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FRACTIONAL DIFFUSION EQUATION WITH REACTION TERM DESCRIBED BY THE CAPUTO-LIOUVILLE GENERALIZED FRACTIONAL DERIVATIVE

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Abstract. In this present paper, we investigate a new model for fluids in the context of fractional calculus. We study the fractional diffusion reaction equations. In our model, the reaction term describes a fractional heat equation. We present the qualitative properties of the introduced models. We propose the solution of the fractional diffusion reaction equations represented by the Caputo-Liouville generalized fractional derivative. We combine in this work the Laplace transformation and the Fourier transformation for getting the solutions of the introduced models. Our contributions are to analyze the impact of the fractional-order derivative and the reaction term on the diffusion processes. We also interpret the effect of the Prandtl number Pr on the diffusion processes.

1. Introduction

The integer order derivative doesn't take into account many types of diffusion processes. The importance of fractional derivatives in the diffusion equations is to introduce into the literature new class of diffusion processes [23, 24, 25, 29] as the super diffusive process, the subdiffusion process, ballistic diffusion, Richardson diffusion, and hyper-diffusive process. Note that, we obtain the subdiffusion process when the order of the fractional derivative is into $(0, 1)$, see in [25, 29]. And we get the super diffusion process when the order of the fractional derivative is outside $(0, 1)$. There exist many other examples like hyper-diffusion, ballistic diffusion and others. The fractional diffusion equations have attracted many researchers in the literature due to the various types of fractional derivatives and the diffusion processes generated by these fractional derivatives.

There exist many types of fractional diffusion equations in the literature; see, in [18, 23, 27]. It is observed in many investigations, the authors have proposed analytical and numerical approaches. In [27], Sene introduces the numerical method for solving the fractional diffusion equation with Caputo-Liouville generalized derivative. Fazio et al. in [4] propose the numerical solution of the time-fractional

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advection-diffusion equations with a Source Term. In [5], Ferras et al. present the numerical method for the solution of the time-fractional diffusion equation described by the Caputo-Liouville derivative. In [6], Garg et al. address the numerical solution of the fractional diffusion wave in two-dimensional spaces using the matrix method. Tasbozan et al. in [30] propose a numerical approach to the fractional diffusion equation for the force-free case. In [29], Sene et al. investigate the Mean Square displacement of the fractional diffusion equation represented by a nonsingular derivative. See also in [4, 7, 12, 15, 18, 19].

In this paper, we introduce new models in fractional calculus via the Caputo-Liouville generalized fractional derivative. We investigate on the fractional diffusion-reaction equations described by fractional-order derivatives. The reaction term satisfies the fractional heat equation. The proposed model takes into account many other models. We will come back to the models in the next sections. The applications of the numerical methods are not trivial for some categories of fractional differential equations. The present issue is to propose a new methodology to find the analytical solutions of the fractional diffusion equations with reaction terms. We use the Fourier sine transformation in both equations to solve the equations of the considered models. The fractional heat equation as a reaction term will have a significant impact on the fractional diffusion-reaction equation. We also focus on the effect of the fractional-order derivative on the diffusion equation and the diffusion-reaction equation. We discuss the effects of the Prandtl number on the diffusion processes. Another advantage of our used method is it's more straightforward for us to give physical interpretations of the obtained solutions. We also propose the mean square displacement of the constructive equations considered in this paper. The Mean Square Displacement plays an important role in physics; it permits in fractional calculus to characterize the type of diffusion processes generated by the values of the order of the fractional derivatives. Note that the types of diffusion processes and the form of the analytical solutions motive our present work.

We organize the paper as follows. In Section 2, we recall the fractional operators. In Section 3, we present the constructive equations related to the fractional diffusion equation with a reaction term and the frictional heat equation. In Section 4, we study the qualitative properties of the fractional diffusion equations. In Section 5, we describe the solution procedures. In Section 6, we give the graphics, we represent, and analyze the obtained solutions for different values of the order of the fractional derivatives. In Section 7, we provide our conclusions and remarks.

2. Fractional calculus operators

This section addresses the definitions of fractional operators used to establish this present work. The Laplace transform and the Mittag-Leffler function will be proposed in this section.

We begin by the Caputo-Liouville fractional derivative. We have the following definition [1, 14, 20].

Suppose the function $h:(0,+\infty)\longrightarrow\mathbb{R}$, the Caputo-Liouville derivative of the function h of order α is described in the following form

$$D_{\alpha}^{c}h(t)=\frac{1}{\Gamma(1-\alpha)}\int_{0^+}^{t}(t-s)^{-\alpha}h'(s)ds, \tag{1}$$

for all $t > 0$, where the order $\alpha \in (0,1)$, and $\Gamma(...)$ is the gamma function.

Another fractional operator introduced in fractional calculus is the Riemann-Liouville fractional derivative. We have the following definition [14, 20].

Let's the function $h : (0, +\infty) \longrightarrow \mathbb{R}$, the Riemann-Liouville derivative of the function h of order α is described by the following form

$$D_{\alpha}^{RL} h(t) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{d}{dt}\right) \int_{0^+}^{t} \frac{h(s)}{(t-s)^{\alpha}} ds, \tag{2}$$

for all $t > 0$, where the order $\alpha \in (0,1)$ and $\Gamma(...)$ is the gamma function.

We introduce the generalized forms of the above fractional derivatives introduced by Thabet et al. in [8]. For the Caputo-Liouville generalized fractional derivative, we have the following definition.

