_2,\beta_2)}[g]<\infty$ *and* $0<\overline{\tau}_{(\alpha_1,\beta_1)}[f]_h<\infty$ *respectively and other conditions remain same, then the conclusion of Theorem 8 remains valid with "*$\overline{\tau}_{(\alpha_2,\beta_2)}[g]$*" and "*$\overline{\tau}_{(\alpha_1,\beta_1)}[f]_h$*" in place of "*$\overline{\sigma}_{(\alpha_2,\beta_2)}[g]$*" and "*$\overline{\sigma}_{(\alpha_1,\beta_1)}[f]_h$*" respectively.*

Remark 41. *In Remark 40, if we replace the conditions "* $\overline{\tau}_{(\alpha_2,\beta_2)}[g]<\infty$*" and "*$0<\overline{\tau}_{(\alpha_1,\beta_1)}[f]_h<\infty$*" by "*$\tau_{(\alpha_2,\beta_2)}[g]<\infty$*" and "*$0<\tau_{(\alpha_1,\beta_1)}[f]_h<\infty$*", then Remark 40 remains valid with "*$\tau_{(\alpha_2,\beta_2)}[g]$*" and "*$\tau_{(\alpha_1,\beta_1)}[f]_h$*" in place of "*$\overline{\tau}_{(\alpha_2,\beta_2)}[g]$*" and "*$\overline{\tau}_{(\alpha_1,\beta_1)}[f]_h$*" respectively.*

Remark 42. *If we replace the conditions* $0<\lambda_{(\alpha_1,\beta_1)}[f]_h<\infty$ *and* $0<\overline{\tau}_{(\alpha_1,\beta_1)}[f]_h<\infty$ *of Remark 40 by* $0<\lambda_{(\alpha_1,\beta_1)}[f]_h\leq\rho_{(\alpha_1,\beta_1)}[f]_h<\infty$ *and* $0<\tau_{(\alpha_1,\beta_1)}[f]_h<\infty$ *respectively, then the conclusion of Theorem 8 remains valid with "*$\rho_{(\alpha_1,\beta_1)}[f]_h$*", "*$\overline{\tau}_{(\alpha_2,\beta_2)}[g]$*" and "*$\tau_{(\alpha_1,\beta_1)}[f]_h$*" in place of "*$\lambda_{(\alpha_1,\beta_1)}[f]_h$*", "*$\overline{\sigma}_{(\alpha_2,\beta_2)}[g]$*" and "*$\overline{\sigma}_{(\alpha_1,\beta_1)}[f]_h$*" respectively.*

Remark 43. *If we replace the condition* $0<\overline{\tau}_{(\alpha_1,\beta_1)}[f]_h<\infty$ *of Remark 40 by* $0<\tau_{(\alpha_1,\beta_1)}[f]_h<\infty$*, then the conclusion of Theorem 8 remains valid with "limit inferior", "*$\overline{\tau}_{(\alpha_2,\beta_2)}[g]$*" and "*$\tau_{(\alpha_1,\beta_1)}[f]_h$*" in place of "limit superior", "*$\overline{\sigma}_{(\alpha_2,\beta_2)}[g]$*" and "*$\overline{\sigma}_{(\alpha_1,\beta_1)}[f]_h$*" respectively.*

Remark 44. *If we replace the conditions* $0<\lambda_{(\alpha_1,\beta_1)}[f]_h\leq\rho_{(\alpha_1,\beta_1)}[f]_h<\infty$, $\rho_{(\alpha_1,\beta_1)}[f]_h=\rho_{(\alpha_2,\beta_2)}[g]$ *and* $0<\overline{\sigma}_{(\alpha_1,\beta_1)}[f]_h<\infty$ *of Theorem 8 by* $0<\lambda_{(\alpha_1,\beta_1)}[f]_h<\infty$, $\lambda_{(\alpha_1,\beta_1)}[f]_h=\rho_{(\alpha_2,\beta_2)}[g]$ *and* $0<\overline{\tau}_{(\alpha_1,\beta_1)}[f]_h<\infty$ *respectively and other conditions remain same, then Theorem 8 remains valid with "*$\overline{\tau}_{(\alpha_1,\beta_1)}[f]_h$*" in place of "*$\overline{\sigma}_{(\alpha_1,\beta_1)}[f]_h$*".*

Remark 45. *In Remark 44, if we replace the conditions "* $\overline{\sigma}_{(\alpha_2,\beta_2)}[g] < \infty$*" and "*$0 < \overline{\tau}_{(\alpha_1,\beta_1)}[f]_h < \infty$*" by "*$\sigma_{(\alpha_2,\beta_2)}[g] < \infty$*" and "*$0 < \tau_{(\alpha_1,\beta_1)}[f]_h < \infty$*", then the conclusion of Theorem 8 remains valid with "*$\sigma_{(\alpha_2,\beta_2)}[g]$*" and "*$\tau_{(\alpha_1,\beta_1)}[f]_h$*" in place of "*$\overline{\sigma}_{(\alpha_2,\beta_2)}[g]$*" and "*$\overline{\sigma}_{(\alpha_1,\beta_1)}[f]_h$*" respectively.*

