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A NOTE ON THE ASYMPTOTIC PROPERTIES OF A GENERALIZED DIFFERENTIAL EQUATIONS

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Abstract. In this paper we present some asymptotic properties of a generalized differential equation, using a derivative notion general which contains as particular cases many of the differential operators reported in the literature. In particular, the stability in the Hyers-Ullam sense of a generalized differential equation of order α is studied.

1. Introduction

Fractional Calculus, a mathematical area as old as Ordinary Calculus, is the general extension to the non-integer case of the known results of classical Differential and Integral Calculus.

In this paper we will consider some non linear generalized differential equations of type

$$F(t, x(t), N^{\alpha}x(t)) = 0, \tag{1}$$

involving a differential operator N^{α}, $0 < \alpha \leq 1$, will be defined later. Although since 2014 (see [1], [11]) there have been several local fractional derivatives, all of these can be considered as particular cases (including the ordinary classic) of the following definition of Generalized Derivative, as we will see later.

In [26] a generalized fractional derivative was defined in the following way (see also [36] and [3], in this last work the generalized derivative presented is defined for all $\alpha > 0$).

Definition 1 Given a function $f : [0, +\infty) \to \mathbb{R}$. Then the N-derivative of f of order α is defined by

$$N_F^{\alpha} f(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon F(t, \alpha)) - f(t)}{\varepsilon} \tag{2}$$

for all $t > 0$, $\alpha \in (0, 1)$ being $F(\alpha, t)$ is some function. Here we will use some cases of F defined in function of $E_{a,b}(.)$ the classic definition of Mittag-Leffler function with $Re(a), Re(b) > 0$. Also we consider $E_{a,b}(t^{-\alpha})_k$ is the k-nth term of $E_{a,b}(.)$.

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If f is $\alpha-$differentiable in some $(0,\alpha)$, and $\lim_{t\to 0^+} N_F^{(\alpha)} f(t)$ exists, then define $N_F^{(\alpha)} f(0) = \lim_{t\to 0^+} N_F^{(\alpha)} f(t)$, note that if f is differentiable, then $N_F^{(\alpha)} f(t) = F(t,\alpha) f'(t)$ where $f'(t)$ is the ordinary derivative.

The original function $E_{\alpha,1}(z) = E_\alpha(z)$ was defined and studied by Mittag-Leffler in the year 1903, that is, a uniparameter function (see [18, 19]). It is a direct generalization of the exponential function. Wiman proposed and studied a generalization of the role of Mittag-Leffler, who we'll call it the Mittag-Leffler function with two parameters $E_{\alpha,\beta}(z)$ (see [34]), Agarwal in 1953 and Humbert and Agarwal in 1953, also made contributions to the final formalization of this function.

We consider the following examples:

I) $F(t,\alpha) \equiv 1$, in this case we have the ordinary derivative.

II) $F(t,\alpha) = E_{1,1}(t^{-\alpha})$. In this case we obtain, from Definition **??**, the derivative $N_1^\alpha f(t)$ defined in [4] (see also [24]).

III) $F(t,\alpha) = E_{1,1}((1-\alpha)t) = e^{(1-\alpha)t}$, this kernel satisfies that $F(t,\alpha) \to 1$ as $\alpha \to 1$.

IV)$F(t,\alpha) = E_{1,1}(t^{1-\alpha})_1 = t^{1-\alpha}$, the linear term ($k = 1$), with this kernel we have $F(t,\alpha) \to 0$ as $\alpha \to 1$ (see [11]).

V)$F(t,\alpha) = E_{1,1}(t^\alpha)_1 = t^\alpha$, in this case, only the linear term is also considered ($k = 1$), so we have $F(t,\alpha) \to t$ as $\alpha \to 1$ (see [27]). It is clear that in this case, the results will differ from those obtained previously, which enhances the study of these cases.

VI) $F(t,\alpha) = E_{1,1}(t^{-\alpha})_1 = t^{-\alpha}$ with this kernel we have $F(t,\alpha) \to t^{-1}$ as $\alpha \to 1$ This is the derivative N_F^α studied in [16]. As in the previous case, the results obtained have not been reported in the literature.

