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N-FRACTIONAL CALCULUS OF THE GENERALIZED HURWITZ-LERCH ZETA FUNCTION AND FOX'S H-FUNCTION

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Abstract. The aim of the present paper is to establish N-fractional calculus of the generalized Hurwitz-Lerch zeta function and Fox's H-function. The main results are general in nature and provide useful extension and unification of a number of known or new results. For illustration, some special cases of the main results are mentioned.

1. Introduction

In this section we give some important definitions.

Definition 1 The author has introduced the generalized Hurwitz-Lerch zeta function [7] defined in the following manner:

$$\phi_{\mu}^{\alpha,\ \beta}(z,\ s,\ a) = \sum_{k=o}^{\infty}(a + \alpha k z^{\beta})^{-s}(\mu)_k \frac{z^k}{k!}, \tag{1}$$

where $a \in \mathbb{C}\backslash\mathbb{Z}_0^-$, $s \in \mathbb{C}$, $\mathrm{Re}(\mu) \geq 1$, $\mathrm{Re}(\alpha) > 0$ and either $|z| \leq 1,\ z \neq 1,\ \beta \geq 0$, $\mathrm{Re}(s) > 0$ or $z = 1$, $\mathrm{Re}(s - \mu) > 0$.

Equivalently, the function $\phi_{\mu}^{\alpha,\ \beta}(z,\ s,\ a)$ has the integral representation

$$\phi_{\mu}^{\alpha,\ \beta}(z,\ s,\ a) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1}\ e^{-at}\ (1 - z e^{-\alpha z^{\beta} t})^{-\mu}\ dt, \tag{2}$$

provided that $\mathrm{Re}(a) > 0$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\mu) \geq 1$ and either $|z| \leq 1,\ z \neq 1$, $\beta \geq 0$, $\mathrm{Re}(s) > 0$ or $z = 1$, $\mathrm{Re}(s - \mu) > 0$.

On substituting $\alpha = 1$ and $\beta = 0$ in (1), we get unified Riemann zeta function [3, p. 100, Eq. (1.5)] which further reduces to Hurwitz-Lerch zeta function [1, p. 27, Eq. (1)] when $\mu = 1$, generalized zeta function [1, p. 24, Eq. (1)] when $\mu = 1$ and $z = 1$, and Riemann zeta function [1, p. 32, Eq. (1)] when $\mu = 1$, $z = 1$ and $a = 1$.

Definition 2 Fox's H-function occurring in this paper is defined and represented

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by means of the following Mellin-Barnes type contour integral [18, p. 10, Eq. (2.1.1)-(2.1.3)]:

$$H[z] = H^{M,\ N}_{P,\ Q}\left[z\,\middle|\begin{matrix}(A_J,\ \alpha_J)_{1,\ P}\\(B_J,\ \beta_J)_{1,\ Q}\end{matrix}\right] = \frac{1}{2\pi i}\int_L \phi\,(\xi)\,z^{\xi}d\xi, \tag{3}$$

where $i=\sqrt{-1}$, L is a contour which goes from $\gamma - i\infty$ to $\gamma + i\infty$, $z \neq 0$. and

$$(\phi\xi) = \frac{\prod\limits_{J=1}^{M}\Gamma(B_J-\beta_J\xi)\prod\limits_{J=1}^{N}\Gamma(1-A_J+\alpha_J\xi)}{\prod\limits_{J=M+1}^{Q}\Gamma(1-B_J+\beta_J\xi)\prod\limits_{J=N+1}^{P}\Gamma(A_J-\alpha_J\xi)}. \tag{4}$$

For the convergence, existence conditions and other details of the above Fox's H-function, one may refer to the books written by Srivastava, Gupta and Goyal [18]; Kilbas and Saigo [6]; Mathai, Saxena and Haubold [14] and Prudnikov, Brychkov and Marichev [17].

