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EIGENVALUE REGION FOR ERDÉLYI-KOBER FRACTIONAL BOUNDARY VALUE PROBLEMS

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Abstract. In this paper we want to obtain distribution region for eigenvalues of the Erdélyi-Kober fractional boundary value problems subject to the Dirichlet boundary conditions. We use Lyapunov inequalities technique to find the region for distrbution of the eigenvalues. In this way, the corresponding Green's functions plays a fundamental role.

1. Introduction

Chasing the origin of the Lyapunov inequalities one has to review a considerable history at mathematical inequalities to arrive at this fact that when A. M. Lyapunov in 1892 was studying the stability of motion, in light of the Floquet theory, he established the following stability criterion.

Theorem 1.1. *(c.f.* [15]*, Chap. 3, Th. 2) Consider the second order differential equation with T-periodic coefficient*

$$u^{''} + q(t)u = 0, \quad -\infty < t < \infty. \tag{1.1}$$

If the function q takes only positive or zero values(without being identically zero), and if further it satisfies the condition

$$\int_0^T q(t) \leq \frac{T}{4}, \tag{1.2}$$

then roots of the characteristic equation corresponding to (1.1) will always be complex and their modulus are equal to 1*.*

Since then, the inequality

$$\int_0^T q(t) > \frac{T}{4},$$

is known as Lyapunov inequality. This fully applicable inequality has been identified as one of the most useful inequalities in theory of the mathematical inequalities. Systematic study of the Lyapunov inequalities has begun by the P. Hartman in

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1964 (see [11]), when he presented the most inspiring Lyapunov inequality of the Dirichlet ordinary boundary value problems as below.

Theorem 1.2. *Assume that $u(t)$ is a nontrivial solution of the second order boundary value problem*

$$u'' + q(t)u = 0, \quad t \in (a,b), \tag{1.3}$$

$$u(a) = u(b) = 0, \tag{1.4}$$

where $q(t) \in C([a,b];\mathbb{R})$. Then the Lyapunov inequality

$$\int_a^b |q(t)| > \frac{4}{b-a}, \tag{1.5}$$

holds.

Considering the Lyapunov inequality (1.5) and its governing Dirichlet problem (1.3)-(1.4), it is a challenging problem that what if the solid integer order 2 in (1.3) is replaced by the non-integer order α sliding throughout the interval $(1,2]$. Immediately, the idea of the fractional-order Dirichlet boundary value problems and their Lyapunov inequalities sparks in mind. On the other hand daily growing heuristic developments in fractional calculus motivate us to make a new investigation about the Lyapunov inequalities within the fractional-order boundary value problems. Since by the use of these inequalities we are able to study qualitative dynamics of nontrivial solutions of the corresponding boundary value problems such as stability, disconjugacy, distribution of the eigenvalues in related eigenvalue problems, nonexistence results, oscillatory properties and maximum number of zeros of these solutions, so this is a reasonable expectation that how many of these qualitative behaviour can be established for fractional-order boundary value problems? In the way of to make the answer, it finds out that none of the Riemann-Liouville based fractional-order operators can give us the aforementioned qualitative dynamics, that is we have to use other approaches for the fractional-order differentiation, namely conformable fractional differential equations. In this case one may arrive at the above mentioned qualitative dynamics. For instance we suggest [6]-[9] and related bibliography therein.

The period of the fractional-order Lyapunov inequalities has been initiated by the following work, where the author [3] considered the following general fractional linear boundary value problem with Dirichlet boundary conditions:

$$D_{a^+}^{\alpha} u(t) + q(t)u(t) = 0, \quad 1 < \alpha \leq 2, \ a < t < b, \tag{1.6}$$

$$u(a) = 0, \quad u(b) = 0, \tag{1.7}$$

in which $D_{a^+}^{\alpha}$ stand for the Riemann-Liouville fractional derivative of order α. The Lyapunov inequality of this boundary value problem was obtained as:

$$\int_a^b |q(t)|dt > \Gamma(\alpha)\left(\frac{4}{b-a}\right)^{\alpha-1}. \tag{1.8}$$

Changing the Riemann-Liouville derivative $D_{a^+}^{\alpha}$ in fractional-order boundary value problem (1.6)-(1.7) with the Caputo fractional derivative ${}^{c}D_{a^+}^{\alpha}$, the author obtained the following Lyapunov inequality [4]:

$$\int_a^b |q(s)|ds > \frac{\Gamma(\alpha)\alpha^{\alpha}}{[(\alpha-1)(b-a)]^{\alpha-1}}. \tag{1.9}$$