Now, we give the definition of a general fractional integral (see [36], [3] and [5]):

Definition 2 Let $\alpha \in (0,1]$ and $0 \le a \le b$. We say that a function $h : [a,b] \to \mathbb{R}$ is α-fractional integrable on $[a,b]$, if the integral

$$ {}_{N_F}J_a^\alpha h(x) = {}_{N_F}J_a^\alpha h(x) = \int_a^x \frac{h(t)}{F(t,\alpha)} dt \tag{3} $$

exists and is finite.

Remark 1 Taking into account the examples of kernels presented above, it is clear that we will have different integral operators. To name just one case, if $F(t,\alpha) \equiv 1$ we will have the classic Riemann integral.

The following statements are analogous to those from the Ordinary Calculus (see [36], [3] and [5]).

Theorem 1 Let f be N-differentiable function in (t_0,∞) with $\alpha \in (0,1]$. Then for all $t > t_0$ we have

a) If f is differentiable ${}_{N_F}J_{t_0}^\alpha \left(N_F^\alpha f(t)\right) = f(t) - f(t_0)$.

b) $N_F^\alpha \left({}_{N_F}J_{t_0}^\alpha f(t)\right) = f(t)$.

An important property, and necessary, in our work is that established in the following result.

Theorem 2 (Integration by parts) Let u and v be N-differentiable function in (t_0, ∞) with $\alpha \in (0,1]$. Then for all $t > t_0$ we have

$$ {}_{N_F}J^{\alpha}_{t_0}\left((uN^{\alpha}_F v)(t)\right) = [uv(t) - uv(t_0)] - {}_{N_F}J^{\alpha}_{t_0}\left((vN^{\alpha}_F u)(t)\right) \tag{4} $$

2. On the global existence of solutions

Consider the following generalized differential equation (see [28]):

$$ N^{\alpha}_F x(t) + B(t)x(t) = A(t), \quad x(t_0) = x_0. \tag{5} $$

The following result is basic in our paper.

Proposition 1 Let A and B be continuous functions on an interval $[a, +\infty)$, $a \geq 0$, and $\alpha \in (0,1]$. Then the solution of the linear fractional equation (5) on I is given by

$$ x(t) = e^{-{}_{N_F}J^{\alpha}_{t_0}(B)(t)}\left({}_{N_F}J^{\alpha}_{t_0}\big(Ae^{{}_{N_F}J^{\alpha}_{t_0}(B)}\big)(t) + C\right). \tag{6} $$

Proof. If x is given by (6), then properties of derivatives N^{α}_F give

$$ \begin{aligned} N^{\alpha}_F x &= N^{\alpha}_F\Big(e^{-{}_{N_F}J^{\alpha}_{t_0}(B)(t)}\left({}_{N_F}J^{\alpha}_{t_0}\big(Ae^{{}_{N_F}J^{\alpha}_{t_0}(B)}\big)(t) + C\right)\Big) \\ &= -B(t)x + A(t), \end{aligned} $$

and x is a solution of (5). Therefore, it suffices to check that for each $t_0 \in I$ and $x_0 \in \mathbb{R}$ there exists $C = C(t_0, x_0)$ such that x is the solution of (5) with $x(t_0) = x_0$. If we choose

$$ C = x_0 e^{{}_{N_F}J^{\alpha}_{t_0}(B)(t_0)} - {}_{N_F}J^{\alpha}_{t_0}\big(Ae^{{}_{N_F}J^{\alpha}_{t_0}(B)}\big)(t_0), $$

then

$$ x(t) = e^{-{}_{N_F}J^{\alpha}_{t_0}(B)(t)}\left({}_{N_F}J^{\alpha}_{t_0}\big(Ae^{{}_{N_F}J^{\alpha}_{t_0}(B)}\big)(t) + x_0 e^{{}_{N_F}J^{\alpha}_{t_0}(B)(t_0)} - {}_{N_F}J^{\alpha}_{t_0}\big(Ae^{{}_{N_F}J^{\alpha}_{t_0}(B)}\big)(t_0)\right) $$

satisfies

$$ x(t_0) = e^{-{}_{N_F}J^{\alpha}_{t_0}(B)(t_0)}\left({}_{N_F}J^{\alpha}_{t_0}\big(Ae^{{}_{N_F}J^{\alpha}_{t_0}(B)}\big)(t_0) + x_0 e^{{}_{N_F}J^{\alpha}_{t_0}(B)(t_0)} - {}_{N_F}J^{\alpha}_{t_0}\big(Ae^{{}_{N_F}J^{\alpha}_{t_0}(B)}\big)(t_0)\right) = x_0, $$

and the proof is finished. □

The study of boundedness of solutions of a differential equation, either fractional or not, plays an important role in qualitative theory (additional details can be found in [21, 22, 23]). In addition, the qualitative behavior of solutions plays an important role in many real-world phenomena related to applied research. Based on the previous results, we can obtain bounds for the solutions of local fractional differential equations.

