ON A NEW SUBCLASS OF ANALYTIC FUNCTIONS INVOLVING KOMATU INTEGRAL OPERATOR

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Abstract. The object of the paper is to study some properties for \( K_c^\delta f(z) \) belonging to some class by applying Jack’s lemma.

1. Introduction

Let \( A \) be denote the class of all analytic functions \( f \) of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad n \in \mathbb{N} = \{1, 2, 3, \ldots\}
\]

(1)

which are analytic in the punctured unit disc \( E = \{ z \in \mathbb{C} : 0 < |z| < 1 \} \). Let \( S \) denote the subclass of \( A \) which consists of functions of the form (1) that are univalent and normalized by conditions \( f(0) = 0 \) and \( f'(0) = 1 \) in \( E \).

Recently Komatu [6] introduced a certain integral operator \( K_c^\delta f(z) \)

\[
K_c^\delta f(z) = c^\delta \frac{1}{\Gamma(\delta)} \int_0^1 t^{c-2} \left( \log \frac{1}{t} \right)^{c-1} f(tz) \, dt,
\]

(2)

\( c > 0, \delta \geq 0 \) and \( z \in E \).

Thus, if \( f \in A \) is of the form (1) then it is easily seen from (2) that

\[
K_c^\delta f(z) = z + \sum_{n=2}^{\infty} \left( \frac{c}{c+n-1} \right)^\delta a_n z^n, \quad a > 0, \delta \geq 0.
\]

(3)

We note that

(i). for \( c = 1 \) and \( \delta = k(k \text{ is any integer}), \) the multiple transformation \( K_1^\delta f(z) = I^k f(z) \) was studied by Flett [1].

(ii). for \( c = 1 \) and \( \delta = -k(k \in \mathbb{N}_0) \), the differential operator \( K_1^{-k} f(z) = D^k f(z) \) was studied by Salagean [7].

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(iii). for $c = 2$ and $\delta = k (k$ is any integer), the operators $K^k_\delta f(z) = K^k f(z)$ was studied by Uralegaddi and Somanatha [9].

(iv). for $c = 2$, the multiple transformation $K^2_\delta f(z) = K^3 f(z)$ was studied by Jung et al. [3].

In the following definition, we introduce a new class of analytic functions containing a integral operator of equation (3).

**Definition 1.1.** Let a function $f \in A$. Then $f \in K^c_\delta f(z)$ if and only if

$$\text{Re} \left\{ \frac{z (K^c_\delta f(z))'}{K^c_\delta f(z)} \right\} > \beta, z \in E, 0 \leq \beta \leq 1. \quad (4)$$

Let $f$ and $g$ be analytic in $E$. Then $f$ is said to be subordinate to $g$ if there exists an analytic function $!$ satisfying $! (0) = 0$ and $! (z) < 1$, such that $f(z) = g(!z)$, $z \in E$. We denote this subordination as $f(z) \prec g(z)$ or $(f \prec g)$, $z \in E$.

The basic idea in proving our result is the following lemma due to Jack [2] (also, due to Miller and Mocanu [4])

**Lemma 1.2.** Let $\omega(z)$ be analytic in $E$ with $\omega(0) = 0$. Then if $|\omega(z)|$ attains its maximum value on the circle $|z| = r$ at a point $z_0$ then we have $z_0 \omega'(z) = k \omega(z_0)$, where $k \geq 1$ is a real number.

**2. Main Results**

In the present paper, we follow similar works done by Shireishi and Owa [8] and Ochiai et al. [5], we derive the following result.

