FEKETE-SZEGÖ INEQUALITY FOR CERTAIN CLASSES OF CLOSE-TO-CONVEX FUNCTIONS

GAGANDEEP SINGH, GURCHARANJIT SINGH, HARJINDER SINGH

Abstract. Close-to-convex functions and quasi-convex functions are of great importance in geometric function theory. In the present investigation, the authors study the subclass $C_1$ of close-to-convex functions and the subclasses $C'$ and $C'_1$ of quasi-convex functions in the open unit disc $E = \{ z : |z| < 1 \}$. The sharp upper bounds of the functional $|a_3 - \mu a_2^2|$, $\mu$ real, for the functions of the form $f(z) = z + \sum_{n=3}^{\infty} a_n z^n$ belonging to these classes are provided. This work will pave the way to investigate the upper bound of the Fekete-Szegö functional for some other subclasses of close-to-convex and quasi-convex functions.

1. Introduction

Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the unit disc $E = \{ z : |z| < 1 \}$. Let $S$ be the class of functions of the form (1) which are analytic univalent in $E$.

We shall concentrate on the coefficient problem for the class $S$ and certain of its subclasses. In 1916, Bieberbach [3] proved that $|a_2| \leq 2$ for $f(z) \in S$ as a corollary to an elementary area theorem. He conjectured that, for each function $f(z) \in S$, $|a_n| \leq n$; equality holds for the Koebe function $k(z) = z/(1-z)^2$, which maps the unit disc $E$ onto the entire complex plane minus the slit along the negative real axis from $-\frac{1}{4}$ to $-\infty$. De Branges [5] solved the Bieberbach conjecture in 1984. The contribution of Löwner [10] in proving that $|a_3| \leq 3$ for the class $S$ was huge.

With the known estimates $|a_2| \leq 2$ and $|a_3| \leq 3$, it was natural to seek some relation between $a_3$ and $a_2^2$ for the class $S$. This thought prompted Fekete and Szegö [6] and they used Löwner’s method to prove the following well-known result for the class $S$:

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If \( f(z) \in S \), then
\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
3 - 4\mu & \text{if } \mu \leq 0, \\
1 + 2\exp\left(-\frac{2\mu}{1-\mu}\right) & \text{if } 0 \leq \mu \leq 1, \\
4\mu - 3 & \text{if } \mu \geq 1.
\end{cases}
\] (2)

The inequality (2) plays a very important role in determining estimates of higher coefficients for some subclasses of \( S \) (see Chichra [4], Babalola [2]).

Next, we define some subclasses of \( S \) and obtain analogous of (2).

We denote by \( S^* \) the class of univalent starlike functions \( g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in A \) and satisfying the condition
\[
\Re\left(\frac{zg'(z)}{g(z)}\right) > 0, \quad z \in E.
\] (3)

We denote by \( K \) the class of convex univalent functions \( h(z) = z + \sum_{n=2}^{\infty} c_n z^n \in A \) which satisfy the condition
\[
\Re\left(\frac{(zh'(z))'}{h'(z)}\right) > 0, \quad z \in E.
\] (4)

A function \( f(z) \in A \) is said to be close to convex if there exists a function \( g(z) \in S^* \) such that
\[
\Re\left(\frac{zf'(z)}{g(z)}\right) > 0, \quad z \in E.
\] (5)

The class of close to convex functions is denoted by \( C \) and was introduced by Kaplan [8], who showed that all close to convex functions are univalent. The immediate shoot of \( C \) are its following subclasses:

\[
C_1 = \left\{ f(z) \in A : \Re\left(\frac{zf'(z)}{h(z)}\right) > 0, \quad h(z) \in K, \quad z \in E \right\},
\] (6)

\[
C' = \left\{ f(z) \in A : \Re\left(\frac{(zf'(z))'}{g'(z)}\right) > 0, \quad g(z) \in S^*, \quad z \in E \right\},
\] (7)

\[
C'_1 = \left\{ f(z) \in A : \Re\left(\frac{(zf'(z))'}{h'(z)}\right) > 0, \quad h(z) \in K, \quad z \in E \right\}.
\] (8)