Let $\alpha \in (0,1]$, $a \geq 0$, $t_0 \geq a$, $x_0 \in \mathbb{R}^n$, and $f : [a, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ in (C, Lip) on $[a, \infty) \times \mathbb{R}^n$. Let us consider the initial value problem

$$ N^{\alpha}_F x(t) = f(t, x), \qquad x(t_0) = x_0. \tag{7} $$

Definition 3 The trivial solution $x \equiv 0$ of (7) is said to be *stable* if for any $\varepsilon > 0$, there exists $\delta = \delta(t_0, \varepsilon) > 0$ such that if $|x_0| < \delta$, then $|x(t)| < \varepsilon$ for every $t \geq t_0$; it is *uniformly stable* if there exists $\delta = \delta(\varepsilon) > 0$ such that if $|x_0| < \delta$, then $|x(t)| < \varepsilon$ for every $t \geq t_0 \geq a$.

The following result is very easy to obtain (see [28])

Proposition 2 Let $\alpha \in (0,1]$, $t_0 \in [a,\infty)$, $x_0 \in \mathbb{R}$, and $f : [a,\infty) \times \mathbb{R} \to \mathbb{R}$ in (C, Lip) on $[a,\infty) \times \mathbb{R}$. Then the problem (7) is equivalent to

$$x(t) = x_0 +_{N_F} J^{\alpha}_{t_0} f(s, x(s))(t). \tag{8}$$

The theorems of existence and local uniqueness are incomplete for the Qualitative Theory because they can not affirm anything when t is sufficiently large, that is why we need results framed in the Definition 1 (for additional results consult [28]).

Theorem 3 Under assumptions on function f, $|f(t, x(t))| < M(t, x)$ and the following condition, there exists a δ_0 such that

$$\left|{}_{N_F} J^{\alpha}_{t_0} M(t, \delta)(+\infty)\right| \leq M_2, M_2 > 0, \tag{9}$$

holds for all for $0 \leq \delta \leq \delta_0$, then the trivial solution $x \equiv 0$ of (8) is uniformly stable.

Proof. Let $\varepsilon > 0$ and let $x = x(t)$ be the solution of the Cauchy problem (7) where x_0 is such that

$$|x_0| < \delta(\varepsilon) = min\left\{\frac{\varepsilon}{2}, \frac{\varepsilon}{2M_2}\right\}. \tag{10}$$

Then we have

$$\begin{aligned} |x(t)| &\leq |x_0| + |x(t) - x_0| \leq |x_0| + \left|{}_{N_F} J^{\alpha}_{t_0} f(s, x(s))(t)\right| \leq \\ &\leq |x_0| + \left|{}_{N_F} J^{\alpha}_{t_0} M(s, \delta)(t)\right| \leq |x_0| + \left|{}_{N_F} J^{\alpha}_{t_0} M(t, \delta)(+\infty)\right| < \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

for all $t \geq t_0$. This implies the fact that the trivial solution of (8) is uniformly stable. □

In a similar way we have the following result:

Theorem 4 Under assumptions of Theorem 3 and the following conditions: there exists a δ_0 such that $0 < M_1 \leq \left|{}_{N_F} J^{\alpha}_{t_0} M(t, \delta)(+\infty)\right| \leq M_2$, holds for all $0 \leq \delta \leq \delta_0$, then the trivial solution $x \equiv 0$ of (8) is uniformly stable.

Proof. As previous result, we have

$$|x_0| - M \leq x(t) \leq |x_0| + M, \quad M = sup\{M_1, M_2\}.$$

This allows us to obtain the desired result. □

The previous results allow us to obtain the following.