**Theorem 2.1.** If $f \in A$ satisfies

$$\text{Re} \left\{ \frac{z (K^c_\delta f(z))'}{K^c_\delta f(z)} \right\} < \frac{\beta - 3}{2(\beta - 1)}, z \in E$$

for some $\beta (-1 < \beta \leq 0)$ then

$$K^c_\delta f(z) < \frac{1 + \beta z}{1 - z}, z \in E.$$ 

This implies that

$$\text{Re} \left\{ \frac{K^c_\delta f(z)}{z} \right\} > \frac{1 - \beta}{2}$$

**Proof.** Let us define the function $\omega(z)$ by

$$K^c_\delta f(z) = \frac{1 - \beta \omega(z)}{1 - \omega(z)}, (\omega(z) \neq 1).$$

Clearly, $\omega(z)$ is analytic in $E$ and $\omega(0) = 0$. We want to prove that $|\omega(z)| < 1$ in $E$. Since

$$z \left( K^c_\delta f(z) \right)' = \frac{-\beta z \omega'(z)}{1 - \beta \omega(z)} + \frac{z \omega'(z)}{1 - \omega(z)} + 1,$$

we see that

$$\text{Re} \left\{ \frac{z \left( K^c_\delta f(z) \right)'}{K^c_\delta f(z)} \right\} = \text{Re} \left\{ \frac{-\beta z \omega'(z)}{1 - \beta \omega(z)} + \frac{z \omega'(z)}{1 - \omega(z)} + 1 \right\} < \frac{\beta - 3}{2(\beta - 1)}, (z \in E)$$
for $-1 < \beta \leq 0$. If there exists a point $z_0 \in E$ such that
\[
\max_{|z| \leq |z_0|} |\omega(z)| = |\omega(z_0)| = 1,
\]
then Lemma 1.2, gives us that $\omega(z_0) = e^{i\theta}$ and $z_0\omega'(z_0) = k\omega(z_0), k \geq 1$.

Thus we have
\[
\frac{z_0 (K_c^\beta f(z_0))'}{K_c^\beta f(z_0)} = -\beta z_0 \omega'(z_0) + \frac{z_0 \omega'(z_0)}{1 - \beta \omega(z_0)} + 1
\]
\[
= 1 + \frac{k}{1 - e^{i\theta}} - \frac{k}{1 - e^{i\theta}}.
\]

It follows that
\[
Re \left\{ \frac{1}{1 - \omega(z_0)} \right\} = Re \left\{ \frac{1}{1 - e^{i\theta}} \right\} = \frac{1}{2}
\]
and
\[
Re \left\{ \frac{1}{1 - \beta \omega(z_0)} \right\} = Re \left\{ \frac{1}{1 - \beta e^{i\theta}} \right\} = \frac{1}{2} - \frac{1 - \beta^2}{2(1 + \beta^2 - 2\beta \cos \theta)}.
\]

Therefore, we have
\[
Re \left\{ \frac{z_0 (K_c^\beta f(z_0))}{K_c^\beta f(z_0)} \right\} = 1 - \frac{k(\beta^2 - 1)}{2(1 + \beta^2 - 2\beta \cos \theta)}.
\]

This implies that $-1 < \beta \leq 0$,
\[
Re \left\{ \frac{z_0 (K_c^\beta f(z_0))'}{K_c^\beta f(z_0)} \right\} \geq 1 + \frac{1 - \beta^2}{2(\beta - 1)^2} = \frac{\beta - 3}{2(\beta - 1)}.
\]

This contradicts the condition in the theorem. Then there is no $z_0 \in E$ such that $|\omega(z_0)| = 1$ for all $z \in E$, that is
\[
\frac{K_c^\beta f(z)}{z} < \frac{1 + \beta z}{1 - z}, \quad z \in E.
\]

Further more, since
\[
\omega(z) = \frac{K_c^\beta f(z)}{K_c^\beta f(z)} - \beta, \quad z \in E
\]
and $|\omega(z)| < 1$, $(z \in E)$, we conclude that
\[
Re \left\{ \frac{K_c^\beta f(z)}{z} \right\} > \frac{1 - \beta}{2}.
\]

Taking $\beta = 0$ in the Theorem 2.1, we have the following corollary.