Some specific examples for the functions belonging to the classes \( C, C_1, C' \) and \( C'_1 \) are
\[
f(z) = \frac{z}{(1-z)^2},
\]
\[
f_1(z) = \int_0^z \left(1 + \frac{19\sqrt{2}}{3} - \frac{15\sqrt{2}}{3} z^2 \right)^{\frac{1}{2}} dz - 1,
\]
\[
f_2(z) = \int_0^z \left(1 + \frac{29}{3\sqrt{3}} z\right)^{\frac{24}{44}} - 1 \right] dz
\]
\[
\text{and}
\]
\[
f_3(z) = \int_0^z \left(1 + \frac{19}{3\sqrt{3}} z\right)^{\frac{28}{75}} - 1 \right] dz
\]

Abdel Gawad and Thomas [1] investigated the class \( C_1 \) and also obtained (2) for \(-\infty < \mu \leq 1 \) (although this result seems to be doubtful).
Let $U$ be the class of analytic bounded functions of the form

$$w(z) = \sum_{n=1}^{\infty} d_n z^n, \quad z \in E,$$

and satisfying the conditions $w(0) = 0$, $|w(z)| < 1$. It is known (see [11]) that

$$|d_1| \leq 1, \quad |d_2| \leq 1 - |d_1|^2. \quad (10)$$

We shall apply the subordination principle due to Rogosinski [12], which states that if $f(z) \prec F(z)$, then $f(z) = F(w(z))$, $w(z) \in U$ (where $\prec$ stands for subordination).

Hummel [7] proved a conjecture of V. Singh that $|c_3 - c_2^2| \leq \frac{1}{3}$ for the class $K$. Keogh and Merkes [9] obtained the estimates (2) for the classes $S^*$, $K$ and $C$. Estimates (2) for the classes $C_1$, $C'$ and $C_1'$ have been waiting to be determined for the last 60 years.

**Lemma 1** Let $g(z) \in S^*$. Then

$$|b_3 - 3\mu b_2^2| \leq \begin{cases} 3(1 - \mu) & \text{if } \mu \leq \frac{2}{3}, \\ 1 & \text{if } \frac{2}{3} \leq \mu \leq \frac{4}{3}, \\ 3(\mu - 1) & \text{if } \mu \geq \frac{4}{3}. \end{cases}$$

This lemma is a direct consequence of the result of Keogh and Merkes [9] which states that for $g(z) \in S^*$,

$$|b_3 - \mu b_2^2| \leq \begin{cases} 3 - 4\mu & \text{if } \mu \leq \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} \leq \mu \leq 1, \\ 4\mu - 3 & \text{if } \mu \geq 1. \end{cases}$$

**Lemma 2** Let $h(z) \in K$. Then

$$|c_3 - \frac{3\mu}{4} c_2^2| \leq \begin{cases} 1 - \frac{3}{2} \mu & \text{if } \mu \leq \frac{8}{9}, \\ \frac{1}{3} & \text{if } \frac{8}{9} \leq \mu \leq \frac{16}{9}, \\ \frac{3}{4} \mu - 1 & \text{if } \mu \geq \frac{16}{9}. \end{cases}$$

This lemma is a direct consequence of a result of Keogh and Merkes [9], which states that for $h(z) \in K$,

$$|c_3 - \mu c_2^2| \leq \begin{cases} 1 - \mu & \text{if } \mu \leq \frac{2}{3}, \\ \frac{1}{3} & \text{if } \frac{2}{3} \leq \mu \leq \frac{4}{3}, \\ \mu - 1 & \text{if } \mu \geq \frac{4}{3}. \end{cases}$$

Unless mentioned otherwise, throughout the paper we assume the following notations:

$w(z) \in U$, $z \in E$.

For $0 < c < 1$, we write $w(z) = z(\frac{c + z}{1 + cz})$ so that $\frac{1 + w(z)}{1 - w(z)} = 1 + 2cz + 2z^2 + \ldots$, $z \in E$. 

2. Main Results

**Theorem 1** Let \( f(z) \in C' \). Then

\[
|a_3 - \mu a_2^2| \leq \begin{cases}
\frac{19}{9} - \frac{9\mu}{4} & \text{if } \mu \leq \frac{16}{27}, \\
\frac{64}{81\mu} - \frac{5}{9} & \text{if } \frac{16}{27} \leq \mu \leq \frac{2}{3}, \\
\frac{5}{9} + \frac{(8 - 9\mu)^2}{81\mu} & \text{if } \frac{2}{3} \leq \mu \leq \frac{8}{9}, \\
\frac{5}{9} + \frac{(9\mu - 9)^2}{16 - 9\mu} & \text{if } \frac{8}{9} \leq \mu \leq \frac{32}{27}, \\
\frac{5\mu}{9} - \frac{7}{9} & \text{if } \frac{32}{27} \leq \mu \leq \frac{4}{3}, \\
\frac{9\mu}{4} - \frac{19}{9} & \text{if } \mu \geq \frac{4}{3}.
\end{cases}
\] (11)

These results are sharp.