Corollary 1 Suppose that equation (5) has a unique continuous solutions $x(t)$, A and B be continuous functions on an interval $[a, +\infty)$, $\quad a \geq 0$ and $\alpha \in (0,1].B(t)$ is non negative on $[a,\infty)$ and $A(t)$ bounded on $[a,\infty)$, then $x(t)$ and its derivative $N^{\alpha}_{F} x(t)$ are both exponentially bounded, and thus, their generalized Laplace transforms exist.

3. Examples and applications

Consider the following particular case of fractional equation (7):

$$N_F^\alpha x(t) = q(t)h(x) - p(t)H(x), x(t_0) = x_0, \tag{11}$$

where p and q are continuous real functions defined over a certain interval $I \subseteq \mathbb{R}$, the functions h and H are continuous $h, H : \mathbb{R} \to \mathbb{R}$, h is a N-differentiable function with $h(x) \neq 0$ and $H(x) = h(x)_{N_F}J_{t_0}^\alpha[h(s)]^{-1}(x)$.

Consider a change of variables $z = \frac{H(x)}{h(x)}$, and taking into account the Chain Rule (see [26] and [3]) we have

$$N_F^\alpha z(t) = N_F^\alpha\left(\frac{H(x)}{h(x)}\right)(t) = \left(\frac{H(x)}{h(x)}\right)' N_F^\alpha x(t) = \frac{1}{h(x)}N_F^\alpha x(t), \tag{12}$$

from equation (11) we have:

$$\frac{1}{h(x)}N_F^\alpha x(t) = q(t) - p(t)\frac{H(x)}{h(x)} \tag{13}$$

from where it is easily obtained

$$N_F^\alpha z + p(t)z = q(t), \tag{14}$$

a fractional linear equation, with known solution (see (6)). Thus the general solution for (14) is given by the expression:

$$z(t) = e^{-{}_{N_F}J_{t_0}^\alpha p(t)}\,{}_{N_F}J_{t_0}^\alpha\left\{e^{{}_{N_F}J_{t_0}^\alpha p(s)}q(s)\right\}(t) + Ce^{-{}_{N_F}J_{t_0}^\alpha p(t)}, \tag{15}$$

from this we have the general solucin of (11) is

$$x(t) = {}_{N_F}J_{t_0}^\alpha\left[h(x(s))N_F^\alpha z(s)\right](t). \tag{16}$$

Remark 2 A remarkable equation covered by (11) is the Fractional Bernoulli Equation $N_F^\alpha x(t) + p(t)x = q(t)x^n$, with $H(x) = x$ and $h(x) = x^n$ (see also [28]). The variable change is $z = \frac{x^{1-n}}{1-n}$, where the general solution is obtained in the form

$$\frac{x^{1-n}}{1-n} = \left\{e^{-(1-n){}_{N_F}J_{t_0}^\alpha p(t)}\,{}_{N_F}J_{t_0}^\alpha\left\{e^{(1-n){}_{N_F}J_{t_0}^\alpha p(s)}q(s)\right\}(t) + Ce^{-(1-n){}_{N_F}J_{t_0}^\alpha p(t)}\right\}.$$

Remark 3 A natural extension of a first order fractional differential equation is the Riccati fractional differential equation $N_F^\alpha x(t) = p(t) + q(t)x + r(t)x^2$, where p, q, r are continuous functions defined over a certain interval $I \subseteq \mathbb{R}$ and $x(t)$ is an unknown function. If a particular solution $x_1(t)$ is known, then the general solution has the form $x(t) = z(t) + x_1(t)$, where $z(t)$ is the general solution of following Bernouilli fractional differential equation

$$N_F^\alpha z(t) - (q(t) + r(t)2x_1(t))\,z(t) = r(t)z^2(t).$$

The necessary and sufficient conditions for the existence and uniqueness of the solution of Bernoulli Equation can be easily obtained by same way in [30].

3.1. **Ulam-Hyers stability in the context of the generalized derivative.** A classic problem of Stanislaw Ulam in the theory of functional equations is the following (dates back to 1940 in the mathematics colloquium of the University of Wisconsin): When is it true that a function that approximately satisfies a functional equation E must be close to an exact solution of E? In 1941, Donald H. Hyers gave a partial affirmative answer to this question in the context of Banach's spaces (see [7]). This was the first significant advance and a step towards more studies in this field of research. Since then, a large number of documents have been published in relation to various generalizations of Ulam's problem and the Hyers' theorem. As for the great influence of Ulam and Hyers in the study of the stability problems of functional equations, this concept is called Hyers-Ulam stability. In [20] the problem of stability in the Ulam-Hyers sense is studied for differential equations with constant coefficients, proving that if the characteristic polynomial does not have pure imaginary roots, then it has Ulam-Hyers stability. Now we will study the Hyers-Ulam stability of linear differential operators generalized with variable coefficients, in the framework of derivative N.