We suppose the function $h : (0, +\infty) \longrightarrow \mathbb{R}$, the Caputo-Liouville generalized derivative of the function h of order α and $\rho > 0$ is described in the following expression

$$D_c^{\alpha,\rho} h(t) = \frac{1}{\Gamma(1-\alpha)} \int_{0^+}^{t} \left(\frac{t^{\rho} - s^{\rho}}{\rho}\right)^{-\alpha} h'(s) ds, \tag{3}$$

for all $t > 0$, where the order $\alpha \in (0,1)$, and $\Gamma(...)$ represents the gamma function. We can observe that when $\rho = 1$, we recover the Caputo-Liouville derivative as in Eq. (1). Another remark is when ρ approaches 0, we get the so-called Caputo-Hadamard fractional derivative, see the definition of this type of derivative in [13].

We continue Riemann-Liouville derivative. For the Riemann-Liouville generalized fractional derivative, we have the following definition.

We suppose the function $h : (0, +\infty) \longrightarrow \mathbb{R}$, the Riemann-Liouville generalized derivative of the function h of order α and $\rho > 0$ is represented by the following expression

$$D^{\alpha,\rho} h(t) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{d}{dt}\right) \int_{0^+}^{t} \left(\frac{t^{\rho} - s^{\rho}}{\rho}\right)^{-\alpha} h(s) \frac{ds}{s^{1-\rho}}, \tag{4}$$

for all $t > 0$, where the order $\alpha \in (0,1)$ and $\Gamma(...)$ denotes the gamma function. We have the same remarks. We can observe when $\rho = 1$, we recover the Riemann-Liouville derivative as in Eq. (2). Another observation is when ρ approaches 0, we get the so-called Hadamard fractional derivative; see the definition of Hadamard derivative in [13]. The generalization of the fractional derivatives is an old problem. Erdelyi and Kobar addressed the first generalizations of the Caputo-Liouville derivative and the Riemann Liouville derivative in 1940 [3, 14, 17]. In general, many fractional derivatives do not satisfy the Leibniz rule. One of the motivations of the introduction of many fractional derivatives is to solve this issue. Note that Eq. (4) give a generalized form of the Riemann-Liouville derivative and the Hadamard derivative.

The Laplace transform of the Caputo-Liouville generalized fractional derivative will play an important role. We recall its definition in the next lines. We represent the ρ-Laplace transform of the Caputo-Liouville generalized derivative as the following form

$$\mathcal{L}_{\rho} \left\{ D_c^{\alpha,\rho} h(t) \right\} = s^{\alpha} \mathcal{L}_{\rho} \left\{ h(t) \right\} - s^{\alpha-1} h(0). \tag{5}$$

We represent the ρ-Laplace transform of a function h as the following relationship

$$\mathcal{L}_{\rho} \left\{ h(t) \right\} (s) = \int_{0}^{\infty} e^{-s\frac{t^{\rho}}{\rho}} h(t) \frac{dt}{t^{1-\rho}}. \tag{6}$$

We finish this section by recalling the definition of the Mittag-Leffler function. The Mittag-Leffler function [2, 16] is essential in fractional calculus, notably in the representation of the solutions of the fractional differential equations. The following equation represents the Mittag-Leffler function with two parameters

$$E_{\alpha,\beta}\left(z\right)=\sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(\alpha k+\beta)}, \tag{7}$$

where $\alpha>0$, $\beta\in\mathbb{R}$ and $z\in\mathbb{C}$.

3. Fractional diffusion equation with reaction term

The diffusion-reaction equation combines the diffusion term plus the reaction term and is represented in general in the form

$$\frac{\partial v}{\partial \tau}=\nu\frac{\partial^{2}v}{\partial x^{2}}+q(x,\tau), \tag{8}$$

where $v=v(x,\tau)$ is the state variable representing the density of the material at $x\in\mathbb{R}^{n}$ and $\tau\in\mathbb{R}^{+}$, or the fluid motion, ν is the diffusion coefficient, and q is a smooth function which describes the type of reaction at $x\in\mathbb{R}^{n}$ and $\tau\in\mathbb{R}^{+}$. In this paper, we address the fractional version of the diffusion-reaction represented by Eq. (8). To this end, we replace the partial derivative with respect to the variable τ by the Caputo-Liouville generalized fractional operator of order α, ρ with respect to the variable τ, too. Then, we represent Eq. (8) as the following form

$$D_{\tau}^{\alpha,\rho}v=\nu\frac{\partial^{2}v}{\partial x^{2}}+q(x,\tau). \tag{9}$$

The motivation of this replacement is to generalize the classical model and to capture more diffusion processes as sub-diffusion, super-diffusion, ballistic diffusion, and hyper-diffusion. An interesting aspect of our modeling is the form of the reaction term. In this paper, we consider the reaction described by $q=\beta u$ where β represents a perturbation parameter and u represents an exogenous input satisfying the solution of the fractional heat equation represented by

$$PrD_{c}^{\alpha,\rho}u=\frac{\partial^{2}u}{\partial x^{2}}, \tag{10}$$

where Pr is the Prandtl number. The following equation represents the fractional model after the introduction of the dimensionless variables considered in our work

$$D_{c}^{\alpha,\rho}v=\nu\frac{\partial^{2}v}{\partial x^{2}}+\beta u(x,\tau), \tag{11}$$

$$PrD_{c}^{\alpha,\rho}u=\frac{\partial^{2}u}{\partial x^{2}}, \tag{12}$$

with the initial conditions given by

$$v(x,0)=u(x,0)=0, \tag{13}$$

and furthermore, the fractional heat equation satisfies the additional boundary condition given by

$$u(0,\tau)=1, \tag{14}$$

and furthermore, we assume that $u(\infty,\tau)=v(\infty,\tau)=0$. In this present paper, we consider both Eq. (11) and Eq. (12). In other words, we consider a fluid model. We propose the solution of the fractional diffusion equation (12) using Fourier transformation. After, we use the solution of Eq. (12) into Eq. (11), and we determine the solution of the fractional diffusion equation with reaction term Eq. (11) using the Fourier transformation again. Before the determination of the solutions, we propose qualitative studies for Eq. (11) and Eq. (12). In other words, the existence and uniqueness have been investigated in the next section to justify the problem consisting of getting the solutions. We can apply our problem in many domains. For example, we recover a non-Newtonian viscoelastic fluid model with $\beta = -\sigma H_0^2/\rho$, where σ denotes the electrical conductivity of fluid, H_0^2 represents the uniform magnetic field, and ρ represents the density of the material. Note the MHD flow near a wall suddenly set in motion. We can consider our problem as a Casson fluid model too and many others.