In continuation, the authors in (see [16]), have substituted the Hadamard fractional derivative ${}^{H}D^{\alpha}_{a+}$ with D^{α}_{a+}, $q(t)$ with $-q(t)$,a and b with 1 and e, respectively, in (1.6)-(1.7) to reach the Lyapunov inequality

$$\int_1^e |q(s)|ds > \Gamma(\alpha)\lambda^{1-\alpha}(1-\lambda)^{1-\alpha}\exp(\lambda). \tag{1.10}$$

At last, the author in (see [17]), has substituted the Hilfer fractional derivative $D^{\alpha,\beta}_{a^+}$ with D^{α}_{a+} in the boundary value problem (1.6)-(1.7) to obtain the Lyapunov inequality

$$\int_a^b |q(s)|ds \geq \frac{\Gamma(\alpha)[\alpha-(2-\alpha)(1-\beta)]^{\alpha-(2-\alpha)(1-\beta)}}{(b-a)^{\alpha-1}[\alpha-1+\beta(2-\alpha)]^{\alpha-1+\beta(2-\alpha)}(\alpha-1)^{\alpha-1}}. \tag{1.11}$$

As a sample-wise collection of research works on the the Lyapunov inequalities of the fractional boundary value problems and particularly, those of the fractional Dirichlet boundary value problems we suggest [1]-[10], [12], [13], [16], [17], [20]-[24] and cited bibliography therein.
Now, by a quick overview on the fractional order derivatives one may recognize lack of the Erdélyi-Kober fractional derivatives in the study of Lyapunov inequalities for the Dirichlet fractional boundary value problems of the form (1.6)-(1.7). This gap encourages us to initiate study of the Lyapunov inequalities of the Erdélyi-Kober fractional linear boundary value problems:

$$D^{\alpha}_{a^+;\sigma,\eta}u(t) + q(t)u(t) = 0, \quad 1 < \alpha \leq 2, \ 0 < a < t < b, \tag{1.12}$$

$$u(a) = 0, \quad u(b) = 0, \tag{1.13}$$

in which $D^{\alpha}_{a^+;\sigma,\eta}$ stands for the Erdélyi-Kober fractional derivative of order $\alpha > 0$, with the parameters $\sigma \in \mathbb{R}^+$ and $\eta \in \mathbb{R}$ that will be defined in the next section. Besides, $q(t) \in C([a,b];\mathbb{R})$ as well as the q in all of the above mentioned fractional boundary value problems. To more convenient comparison of the research works about the fractional Lyapunov inequalities, we present the following table (Table 1) to demonstrate the importance of the current article.

Table 1. Different types of continuous differentiation operators that their Lyapunov inequalities has been investigated

$\mathbb{D}$	References
$\frac{d^2}{dt^2}$: second-order derivative	[1], [11], [15]
${}^{RL}D^{\alpha}_{a+}$: Riemann-Liouville fractional derivative	[3], [24]
${}^{C}D^{\alpha}_{a+}$: Caputo fractional derivative	[2], [4]
$D^{\alpha,\beta}_{a^+}$: Hilfer fractional derivative	[17], [22], [23]
$\mathcal{D}^{\alpha}_{a+}$: Hadamard fractional derivative	[16], [20]
T^{α}_{a+} : conformable fractional derivative	[6], [7],[9]
$D^{\alpha}_{a^+;\sigma,\eta}$: Erdélyi-Kober fractional derivative fractional derivative	- - - - - - - -

At the and of this section, we state the organization of this paper. In Section 2, we present the Erdélyi-Kober fractional integrals and derivatives and some of their technical properties. Section 3 is devoted to obtain the Lyapunov inequality of the Erdélyi-Kober fractional linear boundary value problem (1.12)-(1.13). To this aim, we make use of the Green function technique to reach our desired Lyapunov inequality. Next, we have Section 4, where by the use of the obtained Lyapunov inequality in Section 3, we identify distribution region of the real zeros of the Mittag-Leffler functions corresponding to the Erdélyi-Kober fractional eigenvalue problems. At the end, the Section 5 summarizes outcome of this investigation.

2. Technical Background

We begin this section with Erdélyi-Kober fractional integrals and derivatives. We point out this fact that all of the parameters considered in this paper belong to $\mathbb{R}$.