Definition 3 The author introduced a general class of functions defined in the following manner [8] (see also [9, 10, 11, 12]):

$$\begin{aligned} V_n(x) &= V_n^{h_m,\ d,\ g_j}[p,\ \tau,\ k,\ w,\ q,\ k_m,\ a_j,\ b_r,\ \alpha,\ \beta,\ \delta;\ x] \\ &= \lambda\sum_{n=0}^{\infty}\frac{(-p)^n\prod\limits_{m=1}^{t}\left[(h_m)_{n+k_m}\right](d+\alpha n+\beta)^{-\tau}(x/2)^{nk+dw+q}}{\prod\limits_{j=1}^{s}\left[(g_j)_{n+a_j}\right]\prod\limits_{r=1}^{u}\left[(d)_{\alpha n\delta+b_r}\right]}, \end{aligned} \tag{5}$$

where

(i) $p,\ k,\ w,\ q,\ \beta,\ \delta,\ k_m,\ a_j,\ b_r\ (m=1,\ldots,t;\ j=1,\ldots,s;\ r=1,\ldots,u)$ are real numbers.

(ii) t, s and u are natural numbers.

(iii) $h_m,\ g_j \geq 1\ (m=1,\ldots,t;\ j=1,\ldots,s)$ and d may be real or complex.

(iv) $\alpha > 0$, $\mathrm{Re}(\tau) > 0$, $\mathrm{Re}(d) > 0$, x is a variable and λ is an arbitrary constant.

(v) The series on the right hand side of (5) converges absolutely if $t < s$ or $t = s$ with $|p(x/2)^k| \leq 1$.

For details of convergence conditions of the series on the right hand side of (5) one may refer to the paper [9].

Remark 1 The general class of functions defined by (5) is quite general in nature as it unifies and extends a number of useful functions such as unified Riemann-zeta function [3], generalized hypergeometric function [1], Bessel function [2], Wright's generalized Bessel function [21], Struve's function [2], Lommel's function [2], generalized Mittag-Leffler function [19], exponential function, sine function, cosine function and MacRobert's $E-$function [2] etc. (see, e.g. 8, 10]).

Definition 4 Nishimoto [15] gave the following definition of N-fractional calculus:

Let $D=\{D_-,\ D_+\}$, $C=\{C_-,\ C_+\}$, C_- be a curve along the cut joining two points z and $-\infty + i\ Im(z)$, C_+ be a curve along the cut joining two points z and $\infty + i\ Im(z)$. D_- be a domain surrounded by C_- and D_- contains the two points over the curve C_-, D_+ be a domain surrounded by C_+ and D_+ contains the two points over the curve C_+.

Furthermore, let $f = f(z)$ be a regular function in $D(z \in D)$. Then

$$f_v = (f)_v = \frac{\Gamma(v+1)}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{v+1}}d\zeta,\ (v \notin Z^- := \{-1,\ -2,\ -3, ...\}). \tag{6}$$

$$(f)_{-m'} = \lim_{v \to -m'} (f)_v,\ (m' \in Z^+ := \{1,\ 2,\ 3, ...\}). \tag{7}$$

$$(z^s)_v = e^{-i\pi v}\frac{\Gamma(v-s)}{\Gamma(-s)}z^{s-v},\ \left[0 \neq \left|\frac{\Gamma(v-s)}{\Gamma(-s)}\right| < \infty\right]. \tag{8}$$

where $-\pi \leq arg(\zeta - z) \leq \pi$ for C_-, $0 \leq arg(\zeta - z) \leq 2\pi$ for C_+, $\zeta \neq z$, $z \in C$ and $v \in C$, then for $Re(v) > 0$, $(f)_v$ is derivative of arbitrary order v and for $Re(v) < 0$, integral of arbitrary order $-v$ with respect to z of the function $f(z)$.

2. Main theorems

In this section we establish five theorems for N-fractional calculus involving the generalized Hurwitz-Lerch zeta function, Fox's H-function and general class of functions.

Theorem 1 Let the following conditions be satisfied:
(i) $0 \neq \left|\frac{\Gamma(v-\rho-k-\beta n)}{\Gamma(-\rho-k-\beta n)}\right| < \infty$,
(ii) $\left|\frac{\alpha k z^\beta}{a}\right| < 1$,
(iii) The conditions mentioned with (1) are satisfied. Then the following result holds:

$$\left(z^\rho \phi_\mu^{\alpha,\ \beta}(z,\ s,\ a)\right)_v = \frac{(-1)^v}{a^s}\sum_{n,\ k=0}^{\infty} \frac{(s)_n(\mu)_k \Gamma(v-\rho-k-\beta n)}{n!\ k!\ \Gamma(-\rho-k-\beta n)}\left(-\frac{\alpha k z^\beta}{a}\right)^n z^{\rho+k-v}. \tag{9}$$