Around 20 years ago, the stability of Ulam-Hyers for linear differential equations is studied for the first order (see [2], [9], [32] and the references cited there). In fact, they proved that if a differentiable function $y : I \to \mathbb{R}$ satisfies $|y'(t) - y(t)| \leq \varepsilon$ for all $t \in I$, then there exists a differentiable function $g : I \to \mathbb{R}$ satisfying $g'(t) = g(t)$ for any $t \in I$ such that $|y'(t) - y(t)| \leq 3\varepsilon$ for every $t \in I$. Some extensions can be found in [8], [31] and [33], results stated in the case of ordinary differential equations of the first order. We consider the following generalized equation

$$p(t)N_F^\alpha y(t) = q(t)y(t) + r(t). \tag{17}$$

Definition 4 We say that equation (17) has the Hyers-Ullam stability if there exists a constant $K > 0$ satisfying that

for every $\varepsilon \geq 0$, y is α-differentiable such that $|p(t)N_F^\alpha y(t) - q(t)y(t) - r(t)| \leq \varepsilon$ then there exists some function z, N-differentiable, satisfying $p(t)N_F^\alpha z(t) = q(t)z(t)+ r(t)$ such that $|y(t) - z(t)| \leq K\varepsilon$.

The constant K is called a HU's constant for equation (17).

First we will prove the following auxiliary result for (17) with $q \equiv 1, r \equiv 0$ i.e. for the equation:

$$p(t)N_F^\alpha y(t) = y(t). \tag{18}$$

Lemma 1 Let z be a differentiable function $z : I \to \mathbb{R}$.

a) The inequality $p(t)N_F^\alpha z(t) \geq z(t)$ is true for all $t \in I$ if and only if there exists a differentiable function $g : I \to \mathbb{R}$ such that $p(t)N_F^\alpha g(t) \geq 0$ and $z(t) = g(t)E_{1,1}\left\{ {}_{N_F}J_a^\alpha \left[\frac{1}{p}\right](t) \right\}$ for all $t \in I$.
b) The inequality $p(t)N_F^\alpha z(t) \leq z(t)$ is true for all $t \in I$ if and only if there exists a differentiable function $h : I \to \mathbb{R}$ such that $p(t)N_F^\alpha h(t) \leq 0$ and $z(t) = g(t)E_{1,1}\left\{ {}_{N_F}J_a^\alpha \left[\frac{1}{p}\right](t) \right\}$ for all $t \in I$.

Proof. It is sufficient to define the function g as $g(t) = z(t)e\left\{ -{}_{N_F}J_a^\alpha \left[\frac{1}{p}\right](t) \right\}$ and follow the ideas of [8]. $\square$

Remark 4 The use of the $E_{1,1}(.)$ function in the previous lemma suggests that new results can be derived if other expressions of the Mittag-Leffler Function are used.

Theorem 5 Given $\varepsilon > 0$ a N-differentiable function $y : I \to \mathbb{R}$ is a solution of the following inequality

$$|p(t)N_F^\alpha y(t) - y(t)| \leq \varepsilon, \tag{19}$$

for all $t \in I$, if and only if there exists a N-differentiable function $g : I \to \mathbb{R}$ satisfying

$$y(t) = \varepsilon + g(t)E_{1,1}\left\{{}_{N_F}J_a^\alpha\left[\frac{1}{p}\right](t)\right\} \tag{20}$$

and

$$0 \leq p(t)N_F^\alpha g(t) \leq 2\varepsilon E_{1,1}\left\{-{}_{N_F}J_a^\alpha\left[\frac{1}{p}\right](t)\right\}, \tag{21}$$

for all $t \in I$.