Definition 2.1. *([14],[21]). Let $(a,b),(0 \leq a < t < b \leq +\infty)$ be a finite or infinite interval of the half-axis $\mathbb{R}^+$. Also let $\alpha > 0,\ \sigma > 0$ and $\eta \in \mathbb{R}$. Then, the left-sided Erdélyi-Kober fractional integral of order α is defined as*

$$\left(I^{\alpha}_{a^+;\sigma,\eta}f\right)(t) := \frac{\sigma t^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \int_a^t \frac{s^{\sigma\eta+\sigma-1}}{(t^\sigma - s^\sigma)^{1-\alpha}} f(s)ds. \tag{2.1}$$

Definition 2.2. *([14],[21]). Let $(a,b),(0 \leq a < t < b \leq +\infty)$ be a finite or infinite interval of the half-axis $\mathbb{R}^+$. Also let $\alpha > 0,\ \sigma > 0,\ \eta \in \mathbb{R}$ and $n = [\alpha]+1$. Then, the left-sided Erdélyi-Kober fractional derivative of order α is defined as*

$$\left(D^{\alpha}_{a^+;\sigma,\eta}f\right)(t) := t^{-\sigma\eta} \left(\frac{1}{\sigma t^{\sigma-1}} D\right)^n t^{\sigma(n+\eta)} \left(I^{n-\alpha}_{a^+;\sigma,\eta+\alpha}f\right)(t), \tag{2.2}$$

where $D := \frac{d}{dt}$.

Next, we present inversion formula for the Erdélyi-Kober fractional operators as follows.

Theorem 2.3. *([21]). Let $n-1 < \alpha \leq n,\ n \in \mathbb{N},\ \alpha \geq -\sigma(\eta+1)$ and $f \in C^n_\alpha$. Then, the following relationship between the Erdélyi-Kober fractional derivative and Erdélyi-Kober fractional integral of order α holds true:*

$$\left(I^{\alpha}_{0^+;\sigma,\eta} D^{\alpha}_{0^+;\sigma,\eta}f\right)(t) := f(t) - \sum_{k=0}^{n-1} c_k t^{-\sigma(1+\eta+k)}, \tag{2.3}$$

in which

$$c_k := \frac{\Gamma(n-k)}{\Gamma(\alpha-k)} \lim_{t\to 0} t^{\sigma(1+\eta+k)} \prod_{i=k+1}^{n-1} \left(1+\eta+i+\frac{1}{\sigma} t \frac{d}{dt}\right) \left(I^{n-\alpha}_{0^+;\sigma,\eta+\alpha}f\right)(t),$$

and $C^n_\alpha := \left\{ f(t) := t^p \widetilde{f}(t) \,\middle|\, p > \alpha,\ \widetilde{f}(t) \in C^n[0,\infty) \right\}.$

Remark 2.4. *As can be seen in [21], Theorem 3.1, the identity (2.3) has proven in light of the general solution of the hyper Bessel differential equation*

$$\prod_{k=0}^{n-1} \left(1+\eta+k+\frac{1}{\sigma} t \frac{d}{dt}\right) y(x) = 0,$$

that is

$$y(x) := \sum_{k=0}^{n-1} \theta_k t^{\sigma(1+\eta+k)},\ \theta_k \in \mathbb{R}.$$

Consequently, by the use of this fact that the general solution of the hyper-Bessel differential equation

$$\prod_{k=0}^{n-1}\left(1+\eta+k+\frac{1}{\sigma}(t-a)\frac{d}{dt}\right)y(x)=0,$$

is

$$y(x):=\sum_{k=0}^{n-1}\Theta_k(t-a)^{\sigma(1+\eta+k)},\ \Theta_k\in\mathbb{R},$$

it follows that

$$\left(I^{\alpha}_{a^+;\sigma,\eta}D^{\alpha}_{a^+;\sigma,\eta}f\right)(t):=f(t)-\sum_{k=0}^{n-1}c_k(t-a)^{-\sigma(1+\eta+k)}, \tag{2.4}$$

in which

$$c_k:=\frac{\Gamma(n-k)}{\Gamma(\alpha-k)}\lim_{t\to a}(t-a)^{\sigma(1+\eta+k)}\prod_{i=k+1}^{n-1}\left(1+\eta+i+\frac{1}{\sigma}(t-a)\frac{d}{dt}\right)\left(I^{n-\alpha}_{a^+;\sigma,\eta+\alpha}f\right)(t).$$

Since, in Section 4 we will identify real zeros of the Mittag-Leffler functions corresponding to the Erdélyi-Kober fractional eigenvalue problems, so we have to introduce them formerly.