Theorem 2 Let the following conditions be satisfied:
(i) $Re(v-\rho-k-\beta n) - \sigma \max_{1 \leq J \leq N} \frac{Re(A_J)-1}{A_J} < 0$,
(ii) $\left|\frac{\alpha k z^\beta}{a}\right| < 1$,
(iii) $b \neq 0,\ \sigma > 0,\ |arg\ z| < \frac{1}{2}A\pi$, where

$$A = \sum_{J=1}^{N}\alpha_J - \sum_{J=N+1}^{P}\alpha_J + \sum_{J=1}^{M}\beta_J - \sum_{J=M+1}^{Q}\beta_J > 0 \tag{10}$$

and the conditions mentioned with (1) are satisfied. Then the following result holds:

$$\begin{aligned}\left(z^\rho \phi_\mu^{\alpha,\ \beta}(z,\ s,\ a) H_{P,\ Q}^{M,\ N}\left[bz^\sigma \left|\begin{matrix}(A_J,\ \alpha_J)_{1,\ P}\\(B_J,\ \beta_J)_{1,\ Q}\end{matrix}\right.\right]\right)_v &= \frac{(-1)^v}{a^s}\sum_{n,\ k=0}^{\infty}\frac{(s)_n(\mu)_k}{n!\ k!}\\ \times \left(-\frac{\alpha k z^\beta}{a}\right)^n z^{\rho+k-v} H_{P+1,\ Q+1}^{M+1,\ N}&\left[bz^\sigma \left|\begin{matrix}(-\rho-k-\beta n,\ \sigma),\ (A_J,\ \alpha_J)_{1,\ P}\\(v-\rho-k-\beta n,\ \sigma),\ (B_J,\ \beta_J)_{1,\ Q}\end{matrix}\right.\right].\end{aligned} \tag{11}$$

Theorem 3 Let the following conditions be satisfied:
(i) $0 \neq \left|\frac{\Gamma(v-\rho-\sigma y-b-\epsilon nk-\epsilon dw-\epsilon q)}{\Gamma(-\rho-\sigma y-b-\epsilon nk-\epsilon dw-\epsilon q)}\right| < \infty$,

(ii) $\left|\frac{\gamma b z^\sigma}{a}\right| < 1$,

(iii) $\eta \neq 0,\ \epsilon > 0$ and the conditions mentioned with (1)and (5) are satisfied. Then the following result holds:

$$\begin{aligned}\Big(z^{\rho}\phi_{\mu}^{\gamma,\ \sigma}(z,\ \xi,\ a)\ V_{n}^{h_m,\ d,\ g_j}[p,\ \tau,\ k,\ w,\ q,\ k_m,\ a_j,\ b_r,\ \alpha,\ \beta,\ \delta;\ \eta z^{\epsilon}]\Big)_{v} \\ = \frac{(-1)^{v}}{a^{\xi}}\lambda\sum_{n,\ b\ y=0}^{\infty}\frac{(\xi)_y(\mu)_b}{y!\ b!}\left(-\frac{\gamma b z^{\sigma}}{a}\right)^{y} \\ \times\frac{(-p)^{n}\prod_{m=1}^{t}\left[(h_m)_{n+k_m}\right](d+\alpha n+\beta)^{-\tau}(\eta z^{\epsilon}/2)^{nk+dw+q}}{\prod_{j=1}^{s}\left[(g_j)_{n+a_j}\right]\prod_{r=1}^{u}\left[(d)_{\alpha n\delta+b_r}\right]} \\ \times\frac{\Gamma(v-\rho-\sigma y-b-\epsilon nk-\epsilon dw-\epsilon q)}{\Gamma(-\rho-\sigma y-b-\epsilon nk-\epsilon dw-\epsilon q)}z^{\rho+b-v}.\end{aligned} \tag{12}$$

Theorem 4 Let the following conditions be satisfied:
(i) $0 \neq \left|\frac{\Gamma(v-\rho-\sigma\zeta-\sigma y-c-\epsilon nk-\epsilon dw-\epsilon q)}{\Gamma(-\rho--\sigma\zeta-\sigma y-c-\epsilon nk-\epsilon dw-\epsilon q)}\right| < \infty,$