Proof. Consider that y is the solution of the inequality (23), i.e. we have

$$y(t) - \varepsilon \leq p(t)N_F^\alpha y(t) \leq y(t) + \varepsilon, \tag{22}$$

for all $t \in I$. Putting $z(t) = y(t) - \varepsilon$ we obtain $z(t) \leq p(t)N_F^\alpha z(t)$ for all $t \in I$. From Lemma 1 a) there exists a N-differentiable function $g : I \to \mathbb{R}$ such that the expression in (20) holds for all $t \in I$, where g additionally satisfies

$$p(t)N_F^\alpha g(t) \geq 0 \tag{23}$$

with $t \in I$.

If we now define z as $z(t) = y(t) + \varepsilon$ we obtain $z(t) \geq p(t)N_F^\alpha z(t)$ for all $t \in I$. Then from Lemma 1 b) there exists a N-differentiable function $h : I \to \mathbb{R}$ such that

$$y(t) + \varepsilon = h(t)E_{1,1}\left\{{}_{N_F}J_a^\alpha\left[\frac{1}{p}\right](t)\right\} \tag{24}$$

and

$$p(t)N_F^\alpha h(t) \leq 0 \tag{25}$$

for all $t \in I$. From (20) and (24) we have

$$\begin{aligned} N_F^\alpha y(t) &= N_F^\alpha g(t)E_{1,1}\left\{{}_{N_F}J_a^\alpha\left[\frac{1}{p}\right](t)\right\} + \frac{g(t)}{p(t)}E_{1,1}\left\{{}_{N_F}J_a^\alpha\left[\frac{1}{p}\right](t)\right\} \\ &= N_F^\alpha h(t)E_{1,1}\left\{{}_{N_F}J_a^\alpha\left[\frac{1}{p}\right](t)\right\} + \frac{1}{p(t)}\left(\frac{g(t)}{p(t)}E_{1,1}\left\{{}_{N_F}J_a^\alpha\left[\frac{1}{p}\right](t)\right\} + 2\epsilon\right). \end{aligned}$$

Where it is obtained

$$N_F^\alpha h(t) = N_F^\alpha g(t) - \frac{2\varepsilon}{p(t)}E_{1,1}\left\{-{}_{N_F}J_a^\alpha\left[\frac{1}{p}\right](t)\right\}$$

which, together with (20) and (24) implies (21). Now, let's consider that there is a function y that satisfying (20) and (21). N-differentiating to y, multiplying by

$p(t)$ and subtracting the function itself we get $-\varepsilon \leq p(t)N_F^\alpha y(t) - y(t) \leq \varepsilon$, which is equivalent to the result sought. This completes the proof of Theorem. □

We are now able to establish the stability of Ulam-Hyers for the equation (18).

Theorem 6 Assume that $p(t) > 0$ (or $p(t) < 0$) for all $t \in I$ and if there is an N-differentiable function and $y : I \to \mathbb{R}$ that satisfies (23) for all $t \in I$, then there is a real number c such that

$$\left| y(t) - cE_{1,1}\left({}_{N_F}J_a^\alpha \left[\frac{1}{p}\right](t) \right) \right| \leq \varepsilon \tag{26}$$

for all $t \in I$.

Proof. Consider first that $p(t) > 0$ and that there is an N-differentiable function that satisfies (23) and let's define the number c as follows, $c = \underset{t\to b^-}{Lim} \quad g(t)$ where g is the function defined in the previous Theorem. Dividing (21) by $-p(t)$ and N-integrating from a to t we obtaing

$$0 \geq g(t) - c \geq 2\varepsilon E_{1,1}\left\{ -{}_{N_F}J_a^\alpha \left[\frac{1}{p}\right](b) \right\} - 2\varepsilon E_{1,1}\left\{ -{}_{N_F}J_a^\alpha \left[\frac{1}{p}\right](t) \right\},$$

for all $t \in I$. From here you get

$$\varepsilon \geq \varepsilon + (g(t) - c)E_{1,1}\left\{ -{}_{N_F}J_a^\alpha \left[\frac{1}{p}\right](t) \right\} \geq 2\varepsilon E_{1,1}\left\{ -{}_{N_F}J_t^\alpha \left[\frac{1}{p}\right](b) \right\} - \varepsilon \geq -\varepsilon,$$

this with (20) takes us to (26). In the case $p(t) < 0$ just define c as $c = \underset{t\to a^+}{Lim} \quad g(t)$ and proceed in a similar way. This completes the proof. □

The following result establishes the conditions under which the equation (17) is stable in the sense of Ulam-Hyers.