Definition 2.5. *(*[14]*). The Mittag-Leffler function $E_{\alpha,\beta}(t)$ is defined as*

$$E_{\alpha,\beta}(t):=\sum_{k=0}^{\infty}\frac{t^k}{\Gamma(\alpha k+\beta)},\quad \alpha>0,\ t,\beta\in\mathbb{R}. \tag{2.5}$$

Now, we can identify the general solution of the Erdélyi-Kober fractional eigenvalue problems as follows.

Theorem 2.6. *(* [19]*). The general solution of the Erdélyi-Kober fractional differential equation with $f(t)\in C^0_{\sigma\mu},\ \mu\geq\max\{0,-\eta-\alpha\}-1$,*

$$t^{-\sigma\alpha}\left(D^{\alpha}_{0^+;\sigma,\eta}y\right)(t)-\lambda y(t)=f(t), \tag{2.6}$$

in the space $C^n_{\beta\mu}$, with $\mu\geq\max\{0,-\eta-\alpha\}-1$ and $n-1<\alpha\leq n,\ n\in\mathbb{N}$ has the form:

$$\begin{aligned} y(t):=&\sum_{j=1}^{n}b_jt^{\sigma(\alpha-j)}E_{\alpha,\eta+2\alpha-j+1}\left(\lambda t^{\sigma\alpha}\right)\\ &+t^{-\sigma(\alpha+\eta)}\int_0^t(t^\sigma-s^\sigma)^{\alpha-1}s^{\sigma(\alpha+\eta)}E_{\alpha,\alpha}\left[\lambda(t^\sigma-s^\sigma)^\alpha\right]f(s)d(s^\sigma). \end{aligned} \tag{2.7}$$

3. Main Result

Let us recall the Erdélyi-Kober fractional boundary value problem (1.12)-(1.13) as follows.

$$D^{\alpha}_{a^+;\sigma,\eta}u(t)+q(t)u(t)=0,\quad 1<\alpha\leq 2,\ a<t<b, \tag{3.1}$$

$$u(a)=0,\quad u(b)=0. \tag{3.2}$$

In the following lemma we present the Green function approach of the boundary value problem (3.1)-(3.2). As will be seen, the Green function of the boundary value problem (3.1)-(3.2) strongly depends on the parameters $\alpha,\ \sigma,\ \eta,\ a$ and b. So, we need an accurate setting of these parameters to have an appropriate Green function and related calculations to reach its maximum. More precisely, since in definitions of the Erdélyi-Kober fractional operators the parameter η freely can take any real value, considering the parameter $\eta \in \mathbb{R}_+$ dos not yield Green function of the boundary value problem (3.1)-(3.2), since in this case all powers $-\sigma(1+\eta+k)$ in the identity (2.3) are negative. In light of this key point, we find out that the parameter η must have a controlled negative value. As an appropriate controlled pattern, we suggest $-n < \eta < -n+1$ when we have $n-1 < \alpha \leq n,\ n \in \mathbb{Z}^+$. Now, since in the Erdélyi-Kober fractional boundary value problem (3.1)-(3.2) we have $1 < \alpha \leq 2$, so we must consider the setting $-2 < \eta < -1$. In the next lemma besides obtaining the Green function of the Erdélyi-Kober fractional boundary value problem (3.1)-(3.2), we will examine usefulness of the our suggested pattern for the parameter η to solve Erdélyi-Kober fractional boundary value problems by the use Green function technique.

Lemma 3.1. *Assume $-2 < \eta < -1$ and $\sigma = 1$. Then the integral equation*

$$u(t) := \int_a^b \mathcal{G}(t,s)q(s)u(s)ds, \tag{3.3}$$

with the Green function

$$\mathcal{G}(t,s) := \frac{1}{\Gamma(\alpha)} \begin{cases} \left(g(t,s) - (t-s)^{\alpha-1}.t^{-(\alpha+\eta)}\right) s^{\eta} \ ; & a < s \leq t < b, \\ \\ g(t,s)s^{\eta} \ ; & a < t \leq s < b, \end{cases} \tag{3.4}$$

for which

$$g(t,s) = \frac{(t-a)^{-(\eta+1)}(b-s)^{\alpha-1}}{(b-a)^{-(\eta+1)}}.b^{-(\alpha+\eta)}$$

uniquely solves the Erdélyi-Kober fractional boundary value problem (3.1)-(3.2).