(ii) $\left|\frac{\gamma c z^{\sigma}}{a}\right| < 1,$
(iii) $\eta \neq 0,\ b \neq 0,\ \epsilon > 0,\ \sigma > 0$ and the conditions mentioned with (1)and (5) are satisfied. Then the following result holds:

$$\begin{aligned}\Big(z^{\rho}\phi_{\mu}^{\gamma,\ \sigma}(z,\ \xi,\ a)V_{n}^{h_m,\ d,\ g_j}[p,\ \tau,\ k,\ w,\ q,\ k_m,\ a_j,\ b_r,\ \alpha,\ \beta,\ \delta;\ \eta z^{\epsilon}] \\ \times H_{P,\ Q}^{M,\ N}\left[bz^{\sigma}\left|\begin{matrix}(A_J,\ \alpha_J)_{1,\ P}\\(B_J,\ \beta_J)_{1,\ Q}\end{matrix}\right.\right]\Big)_{v} \\ = \frac{(-1)^{v}}{a^{\xi}}\lambda\sum_{n,\ c,\ y=0}^{\infty}\frac{(\xi)_y(\mu)_c}{y!\ c!}\left(-\frac{\gamma c z^{\sigma}}{a}\right)^{y}z^{\rho+c+\epsilon nk+\epsilon dw+\epsilon q-v} \\ \times\frac{(-p)^{n}\prod_{m=1}^{t}\left[(h_m)_{n+k_m}\right](d+\alpha n+\beta)^{-\tau}(\eta/2)^{nk+dw+q}}{\prod_{j=1}^{s}\left[(g_j)_{n+a_j}\right]\prod_{r=1}^{u}\left[(d)_{\alpha n\delta+b_r}\right]} \\ \times H_{P+1,\ Q+1}^{M+1,\ N}\left[bz^{\sigma}\left|\begin{matrix}\{-\rho-\sigma y-c-\epsilon(nk+dw+q),\ \sigma\},\ (A_J,\ \alpha_J)_{1,\ P}\\\{v-\rho-\sigma y-c-\epsilon(nk+dw+q),\ \sigma\},\ (B_J,\ \beta_J)_{1,\ Q}\end{matrix}\right.\right].\end{aligned} \tag{13}$$

Theorem 5 Let the following conditions be satisfied:
(i) $0 \neq \left|\frac{\Gamma(v+v'-\rho-k-\beta n)}{\Gamma(v-\rho-k-\beta n)}\right| < \infty,\ 0 \neq \left|\frac{\Gamma(v+v'-\rho-k-\beta n)}{\Gamma(v'-\rho-k-\beta n)}\right| < \infty,\ 0 \neq \left|\frac{\Gamma(v+v'-\rho-k-\beta n)}{\Gamma(-\rho-k-\beta n)}\right| < \infty,$
(ii) $\left|\frac{\alpha k z^{\beta}}{a}\right| < 1,$
(iii) The conditions mentioned with (1) are satisfied. Then the following result holds:

$$\left(\left(z^{\rho}\phi_{\mu}^{\alpha,\ \beta}(z,\ s,\ a)\right)_{v}\right)_{v'} = \left(\left(z^{\rho}\phi_{\mu}^{\alpha,\ \beta}(z,\ s,\ a)\right)_{v'}\right)_{v} = \left(z^{\rho}\phi_{\mu}^{\alpha,\ \beta}(z,\ s,\ a)\right)_{v+v'}. \tag{14}$$

Proof To prove (9), we express the generalized Hurwitz-Lerch zeta function occurring in the left hand side of (9) in series form using (1) and then collecting the powers of z, we get

$$\left(z^{\rho}\phi_{\mu}^{\alpha,\ \beta}(z,\ s,\ a)\right)_{v}=\left(z^{\rho}a^{-s}\sum_{k=o}^{\infty}\left(1+\frac{\alpha kz^{\beta}}{a}\right)^{-s}(\mu)_{k}\frac{z^{k}}{k!}\right)_{v}. \tag{15}$$