**Proof.** By definition of \( C' \),

\[
\frac{(zf'(z))'}{g'(z)} = \frac{1 + w(z)}{1 - w(z)},
\]

which on expansion yields

\[
1 + 4a_2z + 9a_3z^2 + \cdots = (1 + 2b_2z + 3b_3z^2 + \cdots)(1 + 2d_1z + 2(d_2 + d_1^2)z^2 + \cdots).
\]

Identifying terms in the above expansion,

\[
a_2 = \frac{1}{2}(b_2 + d_1),
\]

\[
a_3 = \frac{b_3}{3} + \frac{4}{9}b_2d_1 + \frac{2}{9}(d_2 + d_1^2).
\]

From (12) and (13) and using (10), it is easily established that

\[
|a_3 - \mu a_2^2| \leq \frac{1}{3}[|b_3 - 3\mu b_2^2| + \frac{1}{18}|8 - 9\mu||b_2||d_1| + \frac{1}{36}(8(1 - |d_1|^2) + |8 - 9\mu||d_1||b_2|). \quad (14)
\]

\[
|a_3 - \mu a_2^2| \leq \frac{2}{9} + \frac{1}{3}[b_3 - 3\mu b_2^2] + \frac{1}{18}|8 - 9\mu|xy + \frac{1}{36}(|8 - 9\mu| - 8)x^2, \quad (15)
\]

where \( x = |d_1| \leq 1 \) and \( y = |b_2| \leq 2 \).

**Case I.** Suppose that \( \mu \leq \frac{2}{3} \). By Lemma 1, (15) can be written as

\[
|a_3 - \mu a_2^2| \leq \frac{2}{9} + (1 - \mu) + \frac{1}{9}(8 - 9\mu)x - \frac{\mu}{4}x^2 = H_0(x), \quad \text{say},
\]

and

\[
H'_0(x) = \frac{1}{9}(8 - 9\mu) - \frac{\mu}{2}x, \quad H''_0(x) = -\frac{\mu}{2}.
\]

**Subcase I(i).** For \( \mu \leq 0 \), since \( x \geq 0 \) we have \( H'_0(x) > 0 \). \( H_0(x) \) is an increasing function in \([0, 1]\) and \( \max H_0(x) = H_0(1) = \frac{19}{9} - \frac{9\mu}{4} \).

**Subcase I(ii).** Suppose \( 0 < \mu \leq \frac{2}{3} \). \( H'_0(x) = 0 \) when \( x = \frac{2(8-9\mu)}{9\mu} = x_0 \).

\( x_0 > 1 \) if and only if \( \mu < \frac{16}{27} \) and we have \( \max H_0(x) = H_0(1) = \frac{19}{9} - \frac{9\mu}{4} \).

Combining the above two subcases, we obtain first result of (11).
Subcase I(iii). For $\frac{16}{27} \leq \mu \leq \frac{2}{3} (x_0 < 1), since \ H''_0 (x) < 0, therefore we have
\[
\max H_0 (x) = H_0 (x_0) = \frac{64}{81\mu} - \frac{5}{9}.
\]

Case II. Suppose that $\frac{2}{3} \leq \mu \leq \frac{5}{9}, then by Lemma 1, (15) takes the form
\[
|a_3 - \mu a_2 | \leq \frac{2}{9} + \frac{1}{3} + \frac{1}{9} (8 - 9\mu) x - \frac{\mu}{4} x^2.
\]

Subcase II(i). $\frac{2}{3} < \mu < \frac{5}{9}.
Under the above condition, from (13), we get
\[
|a_3 - \mu a_2 | \leq \frac{2}{9} + \frac{1}{3} + \frac{1}{9} (8 - 9\mu) x - \frac{\mu}{4} x^2 = H_1 (x), \text{ say}.
\]
\[
H'_1 (x) = \frac{1}{9} (8 - 9\mu) - \frac{\mu}{2} x, H''_1 (x) = -\frac{\mu}{2} < 0
\]
\[
H'_1 (x) = 0 \text{ implies that } x = \frac{2(8 - 9\mu)}{9\mu} = x_1 \text{ and } \max H_1 (x_1) = \frac{5}{9} + \frac{(8 - 9\mu)^2}{8\mu}.
\]