Proof. Taking $I^{\alpha}_{a^+;1,\eta}$ on both sides of the governing equation (3.1) and then applying the inversion formula (2.4) in Remark 2.4, it follows that

$$u(t) := c_0(t-a)^{-(\eta+1)} + c_1(t-a)^{-(\eta+2)} - \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}.t^{-(\alpha+\eta)}.s^{\eta}q(s)u(s)ds. \tag{3.5}$$

Now we first impose the boundary condition $u(a) = 0$ into the equality (3.5), and come to the conclusion that $c_1 := 0$. Next, if we apply the boundary condition $u(b) = 0$, we get that

$$c_0 := \frac{1}{\Gamma(\alpha)}\int_a^b \frac{(b-s)^{\alpha-1}}{(b-a)^{-(\eta+1)}}.b^{-(\alpha+\eta)}.s^{\eta}q(s)u(s)ds. \tag{3.6}$$

Substituting uniquely determined coefficients c_0 and c_1 in the equality (3.5), directly leads us to the integral equation (3.3)-(3.4). This completes the proof. □

It will be of great interest to illustrate the Green function $\mathcal{G}(t,s)$ given by (3.4). So, we have to choose an appropriate setting. Based on the statement of Lemma 3.1 and $a := 1,\ b := 2,\ \eta := -1.5$ and $s := 1.5$, we have the simulation of $\mathcal{G}(t,s)$ in Figure 1 as follows.

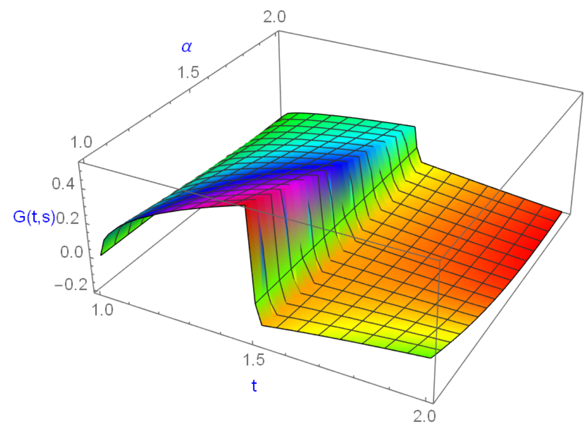

FIGURE 1. Green function $\mathcal{G}(t,s)$ given by (3.4).

Lemma 3.2. *The Green function $\mathcal{G}(t,s)$, defined by (3.4) satisfies the following properties:*

(G_1) *$\mathcal{G}(t,s)$ is continuous for $(t,s)\in(a,b)\times(a,b)$.*

(G_2)

$$\max \mathcal{G}(t,s)_{(t,s)\in(a,b)\times(a,b)} := \frac{1}{\Gamma(\alpha)}\frac{(-(\eta+1))^{-(\eta+1)}(\alpha-1)^{\alpha-1}}{(\alpha-\eta-2)^{\alpha-\eta-2}}.\frac{a^{\eta}}{b^{\alpha+\eta}}(b-a)^{\alpha-1}. \tag{3.7}$$

Proof. Since the property (G_1) is clear, so we prove here the assertion (G_2) as follows. Considering the Green function $\mathcal{G}(t,s)$, one may rewrite it as follows:

$$\mathcal{G}(t,s) := \mathcal{H}(t,s)s^{\eta}, \quad a<s<b,$$

such that

$$\mathcal{H}(t,s) := \frac{1}{\Gamma(\alpha)}\begin{cases} g(t,s)-h(t,s); & a<s\le t<b,\\ g(t,s) \quad ; & a<t\le s<b,\end{cases} \tag{3.8}$$

with

$$g(t,s) := \frac{(t-a)^{-(\eta+1)}}{(b-a)^{-(\eta+1)}}.(b-s)^{\alpha-1}.b^{-(\alpha+\eta)}, \quad h(t,s) := (t-s)^{\alpha-1}.t^{-(\alpha+\eta)}.$$