Using the following result in (15):

$$(1+z)^{-\lambda}=\sum_{n=0}^{\infty}\frac{(\lambda)_{n}(-z)^{n}}{n!},\ |z|<1, \tag{16}$$

we get

$$\left(z^{\rho}\phi_{\mu}^{\alpha,\ \beta}(z,\ s,\ a)\right)_{v}=a^{-s}\sum_{n,\ k=o}^{\infty}\frac{(s)_{n}(\mu)_{k}}{n!\ k!}\left(-\frac{\alpha k}{a}\right)^{n}\left(z^{\rho+k+\beta n}\right)_{v}, \tag{17}$$

where $\left|\frac{\alpha kz^{\beta}}{a}\right|<1$. Using (8) in (17), we get

$$\left(z^{\rho}\phi_{\mu}^{\alpha,\ \beta}(z,\ s,\ a)\right)_{v}=a^{-s}\sum_{n,\ k=o}^{\infty}\frac{(s)_{n}(\mu)_{k}}{n!\ k!}\left(-\frac{\alpha k}{a}\right)^{n}e^{-i\pi v}\frac{\Gamma(v-\rho-k-\beta n)}{\Gamma(-\rho-k-\beta n)}\times z^{\rho+k+\beta n-v}, \tag{18}$$

where $0\neq\left|\frac{\Gamma(v-\rho-k-\beta n)}{\Gamma(-\rho-k-\beta n)}\right|<\infty$.

After a little simplification, we arrive at the result (9).

To prove (11), we express the generalized Hurwitz-Lerch zeta function and Fox's H-function occurring in the left hand side of (11) in series form using (1) and (3), apply the result (16), collect the powers of z and then apply the result (8). Now, interpreting the contour integral in terms of H-function, we arrive at the desired result (11).

To prove (12), we express the generalized Hurwitz-Lerch zeta function and general class of functions occurring in the left hand side of (12) in series form using (1) and (5), apply the result (16), collect the powers of z, apply the result (8) and we arrive at the desired result (12).

To prove (13), we express the generalized Hurwitz-Lerch zeta function, general class of functions and Fox's H-function occurring in the left hand side of (13) in series form using (1), (3) and (5), apply the result (16), collect the powers of z and then apply the result (8). Now, interpreting the contour integral in terms of H-function, we arrive at the desired result (13).

To prove (14), from (9) we have

$$\left(\left(z^{\rho}\phi_{\mu}^{\alpha,\ \beta}(z,\ s,\ a)\right)_{v}\right)_{v'}=\frac{(-1)^{v}}{a^{s}}\sum_{n,\ k=0}^{\infty}\frac{(s)_{n}(\mu)_{k}\Gamma(v-\rho-k-\beta n)}{n!\ k!\ \Gamma(-\rho-k-\beta n)}\left(-\frac{\alpha k}{a}\right)^{n}\times\left(z^{\rho+k+\beta n-v}\right)_{v'}. \tag{19}$$

Using (8) in (19), we get

$$\left(\left(z^{\rho}\phi_{\mu}^{\alpha,\ \beta}(z,\ s,\ a)\right)_{v}\right)_{v'}=\frac{(-1)^{v+v'}}{a^{s}}\sum_{n,\ k=0}^{\infty}\frac{(s)_{n}(\mu)_{k}\Gamma(v+v'-\rho-k-\beta n)}{n!\ k!\ \Gamma(-\rho-k-\beta n)}\left(-\frac{\alpha kz^{\beta}}{a}\right)^{n}z^{\rho+k-v-v'}. \tag{20}$$

In a similar manner, we get

$$\left(\left(z^{\rho}\phi_{\mu}^{\alpha,\ \beta}(z,\ s,\ a)\right)_{v'}\right)_{v} = \frac{(-1)^{v+v'}}{a^{s}} \sum_{n,\ k=0}^{\infty} \frac{(s)_n(\mu)_k\Gamma(v+v'-\rho-k-\beta n)}{n!\ k!\ \Gamma(-\rho-k-\beta n)} \left(-\frac{\alpha k z^{\beta}}{a}\right)^{n} z^{\rho+k-v-v'} \tag{21}$$

and replacing v by $v+v'$ in (9), we get

$$\left(z^{\rho}\phi_{\mu}^{\alpha,\ \beta}(z,\ s,\ a)\right)_{v+v'} = \frac{(-1)^{v+v'}}{a^{s}} \sum_{n,\ k=0}^{\infty} \frac{(s)_n(\mu)_k\Gamma(v+v'-\rho-k-\beta n)}{n!\ k!\ \Gamma(-\rho-k-\beta n)} \left(-\frac{\alpha k z^{\beta}}{a}\right)^{n} z^{\rho+k-v-v'}. \tag{22}$$

Thus, (20), (21) and (22) prove (14).

Remark 2 Results like (14) may be obtained for the results (11), (12) and (13).

3. Special cases

In this section we mention some special cases of our main results.