Subcase II(ii). For $\frac{8}{9} \leq \mu \leq \frac{32}{27}$, by Lemma 1, (15) reduces to
\[
|a_3 - \mu a_2 | \leq \frac{5}{9} + (9\mu - 8) x + \frac{16 - 9\mu}{36} x^2 = H_2 (x), \text{ say}.
\]
\[
H'_2 (x) = (9\mu - 8) - \frac{1}{18} (9\mu - 16) x, H''_2 (x) < 0.
\]

Subcase II(iii). $\frac{32}{27} \leq \mu \leq \frac{4}{3}$. (19) can be expressed as
\[
|a_3 - \mu a_2 | \leq \frac{5}{9} + \frac{1}{9} (9\mu - 8) x - \frac{16 - 9\mu}{36} x^2 = H_3 (x), \text{ say}.
\]
\[
H'_3 (x) = \frac{1}{9} (9\mu - 8) - \frac{1}{18} (16 - 9\mu) x.
\]
\[
H'_3 (x) = 0 \text{ yields } x = \frac{2(9\mu - 8)}{16 - 9\mu} = x_3 \geq 1 \text{ and } \max H_3 (x) = H_3 (1) = \frac{5\mu}{4} - \frac{7}{9}.
\]

Case III. $\mu \geq \frac{4}{3}$. By Lemma 1, (15) can be put in the form
\[
|a_3 - \mu a_2 | \leq \frac{2}{9} + (\mu - 1) + \frac{1}{9} (9\mu - 8) x - \frac{16 - 9\mu}{36} x^2 = H_4 (x), \text{ say}.
\]
\[
H'_4 (x) = \frac{1}{9} (9\mu - 8) - \frac{1}{18} (16 - 9\mu) x
\]
which vanishes at $x = \frac{2(9\mu - 8)}{16 - 9\mu} = x_4 \geq 1$ and therefore $\max H_4 (x) = H_4 (1) = \frac{9\mu}{4} - \frac{10\mu}{9}.
\]

The first and second inequalities of (11) coincide at $\mu = \frac{16}{27}$ and each is equal to $\frac{7}{9}$.
The second and third inequalities of (11) coincide at $\mu = \frac{2}{3}$ and each is equal to $\frac{17}{27}$.
The third and fourth inequalities of (11) coincide at $\mu = \frac{5}{9}$ and each is equal to $\frac{2}{9}$.
The fourth and fifth inequalities of (11) coincide at $\mu = \frac{32}{27}$ and each is equal to $\frac{19}{27}$.
The fifth and last inequalities of (11) coincide at $\mu = \frac{4}{3}$ and each is equal to $\frac{8}{9}$.
Proof. Proceeding as in Theorem 1, we have

\[ f_1(z) = \frac{1}{z} \left( \int_0^z \frac{(e^{t} + 1 + 2t + 2t^2 + \cdots)}{(1-t)^3} \, dt \right). \]

\[ f'_2(z) = \frac{1}{z} \left[ \left( \int_0^z \frac{(e^{t} + 1 + 2t + 2t^2 + \cdots)}{(1-t)^3} \, dt \right) \right] \text{ where } c = \frac{2(8-9\mu)}{9\mu}. \]

\[ f'_3(z) = \frac{1}{z} \left[ \left( \int_0^z \frac{(e^{t} + 1 + 2t + 2t^2 + \cdots)}{(1-t)^3} \, dt \right) \right] \text{ where } d = \frac{2(8-9\mu)}{9\mu}. \]

\[ f'_4(z) = \frac{1}{z} \left[ \left( \int_0^z \frac{(e^{t} + 1 + 2t + 2t^2 + \cdots)}{(1-t)^3} \, dt \right) \right] \text{ where } e = \frac{2(9\mu-8)}{16-9\mu}. \]

\[ f'_5(z) = \frac{1}{z} \left[ \left( \int_0^z [1 + \frac{29}{3\sqrt{6}} \frac{|t|}{a^2}] \, dt \right) \right] \text{ where } |t| < \frac{3\sqrt{5}}{29}. \]