So, to find maximum of the Green function $\mathcal{G}(t,s)$, we need only to find the maximum of the function $\mathcal{H}(t,s)$. Since both of the functions $g(t,s)$ and $h(t,s)$ are positive, we immediately conclude that

$$\max_{(t,s)\in(a,b)\times(a,b)} \mathcal{H}(t,s) := \frac{g(t,s)}{\Gamma(\alpha)}, \quad a<t\le s<b.$$

It is clear that $\dfrac{\partial g}{\partial t}>0$. So, we get that

$$\max_{t\le s}\mathcal{H}(t,s) = \frac{g(s,s)}{\Gamma(\alpha)} := \frac{(s-a)^{-(\eta+1)}}{\Gamma(\alpha)(b-a)^{-(\eta+1)}}.(b-s)^{\alpha-1}.b^{-(\alpha+\eta)}, \quad a<s<b.$$

On the other hand, if we define

$$f(s) := (s-a)^{-(\eta+1)}.(b-s)^{\alpha-1},$$

then, we have

$$\frac{df}{ds} := (s-a)^{-\eta-2}.(b-s)^{\alpha-2}\left\{-(\eta+1)(b-s)-(\alpha-1)(s-a)\right\}.$$

Accordingly, $\frac{df}{ds} = 0$ if and only if

$$s_M := \frac{a(\alpha-1)-(\eta+1)b}{\alpha-\eta-2}.$$

Since

$$\begin{cases} \frac{df}{ds} > 0; & s < s_M, \\ \frac{df}{ds} < 0; & s > s_M, \end{cases}$$

we deduce that $f(s)$ admits its maximum at s_M, that is

$$\max_{(t,s)\in(a,b)\times(a,b)} \mathcal{H}(t,s) := \frac{g\left(\frac{a(\alpha-1)-(\eta+1)b}{\alpha-\eta-2}, \frac{a(\alpha-1)-(\eta+1)b}{\alpha-\eta-2}\right)}{\Gamma(\alpha)},$$

and therefore,

$$\max_{(t,s)\in(a,b)\times(a,b)} \mathcal{G}(t,s) := \frac{g\left(\frac{a(\alpha-1)-(\eta+1)b}{\alpha-\eta-2}, \frac{a(\alpha-1)-(\eta+1)b}{\alpha-\eta-2}\right)}{\Gamma(\alpha)} a^{\eta}.$$

This completes the proof. ☐

Remark 3.3. *Considering the Lemma 3.2, we can now describe why we set* $\sigma := 1$ *in the Lemma 3.1. Indeed, taking* $\sigma := 1$ *is the simplest setting to reach the maximum of the Green function* $\mathcal{G}(t,s)$ *defined by (3.4). It is clear that if we set* $\sigma \in \mathbb{R}^+ - \{1\}$*, then finding the maximum of the Green function* $\mathcal{G}(t,s)$ *will be much more complicated than the case* $\sigma := 1$.

Now, we are able to obtain Lyapunov inequality of the Erdélyi-Kober fractional boundary value problem (3.1)-(3.2). To this aim, we present the following theorem.

Theorem 3.4. *Assume that* $u(t)$ *be a nontrivial solution of the Erdélyi-Kober fractional boundary value problem (3.1)-(3.2), then the following Lyapunov inequality is satisfied.*

$$\int_a^b |q(s)|ds \geq \Gamma(\alpha)\frac{(\alpha-\eta-2)^{\alpha-\eta-2}}{(-(\eta+1))^{-(\eta+1)}(\alpha-1)^{\alpha-1}}.\frac{b^{\alpha+\eta}}{a^{\eta}}.\frac{1}{(b-a)^{\alpha-1}}. \tag{3.9}$$

Proof. According to Lemma 3.1, we know that the Erdélyi-Kober fractional boundary value problem (3.1)-(3.2) is equivalent to the integral equation (3.3)-(3.4). So, it is enough to concentrate on the integral equation

$$u(t) := \int_a^b \mathcal{G}(t,s)q(s)u(s)ds.$$

Next considering the Banach space $C[a,b]$ with the standard max norm $\|u\| := \max_{t\in[a,b]}|u(t)|$ and based on the property (G_2), Lemma 3.2 it follows that

$$\begin{aligned}\|u\| = \max_{t\in[a,b]}|u(t)| &:= \max_{t\in[a,b]}\left|\int_a^b \mathcal{G}(t,s)q(s)u(s)ds\right| \\ &\leq \int_a^b \max_{t\in[a,b]}|\mathcal{G}(t,s)||q(s)||u(s)|ds \\ &\leq \frac{1}{\Gamma(\alpha)}\frac{(-(\eta+1))^{-(\eta+1)}(\alpha-1)^{\alpha-1}}{(\alpha-\eta-2)^{\alpha-\eta-2}}.\frac{a^\eta}{b^{\alpha+\eta}}(b-a)^{\alpha-1}\int_a^b|q(s)|ds\|u\|.\end{aligned}$$