(i) If we take $a=1,\ k=0$ and $n=0$ in (11), the generalized Hurwitz-Lerch zeta function reduces to the unity and we get in essence the results due to Goyal and Garg [4].

(ii) If we take $c=0,\ y=0$ and $a=1$ in (13), the generalized Hurwitz-Lerch zeta function reduces to the unity and we get in essence the results due to Kumar [9].

(iii) If we take $p=2,\ m=1,\ j=2,\ r=1,\ h_1=1,\ g_1=1,\ g_2=1,\ \tau=1,\ k=1,\ w=0,\ q=0,\ k_1=0,\ a_1=0,\ a_2=0,\ \beta=0,\ \delta=1,\ b_1=0$ and $\lambda=\frac{1}{\Gamma(d)}$ in (12), the general class of functions reduces to the Wright's generalized Bessel function [21] and we get in essence the following result:

$$\left(z^{\rho}\phi_{\mu}^{y,\ \sigma}(z,\ \xi,\ a)J_{d}^{\alpha}(\eta z^{\epsilon})\right)_{v} = \frac{(-1)^{v}}{a^{\xi}} \sum_{n,\ b\ y=0}^{\infty} \frac{(\xi)_y(\mu)_b(-\eta z^{\epsilon})^{n}}{y!\ b!\ n!\ \Gamma(d+\alpha n+1)} \times \left(-\frac{\gamma b z^{\sigma}}{a}\right)^{y} \frac{\Gamma(v-\rho-\sigma y-b-\epsilon n)}{\Gamma(-\rho-\sigma y-b-\epsilon n)} z^{\rho+b-v}, \tag{23}$$

where $J_{d}^{\alpha}(\eta z^{\epsilon})$ stands for the Wright's generalized Bessel function.

(iv) If we take $p=1,\ m=1,\ j=2,\ r=1,\ h_1=1,\ g_1=3/2,\ g_2=1,\ \tau=1,\ k=2,\ w=1,\ q=1,\ k_1=0,\ a_1=0,\ a_2=0,\ \alpha=1,\ \beta=1/2,\ \delta=1,\ b_1=1/2$ and $\lambda=\frac{1}{\Gamma(d)\Gamma(3/2)}$ in (12), the general class of functions reduces to the Struve's function [2] and we obtain in essence the following result:

$$\left(z^{\rho}\phi_{\mu}^{\gamma,\ \sigma}(z,\ \xi,\ a)H_{d}(\eta z^{\epsilon})\right)_{v} = \frac{(-1)^{v}}{a^{\xi}} \sum_{n,\ b\ y=0}^{\infty} \frac{(\xi)_y(\mu)_b(-1)^{n}}{y!\ b!\ \Gamma(\frac{3}{2}+n)\Gamma(d+\frac{3}{2}+n)} \times \left(-\frac{\gamma b z^{\sigma}}{a}\right)^{y} \left(\frac{\eta z^{\epsilon}}{2}\right)^{2n+d+1} \frac{\Gamma(v-\rho-\sigma y-b-2\epsilon n-\epsilon d-\epsilon)}{\Gamma(-\rho-\sigma y-b-2\epsilon n-\epsilon d-\epsilon)} z^{\rho+b-v}, \tag{24}$$

where $H_{d}(\eta z^{\epsilon})$ stands for the Struve's function.

(v) If we take $p=1,\ m=1,\ j=2,\ r=1,\ h_1=1,\ g_1=(\mu'+v'+3)/2,\ g_2=(\mu'-v'+3)/2,\ \tau=1,\ k=2,\ w=\mu',\ q=1,\ k_1=0,\ a_1=0,\ a_2=0,\ \alpha=1,\ \beta=-1,\ \delta=1,\ b_1=-1,\ d=1$ and $\lambda=2^{\mu'+1}/\{(\mu'+v'+1)(\mu'-v'+1)\}$ in (12), the general class of functions reduces to the Lommel's function [2] and we get in essence the following result:

$$\left(z^{\rho}\phi_{\mu}^{\gamma,\ \sigma}(z,\ \xi,\ a)s_{u',\ v'}(\eta z^{\epsilon})\right)_{v} = \frac{(-1)^{v}}{a^{\xi}}\frac{2^{u'+1}}{\Gamma(u'\pm v'+1)}\sum_{n,\ b\ y=0}^{\infty}\frac{(\xi)_{y}(\mu)_{b}(-1)^{n}}{y!\ b!\ \left(\frac{u'\pm v'+3}{2}\right)_{n}} \times\left(-\frac{\gamma b z^{\sigma}}{a}\right)^{y}\left(\frac{\eta z^{\epsilon}}{2}\right)^{2n+u'+1}\frac{\Gamma(v-\rho-\sigma y-b-2\epsilon n-\epsilon u'-\epsilon)}{\Gamma(-\rho-\sigma y-b-2\epsilon n-\epsilon u'-\epsilon)}z^{\rho+b-v}, \tag{25}$$

where $s_{u',\ v'}(\eta z^{\epsilon})$ stands for the Lommel's function.
(vi) If we take $p=-2,\ m=1,\ j=1,\ r=1,\ h_1=h,\ g_1=g,\ \tau=1,\ k=1,\ w=0,\ q=0,\ k_1=0,\ a_1=0,\ \beta=-1,\ \delta=1,\ b_1=-1$ and $\lambda=1/\Gamma(d)$ in (12), the general class of functions reduces to the generalized Mittag-Leffler function [19] and we obtain in essence the following result:

$$\left(z^{\rho}\phi_{\mu}^{\gamma,\ \sigma}(z,\ \xi,\ a)E_{\alpha,\ d}^{h,\ g}(\eta z^{\epsilon})\right)_{v} = \frac{(-1)^{v}}{a^{\xi}}\sum_{n,\ b\ y=0}^{\infty}\frac{(\xi)_{y}(\mu)_{b}(h)_{n}(\eta z^{\epsilon})^{n}}{y!\ b!\ (g)_{n}\Gamma(d+\alpha n)} \times\left(-\frac{\gamma b z^{\sigma}}{a}\right)^{y}\frac{\Gamma(v-\rho-\sigma y-b-\epsilon n)}{\Gamma(-\rho-\sigma y-b-\epsilon n)}z^{\rho+b-v}, \tag{26}$$

where $E_{\alpha,\ d}^{h,\ g}(\eta z^{\epsilon})$ stands for the generalized Mittag-Leffler function.
Remark 3 If we substitute $g=1$ in (26), the generalized Mittag-Leffler function reduces to the generalized Mittag-Leffler function $E_{\alpha,\ d}^{h}(\eta z^{\epsilon})$ introduced by Prabhakar [16].

If we substitute $h=1$ and $g=1$ in (26), the generalized Mittag-Leffler function reduces to the generalized Mittag-Leffler function $E_{\alpha,\ d}(\eta z^{\epsilon})$ introduced by Wiman [20].

If we substitute $h=1,\ g=1$ and $d=1$ in (26), the generalized Mittag-Leffler function reduces to the Mittag-Leffler function $E_{\alpha}(\eta z^{\epsilon})$ [5, 13]. (vii) If we take $p=-2,\ t=P,\ s=Q,\ r=1,\ d=1,\ \tau=1,\ k=1,\ w=0,\ q=0,\ k_m=0,\ a_j=0,\ b_1=-1,\ \alpha=1,\ \beta=-1,\ \delta=1,$ and $\lambda=1$ in (12), the general class of functions reduces to the generalized hypergeometric function [1] and we obtain in essence the following result:

$$\left(z^{\rho}\phi_{\mu}^{\gamma,\ \sigma}(z,\ \xi,\ a)_{P}F_{Q}(h_P;\ g_Q;\ \eta z^{\epsilon})\right)_{v} = \frac{(-1)^{v}}{a^{\xi}}\sum_{n,\ b\ y=0}^{\infty}\frac{(\xi)_{y}(\mu)_{b}\prod_{m=1}^{P}(h_m)_{n}(\eta z^{\epsilon})^{n}}{y!\ b!\prod_{j=1}^{Q}(g_j)_{n}\ n!} \times\left(-\frac{\gamma b z^{\sigma}}{a}\right)^{y}\frac{\Gamma(v-\rho-\sigma y-b-\epsilon n)}{\Gamma(-\rho-\sigma y-b-\epsilon n)}z^{\rho+b-v}, \tag{27}$$

where ${}_{P}F_{Q}(h_P;\ g_Q;\ \eta z^{\epsilon})$ stands for the generalized hypergeometric function.
Remark 4 Several other Special cases may be obtained from the results (11) and (13) by reducing the H-function to several special functions with the help of the known results available in [14] and [18].

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