4. Qualitative properties of the fractional diffusion reaction equation

This section treats the existence and uniqueness of the proposed model described by Eq. (11) and Eq. (12). The procedure of demonstration is not new but necessary in our problem. Because we investigate the analytical solution of the fractional diffusion equation with reaction. Note that it is not needed to study the analytical solutions when we are not sure the solution of the given model exists.

We begin the study with the fractional heat equation defined by Eq. (12). Let's the function

$$\eta\left(u,x\right) = \frac{\partial^2 u}{\partial x^2}. \tag{15}$$

In our study, we omitted the parameter Pr, which hasn't any impact on the problem of existence and uniqueness. Or we can reason with $Pr = 1$. We use the assumptions, that the functions u and v are continuous, and for the rest of the paper we assume that there exists a constant m such that the following relationship is held

$$\left\|\eta\left(u,x\right) - \eta\left(v,x\right)\right\| \quad \leq \quad m\left\|u - v\right\|. \tag{16}$$

From which we can deduce the function, η is Lipschitz continuous. The constant m represents the Lipschitz constant. We apply the generalized fractional integral into Eq. (15). It gives us the solution of the fractional diffusion equation defined by Eq. (12), we have that

$$u(x,\tau) - u(x,0) = I^{\alpha,\rho}\eta\left(u,x\right). \tag{17}$$

We define from Eq. (17), the Picard's operator is given by the following form

$$Mu(x,\tau) = I^{\alpha,\rho}\psi\left(x,u\right). \tag{18}$$

For precision, note that we have replaced $u(x,0) = 0$ into Eq. (17). From which we obtain the relation in Eq. (18). Our objective is now to prove the function M is well defined. We apply the Euclidean norm into Eq. (18), we obtain the relationship described as follows

$$\begin{aligned} \left\|Mu(x,\tau) - u(x,0)\right\| &= \left\|I^{\alpha,\rho}\eta\left(u,x\right)\right\|, \\ &\leq I^{\alpha,\rho}\left\|\eta\left(u,x\right)\right\|, \\ &\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\left\|\eta\left(u,x(\tau)\right)\right\| \int_0^\tau \left(\frac{\tau^\rho - s^\rho}{\rho}\right)^{\alpha-1} \frac{ds}{s^{1-\rho}}. \end{aligned} \tag{19}$$

Using the Eq. (19), the function η is bounded. The condition $t \leq T$, Eq. (19), we obtain the following relation

$$\|Mu(x,\tau) - u(x,0)\| \leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\left(\frac{T^{\rho}}{\rho}\right)^{\alpha} c. \tag{20}$$

That is, the function M is well defined. We now provide a condition under which the function M is a contraction. We have the following relations

$$\begin{aligned} \|Mu(x,\tau) - Mv(x,\tau)\| &= \left\|I^{\alpha,\rho}\left(\eta\left(u,x\right) - \eta\left(x,v\right)\right)\right\|, \\ &\leq I^{\alpha,\rho}\left\|\left(\eta\left(u,x\right) - \eta\left(v,x\right)\right)\right\|, \\ &\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\left\|\eta\left(u,x\right) - \eta\left(v,x\right)\right\| I^{\alpha,\rho}(1). \end{aligned} \tag{21}$$

From Eq. (21), we have the relationship described in the following form

$$\|Mu(x,\tau) - Mv(x,\tau)\| \leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha+1)}\left(\frac{T^{\rho}}{\rho}\right)^{\alpha} m\left\|u - v\right\|. \tag{22}$$

We conclude that the function M is a contraction when the following condition is held

$$\frac{\rho^{1-\alpha}}{\Gamma(\alpha+1)}\left(\frac{T^{\rho}}{\rho}\right)^{\alpha} k < 1. \tag{23}$$

From the Banach fixed Theorem, the $Mu = u$ admits a unique solution. Thus, the solution of the fractional diffusion equation is represented in Eq. (12) exists. Therefore, we can investigate the solution of the fractional heat diffusion equation defined by Eq. (12).

Let's prove now the existence and the uniqueness of the Eq. (11). We consider the function

$$\sigma\left(u,x\right) = \frac{\partial^2 u}{\partial x^2} + \beta u(x,\tau). \tag{24}$$

The objective is to prove the function σ is Lipschitz continuous. We apply the Euclidean norm into Eq. (24). Using the assumptions, that the functions u and v are continuous, and for the rest of the paper we assume that there exists a constant κ such that the following relationships are held

$$\left\|\sigma\left(u,x\right) - \sigma\left(v,x\right)\right\| \leq \kappa\left\|u - v\right\|. \tag{25}$$

We deduce from Eq. (25) the function σ is Lipschitz continuous. We have as Lipschitz constant κ. We apply the generalized fractional integral into Eq. (24). We get the following solution of the fractional diffusion equation (11), we have that

$$u(x,\tau) - u(x,0) = I^{\alpha,\rho}\sigma\left(u,x\right). \tag{26}$$

We define from Eq. (26), the Picard's operator is given by the following form

$$Nu(x,\tau) = I^{\alpha,\rho}\sigma\left(x,u\right). \tag{27}$$

For precision, note that $u(x,0) = 0$, into Eq. (26). We obtain the relation in Eq. (27). Our objective is now to prove the function M is well defined. We apply the Euclidean norm into Eq. (27), we obtain the relation described as follows

$$\begin{aligned} \|Nu(x,\tau) - u(x,0)\| &= \left\|I^{\alpha,\rho}\sigma\left(u,x\right)\right\|, \\ &\leq I^{\alpha,\rho}\left\|\sigma\left(u,x\right)\right\|, \\ &\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\left\|\sigma\left(u,x\right)\right\| I^{\alpha,\rho}(1). \end{aligned} \tag{28}$$