Now, by the use of this this fact that $u(t)$ is a nontrivial solution of the boundary value problem (3.1)-(3.2), the Lyapunov inequality (3.9) appears itself. This completes the proof. □

Remark 3.5. *Let us consider the case we are going to obtain a strict Lyapunov inequality from (3.9). To this aim we set the parameters α and η for which $\alpha+\eta>0$. In this case since $0<a<b$, so it follows that*

$$\frac{b^{\alpha+\eta}}{a^\eta} > 1,$$

and consequently, we arrive at the following strict Lyapunov inequality

$$\int_a^b|q(s)|ds > \Gamma(\alpha)\frac{(\alpha-\eta-2)^{\alpha-\eta-2}}{(-(\eta+1))^{-(\eta+1)}(\alpha-1)^{\alpha-1}}.\frac{1}{(b-a)^{\alpha-1}}. \tag{3.10}$$

Remark 3.6. *An another interesting refinement is when we consider the case $\alpha := 2$. We think whether or not we have to check this point out that can the Lyapunov inequality (1.5) be recoverd by this setting or not ? Handling this situation we have two cases.*

Case 1. *In the first case we consider the Lyapunov inequality (3.9). In this case choosing the setting*

$$\sigma := 1, \quad \eta \longrightarrow -2,$$

it follows that

$$\int_a^b|q(s)|ds \geq \frac{4}{b-a}a^2. \tag{3.11}$$

Now, if we take the additional assumtion $a \longrightarrow 1^+$, so, the inequality (3.11) is coincided on the Lypunov inequality (1.5).

Case 2. *In the second case, let us consider the strict Lyapunov inequality (3.10). In this case, Just we need to assume $\eta \longrightarrow -2$ to reach the Lyapunov inequality (1.5).*

4. Application

In this section we present an interesting application of the Lyapunov inequality (3.9). More precisely, we consider a special Erdélyi-Kober fractional eigenvalue problem of the form

$$D^\alpha_{a^+;1,\eta}u(t) - \lambda(t-a)^\alpha u(t) = 0, \quad 1<\alpha\leq 2,\ a<t<b, \tag{4.1}$$

$$u(a) = 0, \quad u(b) = 0. \tag{4.2}$$

Equivalently, we have the following eigenvalue problem

$$(t-a)^{-\alpha}D^{\alpha}_{a^+;1,\eta}u(t)-\lambda u(t)=0,\quad 1<\alpha\leq 2,\ a<t<b, \tag{4.3}$$

$$u(a)=0,\quad u(b)=0. \tag{4.4}$$

According to Theorem 2.6 and based on Remark. 5.10, ([14], Chap. 5, Sec. 5.3, Page 311), the general solution of the eigenvalue problem (4.3) is as below:

$$u(t):=c_1(t-a)^{\alpha-1}E_{\alpha,\eta+2\alpha}\left(\lambda(t-a)^{\alpha}\right)+c_2(t-a)^{\alpha-2}E_{\alpha,\eta+2\alpha-1}\left(\lambda(t-a)^{\alpha}\right). \tag{4.5}$$

Imposing the boundary condition $u(a)=0$ in (4.5), implies that

$$u(t):=c_1(t-a)^{\alpha-1}E_{\alpha,\eta+2\alpha}\left(\lambda(t-a)^{\alpha}\right).$$

Implementing the boundary condition $u(b)=0$, yields this fact that

$$E_{\alpha,\eta+2\alpha}\left(\lambda(b-a)^{\alpha}\right)=0. \tag{4.6}$$

Now, it is time to apply the Lyapunov inequality (3.9) to identify spreading region of the real zeros of the transcendental equation (4.6). To this aim, we return to the Erdélyi-Kober fractional eigenvalue problem (4.1)-(4.2). In this case, replacing $-\lambda(t-a)^{\alpha}$ with $q(t)$ in the Lyapunov inequality (3.9) it follows that

$$|\lambda|\int_a^b(s-a)^{\alpha}ds\geq\Gamma(\alpha)\frac{(\alpha-\eta-2)^{\alpha-\eta-2}}{(-(\eta+1))^{-(\eta+1)}(\alpha-1)^{\alpha-1}}.\frac{b^{\alpha+\eta}}{a^{\eta}}.\frac{1}{(b-a)^{\alpha-1}}. \tag{4.7}$$