Subcase I(i). Suppose that \( \mu \leq \frac{4}{9} \). By Lemma 2, and putting \( x = |d_1| \leq 1 \) and \( y = |c_2| \leq 1 \), (16) reduces to

\[ |a_3 - \mu a_2^2| \leq \begin{cases} 1 - \mu & \text{if } \mu \leq \frac{4}{9}, \\ \frac{16}{81\mu} + \frac{1}{9} & \text{if } \frac{4}{9} \leq \mu \leq \frac{8}{9}, \\ \frac{1}{3} + \frac{(9\mu - 8)^2}{36(16 - 9\mu)} & \text{if } \frac{8}{9} \leq \mu \leq \frac{4}{3}, \\ \frac{3\mu - 5}{9} & \text{if } \frac{4}{3} \leq \mu \leq \frac{16}{9}, \\ \mu - 1 & \text{if } \mu \geq \frac{16}{9}. \end{cases} \]

The results are sharp.

**Proof.** Proceeding as in Theorem 1, we have

\[ |a_3 - \mu a_2^2| \leq \frac{2}{9} + \frac{1}{3}|c_3 - \frac{3}{4}\mu c_2| + \frac{1}{18}|8 - 9\mu||c_2||d_1| + \frac{1}{18}(|8 - 9\mu| - 8)|d_1|^2. \tag{16} \]

**Case I.** Suppose that \( \mu \leq \frac{8}{9} \). By Lemma 2, and putting \( x = |d_1| \leq 1 \) and \( y = |c_2| \leq 1 \), (16) reduces to

\[ |a_3 - \mu a_2^2| \leq \frac{2}{9} + \frac{1}{3} \left( 1 - \frac{3\mu}{4} \right) + \frac{1}{18} (8 - 9\mu)xy - \frac{\mu}{4} x^2 
= \left( \frac{5}{9} - \frac{\mu}{4} \right) + \frac{1}{18} (8 - 9\mu)x - \frac{\mu}{4} x^2 = H_6(x), \text{ say.} \]

Then

\[ H_6'(x) = \frac{8 - 9\mu}{18} - \frac{\mu}{2} x, \quad H_6''(x) = -\frac{\mu}{2}. \]

When \( H_6'(x) = 0 \), we have \( 8 - 9\mu = 9\mu x = 9\mu x_6 \), say.

**Subcase I(i).** For \( \mu \leq 0 \), since \( x \geq 0 \) we have \( H_6'(x) \geq 0 \). Suppose \( \mu > 0 \). Since \( x \leq 1 \), \( H_6'(x) \geq 4/9 - \mu > 0 \) if and only if \( \mu < 4/9 \). Then for \( \mu < 4/9 \), we have \( H_6(x) \leq H_6(1) = 1 - \mu \).

**Subcase I(ii).** Suppose that \( \frac{4}{9} \leq \mu \leq \frac{8}{9} \). Then \( \max H_6(x) = H_6(x_6) = 16/81\mu + 1/9 \).
Case II. Suppose that \( \frac{8}{9} \leq \mu \leq \frac{15}{9} \). By Lemma 2 and (16),

\[
|a_3 - \mu a_2^2| \leq \frac{1}{3} + \frac{1}{18} (9\mu - 8)x - \frac{1}{36} (16 - 9\mu)x^2 = H_7(x), \text{ say.}
\]

Then \( H_7(x) = 0 \) when \( x = (9\mu - 8)/(16 - 9\mu) = x_7, \text{ say, and } H_7''(x) = -(16 - 9\mu)/18 < 0. \) Since \( x_7 \leq 1, \) this is relevant only for \( \mu \leq \frac{4}{3}. \)

Subcase II(i). Suppose that \( \frac{8}{9} \leq \mu \leq \frac{4}{3}. \) Then

\[
\max H_7(x) = H_7(x_7) = \frac{1}{3} + \frac{(9\mu - 8)^2}{36(16 - 9\mu)}.
\]

Subcase II(ii). If \( \frac{4}{3} \leq \mu \leq \frac{15}{9}, \) then \( H_7'(x) \geq 0, \) so \( H_7(x) \) is a monotonically increasing function of \( x \) and \( \max H_7(x) = H_7(1) = 3\mu/4 - 5/9. \)

Case III. Suppose that \( \mu \geq \frac{16}{9}. \) By Lemma 2, from (16),

\[
|a_3 - \mu a_2^2| \leq \frac{2}{9} + \frac{1}{3} \left( \frac{3\mu}{4} - 1 \right) + \frac{1}{18} (9\mu - 8)x + \frac{1}{36} (9\mu - 16)x^2 = H_8(x), \text{ say.}
\]

We have \( H_8'(x) > 0 \) and \( \max H_8(x) = H_8(1) = \mu - 1. \)

This completes the proof.