Using Eq. (25), in which the function σ is bounded and with the condition $t \leq T$, we obtain the following

$$\|Nu(x,\tau) - u(x,0)\| \leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \left(\frac{T^\rho}{\rho}\right)^\alpha d. \tag{29}$$

That is, the function N is well defined. We now provide a condition under which the function M is a contraction. We have the relations

$$\begin{aligned} \|Nu(x,\tau) - Nv(x,\tau)\| &= \left\|I^{\alpha,\rho}\left(\sigma\left(u,x\right) - \sigma\left(x,v\right)\right)\right\|, \\ &\leq I^{\alpha,\rho}\left\|\left(\sigma\left(u,x\right) - \sigma\left(v,x\right)\right)\right\|, \\ &\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\left\|\sigma\left(u,x\right) - \sigma\left(v,x\right)\right\| I^{\alpha,\rho}(1). \end{aligned}$$

From Eq. (25), we have the relationships described in the following form

$$\|Nu(x,\tau) - Nv(x,\tau)\| \leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha+1)} \left(\frac{T^\rho}{\rho}\right)^\alpha \kappa \left\|u - v\right\|. \tag{30}$$

We conclude that the function N is a contraction when the following condition is held

$$\frac{\rho^{1-\alpha}}{\Gamma(\alpha+1)} \left(\frac{T^\rho}{\rho}\right)^\alpha < \frac{1}{\kappa}. \tag{31}$$

From the Banach fixed Theorem, the $Nu = u$ admit unique solution, which is the solution of the fractional equation represented in Eq. (11). Thus, we can investigate the solution of Eq. (11).

5. Solution procedures and means square displacement

In this section, we propose the analytical solutions and the mean square displacement of the proposed model Eqs. (11)-(12). We use the Fourier and Laplace transformations. The method is useful and practical to get the solutions of the fluid models. We begin our resolution by solving the fractional heat equation (12). The solution of Eq. (12) is already done in previous works, see in [26, 27]. We first apply the Fourier transform, we have

$$D_c^{\alpha,\rho} u(q,s) = \frac{2q}{Pr\pi s} u(0,\tau) - \frac{q^2}{Pr} u(q,s). \tag{32}$$

We recall in the next line, the Laplace transform of Eq. (32), we obtain the following relationship

$$\begin{aligned} s^\alpha \bar{u}(q,s) - s^{\alpha-1}\bar{u}(q,0) + \frac{q^2}{Pr}\bar{u}(q,s) &= \frac{2q}{Pr\pi s}, \\ s^\alpha \bar{u}(q,s) + \frac{q^2}{Pr}\bar{u}(q,s) &= \frac{2q}{Pr\pi s}, \\ \bar{u}(q,s) &= \frac{2q}{Pr\pi s\left(s^\alpha + \frac{q^2}{Pr}\right)}, \\ \bar{u}(q,s) &= \frac{2}{q\pi}\left\{\frac{1}{s} - \frac{s^{\alpha-1}}{s^\alpha + \frac{q^2}{Pr}}\right\}. \end{aligned} \tag{33}$$

Applying the inverse of the Laplace transformation Eq.(33) and its inverse of Fourier transform, thus the solution of the fractional heat equation (12) is given by

$$u(x,\tau) = 1 - \frac{2}{\pi}\int_0^{+\infty} \frac{\sin qx}{q} E_\alpha\left(-\frac{q^2}{Pr}\left(\frac{\tau^\rho}{\rho}\right)^\alpha\right) dq. \tag{34}$$

The proofs are already in [26, 27]. We obtain a particular case when $\alpha = 1$. We represent the solution as the following form

$$u(x,\tau) = 1 - erf\left(\frac{x}{2\sqrt{\frac{1}{Pr}\frac{\tau^\rho}{\rho}}}\right). \tag{35}$$

We will analyze, for this case, the impact of the order ρ in the diffusion processes. The novelty in our paper is the procedure for getting the solution to Eq. (11). The first step consists of applying the Fourier and the Laplace transformation on Eq. (11) as in Eqs.(32)-(33), we get the following relations

$$\begin{aligned} s^\alpha \bar{v}(q,s) - s^{\alpha-1}\bar{v}(q,0) + \nu q^2 \bar{v}(q,s) &= \frac{2\nu q}{\pi s} + \beta \bar{u}(q,s), \\ s^\alpha \bar{v}(q,s) + \nu q^2 \bar{v}(q,s) &= \frac{2\nu q}{\pi s} + \beta \bar{u}(q,s), \\ \frac{2\nu q}{\pi s\left(s^\alpha + \nu q^2\right)} + \frac{2q\beta}{Pr\pi s\left(s^\alpha + \frac{q^2}{Pr}\right)\left(s^\alpha + \nu q^2\right)} &= \bar{v}(q,s). \end{aligned} \tag{36}$$

Note that we rewrite Eq. (36) as the following form

$$\bar{v}(q,s) = \frac{2\nu q}{\pi s\left(s^\alpha + \nu q^2\right)} + \frac{2\beta}{Prq\pi s\left(\nu - \frac{1}{Pr}\right)}\left\{\frac{1}{s^\alpha + \frac{q^2}{Pr}} - \frac{1}{s^\alpha + \nu q^2}\right\}. \tag{37}$$