Equivalently, one has

$$|\lambda|\geq\Gamma(\alpha)\frac{(\alpha-\eta-2)^{\alpha-\eta-2}}{(-(\eta+1))^{-(\eta+1)}(\alpha-1)^{\alpha-1}}.\frac{b^{\alpha+\eta}}{a^{\eta}}.\frac{\alpha+1}{(b-a)^{2\alpha}}. \tag{4.8}$$

The straightforward outcome of the inequality (4.8) is that, the parameter λ is a real eigenvalue of the Erdélyi-Kober fractional eigenvalue problem (4.1)-(4.2), provided that

$$\lambda\geq\Gamma(\alpha)\frac{(\alpha-\eta-2)^{\alpha-\eta-2}}{(-(\eta+1))^{-(\eta+1)}(\alpha-1)^{\alpha-1}}.\frac{b^{\alpha+\eta}}{a^{\eta}}.\frac{\alpha+1}{(b-a)^{2\alpha}},\ \text{or}$$

$$\lambda\leq-\Gamma(\alpha)\frac{(\alpha-\eta-2)^{\alpha-\eta-2}}{(-(\eta+1))^{-(\eta+1)}(\alpha-1)^{\alpha-1}}.\frac{b^{\alpha+\eta}}{a^{\eta}}.\frac{\alpha+1}{(b-a)^{2\alpha}}.$$

In order to illustrate distribution of the eigenvalues of the fractional eigenvalue problem (4.1)-(4.2) as estimated in (4.8), choosing $a:=1$ and $b:=2$, and further considering setting of α and η in the statement of Lemma 3.1 we arrive at the figure 2 below.

Inspiring by the Lyapunov inequality (4.8), we present here a nonexistence criterion for real zeros of the generalized Mittag-Leffler function (4.6) as follows.

Theorem 4.1. *Assume* $1<\alpha\leq 2$ *and* $-2<\eta<-1$. *Then, the generalized Mittag-Leffler function* $E_{\alpha,\eta+2\alpha}\left(\lambda(b-a)^{\alpha}\right)$ *has no real zeros for*

$$|\lambda|<\Gamma(\alpha)\frac{(\alpha-\eta-2)^{\alpha-\eta-2}}{(-(\eta+1))^{-(\eta+1)}(\alpha-1)^{\alpha-1}}.\frac{b^{\alpha+\eta}}{a^{\eta}}.\frac{\alpha+1}{(b-a)^{2\alpha}}.$$

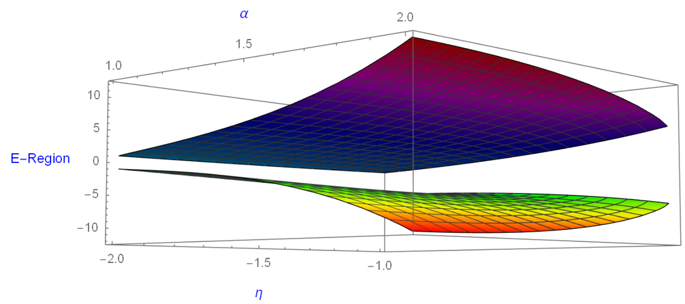

Figure 2. Distribution of eigenvalues of the fractional eigenvalue problem (4.1)-(4.2). Note that there is no any eigenvalue between two surfaces.

5. Concluding Remarks

In this article we studied a special class of the Erdélyi-Kober fractional boundary value problems subject to the Dirichlet boundary conditions of the form (1.12)-(1.13). The backbone of our investigation is to find Lyapunov inequalities for Dirichlet boundary value problems (1.12)-(1.13). To the best of our knowledge this the first time that Lyapunov inequalities of the Erdélyi-Kober fractional boundary value problems have been investigated. Relying on particular setting of types $-2 < \eta < -1$ and $\sigma = 1$ for the Erdélyi-Kober fractional derivatives, the Green function of such fractional-order boundary value problems has obtained and consequently we arrived at the Lyapunov inequality (3.9). As a result, the region for eigenvalues of the Erdélyi-Kober fractional eigenvalue problem (4.1)-(4.2) is identified.

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