Theorem 7 If $p(t), q(t), r(r)$ are continuous real functions for all $t \in I$ satisfying $p(x) \neq 0$ and $|q(t)| > C$ for all $t \in I$ and some $C > 0$ independent of $t \in I$. Then the (17) is stable in the sense of Ulam-Hyers.

Proof. Let $\varepsilon > 0$ and $y : I \to \mathbb{R}$ be a continuously N-differentiable function satisfying

$$|p(t)N_F^\alpha y(t) - q(t)y(t) - r(t)| \leq \varepsilon \tag{27}$$

for all $t \in I$. We will demonstrate that there is the constant K, independent of ε, y, t required in the definition of Ulam-Hyers stability such that $|y(t) - z(t)| \leq \varepsilon$ for certain real function N-differentiable z, defined on I and solution of (17). Without loss of generality we can consider $C \equiv 1$ to simplify the writing. We consider firts $p(t) > 0$ for all $t \in I$, thus from (27) we have $-\varepsilon \leq p(t)N_F^\alpha y(t) - q(t)y(t) - r(t) \leq \varepsilon$ and multiplying this by $\frac{1}{p(t)}E_{1,1}\left\{ -{}_{N_F}J_a^\alpha \left[\frac{q}{p}\right](t) \right\}$ we obtain

$$
\begin{aligned}
& -\varepsilon\frac{1}{p(t)}E_{1,1}\left\{-_{N_F}J_a^{\alpha}\left[\frac{q}{p}\right](t)\right\} \\
\leq\;& N_F^{\alpha}y(t)E_{1,1}\left\{-_{N_F}J_a^{\alpha}\left[\frac{q}{p}\right](t)\right\} \\
-\;& y(t)\frac{q(t)}{p(t)}E_{1,1}\left\{-_{N_F}J_a^{\alpha}\left[\frac{q}{p}\right](t)\right\}-\frac{r(t)}{p(t)}E_{1,1}\left\{-_{N_F}J_a^{\alpha}\left[\frac{q}{p}\right](t)\right\} \\
\leq\;& \varepsilon\frac{1}{p(t)}E_{1,1}\left\{-_{N_F}J_a^{\alpha}\left[\frac{q}{p}\right](t)\right\}
\end{aligned}
$$

From where

$$
\begin{aligned}
& -\varepsilon N_F^{\alpha}\left\{E_{1,1}\left\{-_{N_F}J_a^{\alpha}\left[\frac{q}{p}\right](t)\right\}\right\} \\
\leq\;& N_F^{\alpha}\left\{y(t)E_{1,1}\left\{-_{N_F}J_a^{\alpha}\left[\frac{q}{p}\right](t)\right\}\right\}-\frac{r(t)}{q(t)}N_F^{\alpha}\left\{E_{1,1}\left\{-_{N_F}J_a^{\alpha}\left[\frac{q}{p}\right](t)\right\}\right\} \\
\leq\;& \varepsilon N_F^{\alpha}\left\{E_{1,1}\left\{-_{N_F}J_a^{\alpha}\left[\frac{q}{p}\right](t)\right\}\right\}
\end{aligned}
$$

Taking $b_1 \in I$ such that $y(b_1)$ is finite, so N-integrating from t to b_1 we obtain

$$
\begin{aligned}
& -\varepsilon E_{1,1}\left\{-_{N_F}J_a^{\alpha}\left[\frac{q}{p}\right](t)\right\} \\
\leq\;& (y(b_1)-\varepsilon)E_{1,1}\left\{-_{N_F}J_a^{\alpha}\left[\frac{q}{p}\right](b_1)\right\}-y(t)E_{1,1}\left\{-_{N_F}J_a^{\alpha}\left[\frac{q}{p}\right](t)\right\} \\
-\;& {}_{N_F}J_t^{\alpha}\left[\frac{r}{p}N_F^{\alpha}\quad\left\{-_{N_F}J_a^{\alpha}\left[\frac{q}{p}\right](s)\right\}\right](b_1) \\
\leq\;& \varepsilon E_{1,1}\left\{-_{N_F}J_a^{\alpha}\left[\frac{q}{p}\right](t)\right\}.
\end{aligned}
$$