When we apply the inverse of the Laplace and the Fourier transformations, we get the solution of the fractional equation described by Eq. (11). We have the form

$$v(x,\tau) = m(x,\tau) + n(x,\tau) - p(x,\tau), \tag{38}$$

where the function m is given by the following equation

$$m(x,\tau) = 1 - \frac{2}{\pi}\int_0^{+\infty} \frac{\sin qx}{q} E_\alpha\left(-\nu q^2\left(\frac{\tau^\rho}{\rho}\right)^\alpha\right) dq. \tag{39}$$

Note that when the orders of the fractional derivative satisfy the condition $\alpha = \rho = 1$. We express the function m in the following form

$$m(x,\tau) = 1 - erf\left(\frac{x}{2\sqrt{\nu\tau}}\right) = erfc\left(\frac{x}{2\sqrt{\nu\tau}}\right). \tag{40}$$

The function n is given by the following equation

$$n(x,\tau) = \frac{2\beta}{Pr\pi\left(\nu - \frac{1}{Pr}\right)}\int_0^{+\infty} \frac{\sin qx}{q}\tau^\alpha E_{\alpha,1+\alpha}\left(-\frac{q^2}{Pr}\left(\frac{\tau^\rho}{\rho}\right)^\alpha\right) dq. \tag{41}$$

Note that when the order $\alpha = \rho = 1$, we represent the function n as the following form

$$n(x,\tau) = \frac{2\beta}{Pr\pi\left(\nu - \frac{1}{Pr}\right)}\int_0^{+\infty} \frac{\sin qx}{q}\tau E_{1,2}\left(-\frac{q^2}{Pr}(\tau)\right) dq. \tag{42}$$

The following term represents the similarity variable of the fractional heat equation (12)

$$\theta(x,\tau)=\frac{x}{2\sqrt{\frac{1}{Pr}\tau}}, \tag{43}$$

see in [28]. We make some comments on this function (43). The similarity variable is essential in the diffusion equations because it can be used to express the exact solution of the fractional diffusion equation without using the present method. Note that the solution with the similarity variable is not the subject of research in our paper. Our article provided the form of the similarity variable. We can use it to determine the exact solution of the fractional heat equation in future work. Finally, we represent the function n as the following form

$$p(x,\tau)=\frac{2\beta}{Pr\pi\left(\nu-\frac{1}{Pr}\right)}\int_0^{+\infty}\frac{\sin qx}{q}\tau^{\alpha}E_{\alpha,1+\alpha}\left(-\nu q^2\left(\frac{\tau^{\rho}}{\rho}\right)^{\alpha}\right)dq. \tag{44}$$

We finish this section by recalling the solution of the proposed model in two special cases. Let's be the case $\alpha=1$, and we suppose the order ρ is an arbitrary positive number. Then the solution of the fractional diffusion equation perturbed by the fractional heat equation is represented as the following form

$$v(x,\tau)=erfc\,(\delta)+n(x,\tau)-p(x,\tau),, \tag{45}$$

where the function $\delta(x,\tau)=\frac{x}{2\sqrt{\nu\frac{\tau^{\rho}}{\rho}}}$ represents a similarity variable. In the next section, we will study the impact of the order ρ on the diffusion processes using Eq. (45). Does the fractional-order affect the diffusion processes of the proposed model? We will answer the question in the graphical representation section. Furthermore, the paper brings a new procedure of getting the solutions to the proposed model. We will also analyze the impact of the Prandtl number on the diffusion processes.

The second part of this paper consists of determining the mean square displacement for fractional heat equation (12) and the fractional reaction equation (11), for brief physical interpretations. In this subsection, the initial condition is given in Eq. (13) is reconsidered in the form

$$u(x,0)=v(x,0)=\delta(x), \tag{46}$$

where δ represents the Dirac function, we not consider the condition (13) in this part. Note that the following equation gives the formula from which we obtain the mean square displacement [24, 25] in terms of Laplace transform

$$<x^2>(s)=\lim_{q\to 0}-\frac{d^2\bar{u}(q,s)}{dq^2}, \tag{47}$$

for the fractional heat equation (12). The mean square displacement [24, 25] for the fractional reaction equation (11) in terms of Laplace transform follows from the following relationship

$$<x^2>(s)=\lim_{q\to 0}-\frac{d^2\bar{v}(q,s)}{dq^2}. \tag{48}$$

Combining the Fourier transform of Eq. (12) and Eq. (46), the mean square displacement for the fractional heat equation in the context of Caputo-Liouville generalized fractional derivative is given by

$$<x^2>(\tau)=\mathcal{L}_{\rho}^{-1}\left[\lim_{q\to 0}-\frac{d^2\bar{u}(q,s)}{dq^2}\right]=2Pr\left(\frac{\tau^{\rho}}{\rho}\right)^{\alpha}, \tag{49}$$

where $\bar{u}(q,s)=\frac{s^{\alpha-1}}{s^{\alpha}+Prq^{2}}$. From Eq. (49), we give the following physical interpretation. The fractional heat equation (12) generates the following diffusion processes. We obtain the normal diffusion when the orders respect the relation $\alpha\rho=1$. The sub-diffusion process is obtained with a relationship $\alpha\rho<1$. The super-diffusion process is generated with the relation $1<\alpha\rho<2$. The ballistic diffusion is obtained with the condition $\alpha\rho=2$ and the hyper-diffusion process with $\alpha\rho>2$. We use the Fourier transform of fractional diffusion-reaction (11) and with Eq. (48), thus the main square displacement is given by the following relationship

$$< x^{2} > (\tau)=\mathcal{L}_{\rho}^{-1}\left[\lim_{q\to 0}-\frac{d^{2}\bar{v}(q,s)}{dq^{2}}\right]=2\nu\left(\frac{\tau^{\rho}}{\rho}\right)^{\alpha}+2\beta\left(\nu+Pr\right)\left(\frac{\tau^{\rho}}{\rho}\right)^{2\alpha}, \quad (50)$$

where the details of the Fourier transform is given by

$$\bar{u}(q,s)=\frac{s^{\alpha-1}}{s^{\alpha}+\nu q^{2}}+\frac{s^{\alpha-1}}{s^{2\alpha}+\left(\nu+Pr\right)q^{2}s^{\alpha}+Pr\nu q^{4}}. \quad (51)$$