Remark 46. *If we replace the conditions* $0 < \lambda_{(\alpha_1,\beta_1)}[f]_h < \infty$ *and* $0 < \overline{\tau}_{(\alpha_1,\beta_1)}[f]_h < \infty$ *of Remak 44 by* $0 < \lambda_{(\alpha_1,\beta_1)}[f]_h \leq \rho_{(\alpha_1,\beta_1)}[f]_h < \infty$ *and* $0 < \tau_{(\alpha_1,\beta_1)}[f]_h < \infty$ *respectively, then the conclusion of Theorem 8 remains valid with "*$\rho_{(\alpha_1,\beta_1)}[f]_h$*" and "*$\tau_{(\alpha_1,\beta_1)}[f]_h$*" in place of "*$\lambda_{(\alpha_1,\beta_1)}[f]_h$*" and "*$\overline{\sigma}_{(\alpha_1,\beta_1)}[f]_h$*" respectively.*

Remark 47. *If we replace the condition* $0 < \overline{\tau}_{(\alpha_1,\beta_1)}[f]_h < \infty$ *of Remak 44 by* $0 < \tau_{(\alpha_1,\beta_1)}[f]_h < \infty$*, then the conclusion of Theorem 8 remains valid with "limit inferior" and "*$\tau_{(\alpha_1,\beta_1)}[f]_h$*" in place of "limit superior" and "*$\overline{\sigma}_{(\alpha_1,\beta_1)}[f]_h$*" respectively.*

Remark 48. *If we replace the conditions* $\rho_{(\alpha_1,\beta_1)}[f]_h = \rho_{(\alpha_2,\beta_2)}[g]$ *and* $\overline{\sigma}_{(\alpha_2,\beta_2)}[g] < \infty$ *of Theorem 8 by* $\rho_{(\alpha_1,\beta_1)}[f]_h = \lambda_{(\alpha_2,\beta_2)}[g]$ *and* $\overline{\tau}_{(\alpha_2,\beta_2)}[g] < \infty$ *respectively and other conditions remain same, then Theorem 8 remains valid with "*$\overline{\tau}_{(\alpha_2,\beta_2)}[g]$*" in place of "*$\overline{\sigma}_{(\alpha_2,\beta_2)}[g]$*".*

Remark 49. *In Remark 48, if we replace the conditions "*$\overline{\tau}_{(\alpha_2,\beta_2)}[g] < \infty$*" and "*$0 < \overline{\sigma}_{(\alpha_1,\beta_1)}[f]_h < \infty$*" by "*$\tau_{(\alpha_2,\beta_2)}[g] < \infty$*" and "*$0 < \sigma_{(\alpha_1,\beta_1)}[f]_h < \infty$*", then Remark 48 remains valid with "*$\tau_{(\alpha_2,\beta_2)}[g]$*" and "*$\sigma_{(\alpha_1,\beta_1)}[f]_h$*" in place of "*$\overline{\tau}_{(\alpha_2,\beta_2)}[g]$*" and "*$\overline{\sigma}_{(\alpha_1,\beta_1)}[f]_h$*" respectively.*

Remark 50. *If we replace the conditions* $0 < \lambda_{(\alpha_1,\beta_1)}[f]_h \leq \rho_{(\alpha_1,\beta_1)}[f]_h < \infty$ *and* $0 < \overline{\sigma}_{(\alpha_1,\beta_1)}[f]_h < \infty$ *of Remark 48 by* $0 < \rho_{(\alpha_1,\beta_1)}[f]_h < \infty$ *and* $0 < \sigma_{(\alpha_1,\beta_1)}[f]_h < \infty$ *respectively, then Remark 48 remains valid with "*$\rho_{(\alpha_1,\beta_1)}[f]_h$*" and "*$\sigma_{(\alpha_1,\beta_1)}[f]_h$*" in place of "*$\lambda_{(\alpha_1,\beta_1)}[f]_h$*" and "*$\overline{\sigma}_{(\alpha_1,\beta_1)}[f]_h$*" respectively.*

Remark 51. *If we replace the condition* $0 < \overline{\sigma}_{(\alpha_1,\beta_1)}[f]_h < \infty$ *of Remark 48 by* $0 < \sigma_{(\alpha_1,\beta_1)}[f]_h < \infty$*, then the conclusion of Theorem 8 remains valid with "limit inferior", "*$\overline{\tau}_{(\alpha_2,\beta_2)}[g]$*" and "*$\sigma_{(\alpha_1,\beta_1)}[f]_h$*" in place of "limit superior", "*$\overline{\sigma}_{(\alpha_2,\beta_2)}[g]$*" and "*$\overline{\sigma}_{(\alpha_1,\beta_1)}[f]_h$*" respectively.*

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