Extremal function \( f_1(z) \) for the first and the last results is defined by \( f_1'(z) = \frac{1}{z} \left[ \left( \int_0^z \frac{(1+t)}{(1-t)^2} \, dt \right) \right]. \)

Extremal function \( f_2(z) \) for the second bound is defined by \( f_2'(z) = \frac{1}{z} \left[ \left( \int_0^z \frac{(1+2+t+2+t^2+...)}{(1-t)^2} \, dt \right) \right], \)

where \( c = \frac{(8 - 9\mu)}{9\mu}. \)

Extremal function \( f_3(z) \) for the third bound is defined by \( f_3'(z) = \frac{1}{z} \left[ \left( \int_0^z \frac{(1+2+t+2+t^2+...)}{(1-t^2)^2} \, dt \right) \right], \)

where \( c = \frac{(9\mu - 8)}{16 - 9\mu}. \)

Extremal function \( f_4(z) \) for the fourth bound is defined by \( f_4'(z) = \frac{1}{z} \left[ \left( \int_0^z \frac{1 + \frac{10t}{3\sqrt{3}}}{3\sqrt{3}} \, dt \right) \right], \)

\( |t| \leq \frac{3\sqrt{3}}{9}. \)

Proceeding as in Theorem 2 and using elementary calculus, we can easily prove the following theorem.

**Theorem 3** Let \( f(z) \in C_1. \) Then

\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{5}{3} - \frac{9\mu}{4} & \text{if } \mu \leq \frac{2}{9}, \\
\frac{2}{3} + \frac{1}{9\mu} & \text{if } \frac{2}{9} \leq \mu \leq \frac{2}{3}, \\
\frac{1 - \mu}{4} + \frac{(3\mu - 2)^2}{12(4 - 3\mu)} & \text{if } \frac{2}{3} \leq \mu \leq \frac{8}{9}, \\
\frac{7}{9} + \frac{(3\mu - 2)^2}{12(4 - 3\mu)} & \text{if } \frac{8}{9} \leq \mu \leq \frac{10}{9}, \\
\frac{7}{9} + 2(\mu - 1) & \text{if } \frac{10}{9} \leq \mu \leq \frac{16}{9}, \\
\frac{9\mu}{4} - \frac{5}{3} & \text{if } \mu \geq \frac{16}{9}.
\end{cases}
\]
The results are sharp. Extremal function $f_1(z)$ for the first and the last result is defined by $f_1(z) = \left[ \int_0^z \frac{(1+t)(1-t)}{(1-t)^2} \, dt \right]$. Extremal function $f_2(z)$ for the second bound is defined by $f_2(z) = \left[ \int_0^z \frac{(1+2ct+2t^2+\ldots)}{(1-t)} \, dt \right]$, where $c = \frac{(2-3\mu)}{4\mu}$. Extremal function $f_3(z)$ for the third and fourth bound is defined by $f_3(z) = \left[ \int_0^z \frac{(1+2ct+2t^2+\ldots)}{(1-t)} \, dt \right]$, where $c = \frac{(3\mu-2)}{2(4-3\mu)}$. Extremal function $f_4(z)$ for the fifth bound is defined by $f_4(z) = \left[ \int_0^z (1 + \frac{10\sqrt{2}t^3}{3}) \, dt \right]$, where $|t| \leq \frac{3}{10\sqrt{2}}$.

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GAGANDEEP SINGH
DEPARTMENT OF MATHEMATICS, KHALSA COLLEGE, AMRITSAR, PUNJAB, INDIA
E-mail address: kamboj.gagandeep@yahoo.in

GURCHARANJIT SINGH
DEPARTMENT OF MATHEMATICS, G.N.D.U. COLLEGE, CHUNGH(TARN TARAN), PUNJAB, INDIA
E-mail address: dhillongs82@yahoo.com

HARJINDER SINGH
DEPARTMENT OF MATHEMATICS, GOVT. COLLEGE, MOHALI, PUNJAB, INDIA
E-mail address: harjindpreet@gmail.com