For a short time diffusion process, the value of the mean square displacement described in Eq. (50), is equals to the first term of the sum in Eq. (50) that is

$$< x^{2} > (\tau)\approx 2\nu\left(\frac{\tau^{\rho}}{\rho}\right)^{\alpha}. \quad (52)$$

In a physical view, we obtain the same diffusion processes as in fractional heat equation: the normal diffusion, the subdiffusion, the superdiffusion, the hyper diffusion, the ballistic diffusion processes.

For a long time of the diffusion process, note that Eq. (50) is dominated by the second member into the sum. Thus it can be rewritten as the form

$$< x^{2} > (\tau)\approx 2\beta\left(\nu+Pr\right)\left(\frac{\tau^{\rho}}{\rho}\right)^{2\alpha}. \quad (53)$$

In a physical view, we obtain the normal diffusion when the orders respect the relation $\alpha\rho=1/2$. We have the sub-diffusion process when the relationship $\alpha\rho<1/2$ is held. We get the super-diffusion process in the condition $1/2<\alpha\rho<1$. The ballistic diffusion and hyper-diffusion processes are obtained respectively under the conditions $\alpha\rho=1$ and $\alpha\rho>1$.

6. Graphics and discussions

In this section, we make some analyses related to the presented model. We analyze the impact of the order ρ and the Prandtl number Pr in the diffusion processes. In this section, the order ρ satisfies the conditions $\rho\leq 1$ and $\rho\geq 1$. The Prandtl number describes the values $Pr=5,6,7,8,9$. In this section, we fix the order $\alpha=1$, $\nu=1$, and the time to $\tau=0.1s$. We will also analyze the effect of the fractional heat equation (12) on the fractional diffusion-reaction equation (11).

In Figure 1, we fix the Prandtl number to $Pr=5$. We depict the solutions of the fractional heat equation represented by Eq. (12) for different values of the fractional order ρ. The order ρ satisfies the condition $\rho\leq 1$. We note when the order ρ increases into $(0,1]$, we remark all the curves decrease rapidly following the direction of the arrow with the increase of the state variable x. Physically, we explain these behaviors by the fact we are in the sub-diffusion process context. In conclusion, the order ρ has a significant impact on the diffusion process for the fractional heat equation (12). The order ρ has an acceleration effect in the diffusion

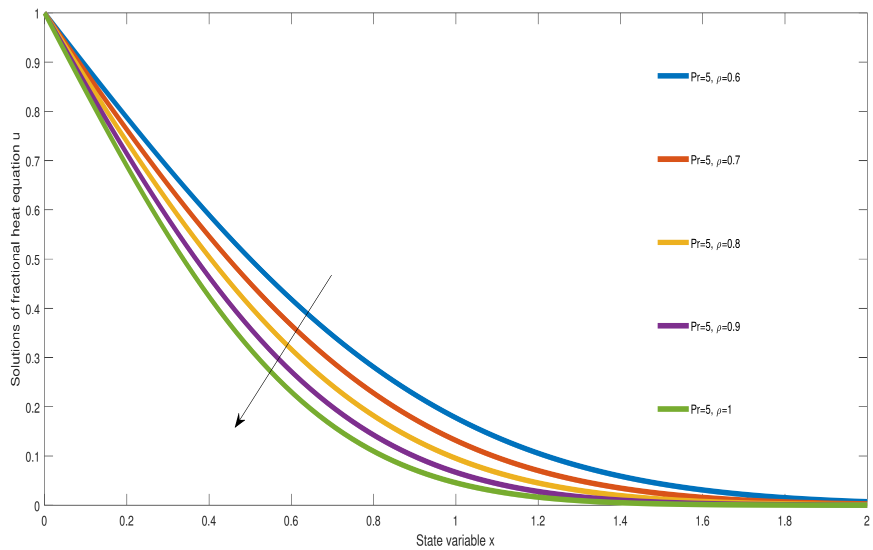

FIGURE 1. Solution of fractional heat equation for $\rho \leq 1$.

processes. When it increases, the solutions decrease and converge rapidly to the normal diffusion.

In Figure 2, we fix the Prandtl number to $Pr = 5$. We depict the solutions of the fractional diffusion equation represented by Eq. (11). The previous fractional heat equation (12) is the reaction term for Eq. (11), for different values of the order ρ. The order ρ satisfies the condition $\rho \leq 1$. Note in the previous graphical representation; we consider the fractional heat equation (12) as the reaction term of Eq. (11). We notice when the order ρ increases into $(0, 1]$, all the curves decrease with retardation, see in Figure 1. In other words, the fractional heat equation (12) as a reaction term decelerates the diffusion process in Eq. (11). Physically, the decrease in the solutions of Eq. (11) is generated by the sub-diffusion process into the fractional diffusion equation (12). The effect of the order ρ does not change.

In Figure 3, we fix the Prandtl number to $Pr = 5$. We depict the solutions of the fractional heat equation represented by Eq. (11) for different values of the order ρ. The order ρ satisfies the condition $\rho \geq 1$. We remark when the order ρ increases and satisfies the condition $\rho \geq 1$, all the curves decrease fastly and do not converge to the normal diffusion. All the trajectories converge to zero. Physically, we explain the fast decrease in the solution of Eq. (12) by the super-diffusion process. The order ρ accelerate the diffusion process fastly in Eq. (12). The arrow indicates the effect ρ, see in Figure 3.