So, we have

$$
\begin{aligned}
& -\varepsilon \\
\leq\;& E_{1,1}\left\{_{N_F}J_a^{\alpha}\left[\frac{q}{p}\right](t)\right\} \qquad (28)\\
& \left((y(b_1)-\varepsilon)E_{1,1}\left\{-_{N_F}J_a^{\alpha}\left[\frac{q}{p}\right](b_1)\right\}-_{N_F}J_t^{\alpha}\left[\frac{r}{p}N_F^{\alpha}\quad\left\{-_{N_F}J_a^{\alpha}\left[\frac{q}{p}\right](s)\right\}(b_1)\right]-y(t)\right) \\
\leq\;& \varepsilon.
\end{aligned}
$$

Analogously, for $t \in [b_1, b)$, we have that

$$\begin{aligned}
& -\varepsilon \\
\leq\ & y(t) - E_{1,1}\left\{ {}_{N_F}J_a^{\alpha}\left[\frac{q}{p}\right](t)\right\} \qquad (29)\\
& \left((y(b_1)-\varepsilon)E_{1,1}\left\{-{}_{N_F}J_a^{\alpha}\left[\frac{q}{p}\right](b_1)\right\} - {}_{N_F}\ J_t^{\alpha}\left[\frac{r}{p}N_F^{\alpha}\quad\left\{-{}_{N_F}J_a^{\alpha}\left[\frac{q}{p}\right](s)\right\}(b_1)\right]\right)\\
\leq\ & \varepsilon(2A-1)
\end{aligned}$$

with $A = E_{1,1}\left\{ {}_{N_F}J_a^{\alpha}\left[\frac{q}{p}\right](b)\right\}$. From these last two inequalities we obtain $|y(t) - z_1(t)| \leq (2A-1)\varepsilon$ with

$$\begin{aligned}
& z_1(t) = E_{1,1}\left\{ {}_{N_F}J_a^{\alpha}\left[\frac{q}{p}\right](t)\right\}\\
& \left((y(b_1)-\varepsilon)E_{1,1}\left\{-{}_{N_F}J_a^{\alpha}\left[\frac{q}{p}\right](b_1)\right\} - {}_{N_F}\ J_t^{\alpha}\left[\frac{r}{p}N_F^{\alpha}\quad\left\{-{}_{N_F}J_a^{\alpha}\left[\frac{q}{p}\right](s)\right\}(b_1)\right]\right).
\end{aligned}$$

So $p(t)N_F^{\alpha}z_1 \equiv q(t)z_1 + r(t)$ for all $t \in I$. Using a similar argument we have, for the case $p(t) < 0$ for all $t \in I$, $|y(t) - z_2(t)| \leq (2B-1)\varepsilon$ being $B = E_{1,1}\left\{-{}_{N_F}J_a^{\alpha}\left[\frac{q}{p}\right](b)\right\}$ and

$$\begin{aligned}
& z_2(t) = E_{1,1}\left\{ {}_{N_F}J_a^{\alpha}\left[\frac{q}{p}\right](t)\right\}\\
& \left((y(b_1)-\varepsilon)E_{1,1}\left\{-{}_{N_F}J_a^{\alpha}\left[\frac{q}{p}\right](b_1)\right\} - {}_{N_F}\ J_t^{\alpha}\left[\frac{r}{q}N_F^{\alpha}\quad\left\{-{}_{N_F}J_a^{\alpha}\left[\frac{q}{p}\right](s)\right\}(b_1)\right]\right),
\end{aligned}$$

clearly $p(t)N_F^{\alpha}z_2 \equiv q(t)z_2 + r(t)$ for all $t \in I$. Thus, we have the desired result. □

Remark 5 In the case $F(t,\alpha) \equiv 1$, that is, in the case of ordinary first order differential equations, our results contain those of [8, 33] and are consistent with those obtained in [31].

4. Epilogue

In this paper we present some asymptotic results referring to generalized differential equations, which are a natural extension of the entire case and of the fractional derivatives, local or global. Specifically, important results of the Qualitative Theory that involve a generalized initial value problem are proposed and tested. In addition, these results are applied to solve generalized differential equations of Bernoulli and Ricatti type. Finally, we have established the conditions under which a generalized linear differential equation of the first order is stable in the Hyers-Ulam sense.

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