In Figure 4, we fix the Prandtl number to $Pr = 5$. We depict the solutions of the fractional diffusion-reaction equation represented by Eq. (11). We conserve the previous fractional heat equation (12) as the reaction term. The order ρ satisfies the following condition $\rho \geq 1$. We notice when the order ρ increases and satisfies the condition posed $\rho \geq 1$, all the curves decrease with a retardation effect and converge to zero. Note here we have a slight decrease in the curves of the solutions

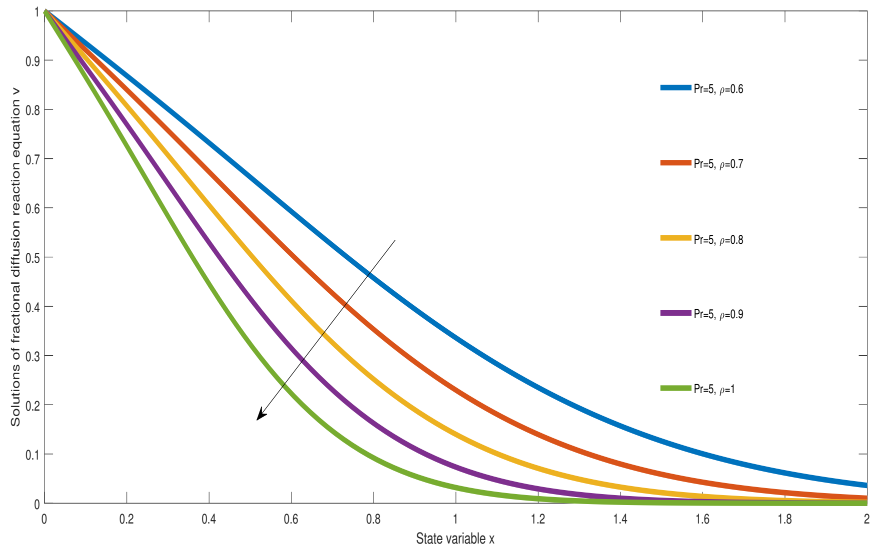

FIGURE 2. Solution of fractional diffusion reaction equation for $\rho \leq 1$.

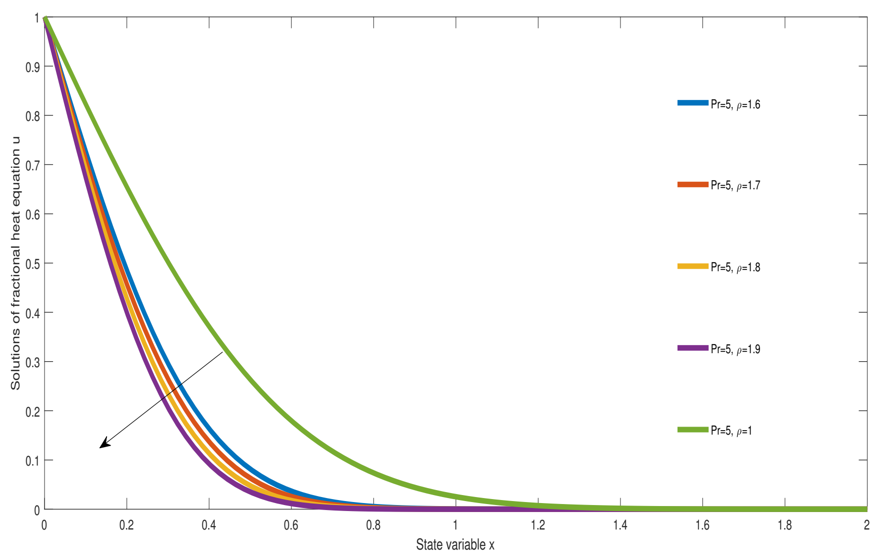

FIGURE 3. Solution of fractional heat equation for $\rho \geq 1$.

of Eq. (11). In other words, when we consider the fractional heat equation (12) as a reaction term, we notice a retardation effect in the diffusion process of Eq. (11). Physically, the decrease in the solutions of Eq. (11) is generated by the

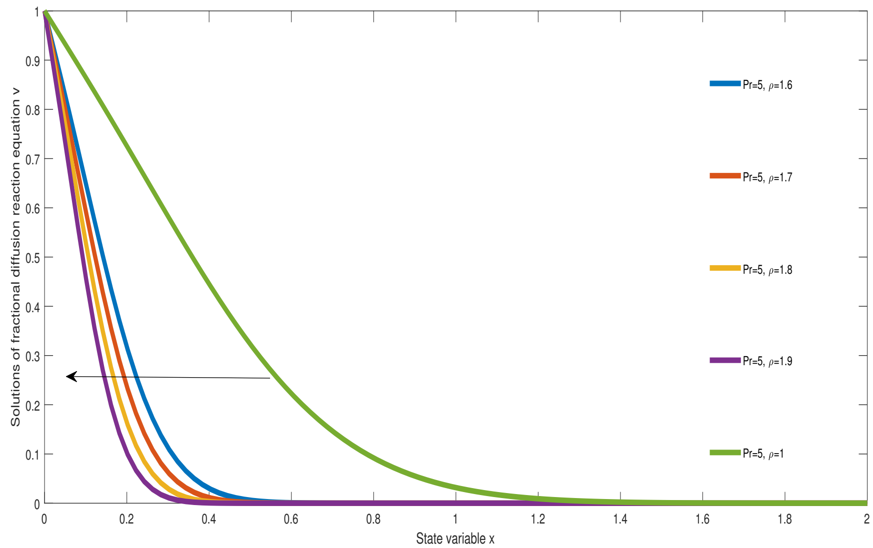

FIGURE 4. Solution of fractional diffusion reaction equation for $\rho \geq 1$.

super-diffusion process of the fractional heat equation (12). See the arrow for the direction of the diffusion process, and the effect of the order ρ does not change.

Consider the order $\rho = 0.6$ fixed and the Prandtl number increase in tolerable values $Pr \in \{5; 6; 7; 8; 9\}$. In Figure 5, we depict the solutions of the fractional heat equation (12) under the previous assumptions. We note all the curves decrease and converge to zero. The Prandtl value has the same effect as the order ρ when it satisfies the condition $\rho \leq 1$. The increase in the Prandtl number generates a decrease in all the solutions of Eq. (12).

Consider the order $\rho = 0.6$ fixed and the Prandtl number increase in tolerable values $Pr \in \{5; 6; 7; 8; 9\}$. In Figure 6, we depict the solutions of the fractional diffusion equation (11) with reaction term (12), under the previous assumptions. We note all the curves decrease and but do not converge to zero. In other words, according to one another, the solutions increase when the Prandtl number increases. We note when we fix the order $\rho = 0.6$, and the Prandtl number increases, we notice after a certain time, the Prandtl number does not impact the diffusion process. The solutions of Eq. (11) become globally asymptotically stable after a certain time. We observe when the Prandtl number converges to his extreme value, all the solutions converge to the normal diffusion.

In conclusion, we note the order ρ has an acceleration effect in the fractional reaction equation when the fractional heat equation generates the sub-diffusion process. The order ρ has a retardation effect into the fractional reaction equation when the fractional heat equation generates the super-diffusion process. The Prandtl number Pr impacts the solution of the fractional diffusion-reaction equation (11) at the beginning of the diffusion process. We notice that after a certain time, it hasn't any impact on the solution of the fractional diffusion-reaction equation (11).

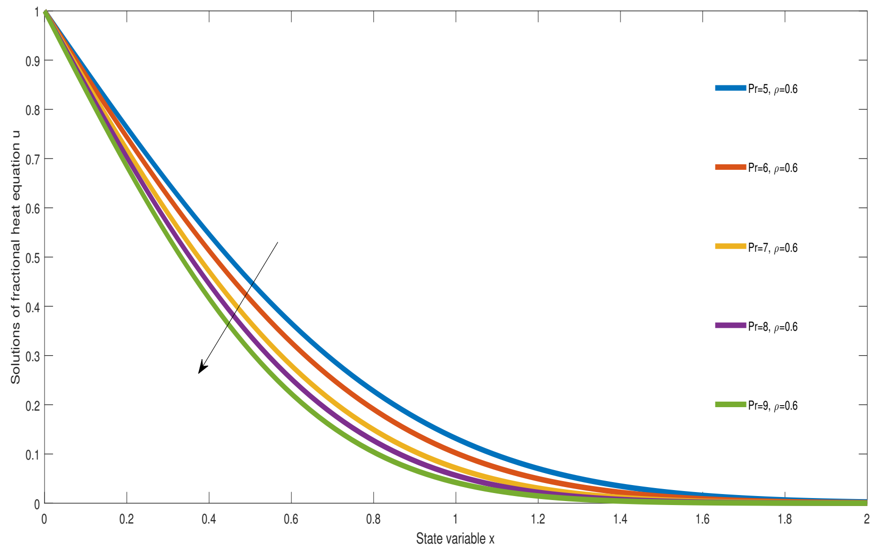

Figure 5. Solution of fractional heat equation for $Pr \in \{5; 6; 7; 8; 9\}$.

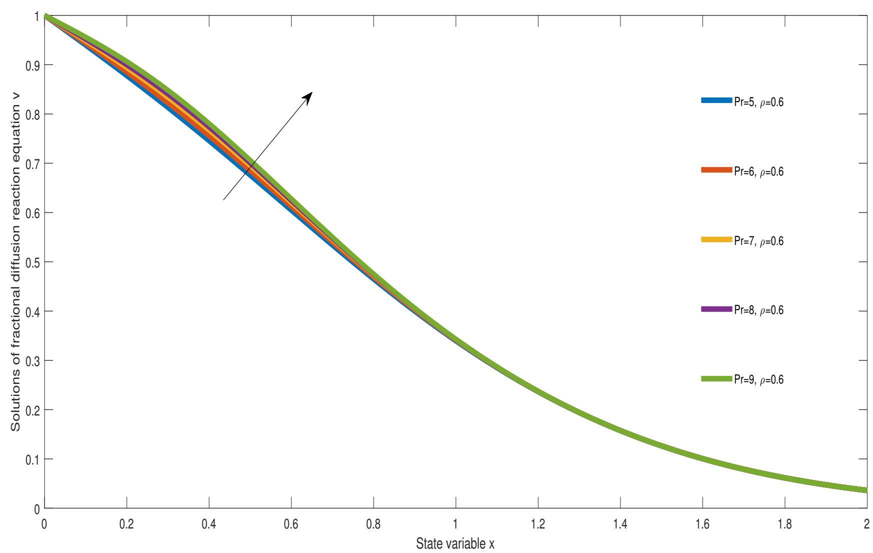

Figure 6. Solution of fractional diffusion reaction equation for $Pr \in \{5; 6; 7; 8; 9\}$.

7. Conclusion

In this paper, we have introduced a new method for getting the solution of the fractional diffusion equation with a reaction term. The reaction term considered in this paper as input satisfies the fractional heat equation. We have analyzed the impact of the order ρ into the diffusion processes. We have noted a retardation effect into the diffusion process when the order ρ satisfies the condition $\rho \leq 1$ and acceleration impact when it satisfies the condition $\rho \geq 1$. We also note the significant effect of the Prandtl number on the diffusion process of the fractional heat equation. It hasn't any impact on the diffusion process of the fractional diffusion-reaction equation after a certain value of the state variable.

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