**Corollary 2.2.** If $f \in A$ satisfies
\[
Re \left\{ \frac{z (K_c^\beta f(z))'}{K_c^\beta f(z)} \right\} > \frac{3}{2}, \quad z \in E
\]
then
\[
\frac{K_c^\beta f(z)}{z} < \frac{1}{1 - z}, \quad z \in E
\]
and
\[
Re \left\{ \frac{K_c^\beta f(z)}{z} \right\} > \frac{1}{2}, \quad z \in E
\]
Theorem 2.3. If $f \in A$ satisfies
\[
\Re \left\{ \frac{z(K_c^β f(z))'}{K_c^β f(z)} \right\} > \frac{3\beta - 1}{2(\beta - 1)}', \ z \in E
\]
for some $\beta (-1 < \beta \leq 0)$ then
\[
\frac{z}{K_c^β f(z)} < \frac{1 + z}{1 - z}, \ z \in E
\]
and
\[
\left| \frac{K_c^β f(z)}{z} - \frac{1}{1 - \beta} \right| < \frac{1}{1 - \beta}, \ z \in E.
\]
This implies that $\Re \left\{ \frac{K_c^β f(z)}{z} \right\} > 0$, $z \in E$.

Proof. Let us define the function $\omega(z)$ by
\[
\frac{z}{K_c^β f(z)} = \frac{1 - \beta \omega(z)}{1 - \omega(z)}, \ \omega(z) \neq 1. \tag{5}
\]
Then, we have $\omega(z)$ is analytic in $E$ and $\omega(0) = 0$. We want to prove that $|\omega(z)| < 1$ in $E$. Differenting equation (5), we obtain
\[
\frac{z(K_c^β f'(z))}{K_c^β f(z)} = \frac{-z \omega'(z)}{1 - \omega(z)} + \frac{\beta \omega'(z)}{1 - \beta \omega(z)} + 1
\]
\[
\Rightarrow \Re \left\{ \frac{z(K_c^β f'(z))}{K_c^β f(z)} \right\} = \Re \left\{ \frac{-z \omega'(z)}{1 - \omega(z)} + \frac{\alpha \omega'(z)}{1 - \beta \omega(z)} + 1 \right\}
\]
\[
> \frac{3\beta - 1}{2(\beta - 1)}, \ z \in E,
\]
for $(-1 < \beta \leq 0)$. If there exists a point $(z_0 \in E)$ such that Lemma 1.2, gives us that $\omega(z_0) = e^{i\theta}$ and $z_0 \omega'(z_0) = k \omega(z_0), k \geq 1$. Thus we have
\[
\frac{z_0(K_c^β f(z_0))'}{K_c^β f(z_0)} = \frac{-z_0 \omega'(z_0)}{1 - \omega(z_0)} + \frac{\beta z_0 \omega'(z_0)}{1 - \beta \omega(z_0)} + 1
\]
\[
= 1 - \frac{k}{1 - e^{i\theta}} + \frac{k}{1 - \beta e^{i\theta}}.
\]
Therefore, we have
\[
\Re \left\{ \frac{z_0(K_c^β f(z_0))'}{K_c^β f(z_0)} \right\} = 1 + \frac{k(\beta^2 - 1)}{2(1 + \beta^2 - 2\beta \cos \theta)}.
\]
This implies that, for $-1 < \alpha \leq 0$,
\[
\Re \left\{ \frac{z_0(K_c^β f(z_0))'}{K_c^β f(z_0)} \right\} = 1 - \frac{k(1 - \alpha^2)}{2(1 + \alpha^2 - 2\alpha \cos \theta)}
\]
\[
\leq \frac{3\alpha - 1}{2(\alpha - 1)}.
\]
This contradicts the condition in the theorem.
Hence, there is no $z_0 \in E$ such that $|\omega(z_0)| = 1$ for all $z \in E$, that is
\[
\frac{z}{K_c^β f(z)} < \frac{1 + z}{1 - z}, \ z \in E.$$
Furthermore, since

\[
\omega(z) = 1 - \frac{K_c^\delta f(z)}{1 - \beta K_c^\delta f(z)}, \quad z \in E
\]

and \(|\omega(z)| < 1, (z \in E)\) we conclude that

\[
\left| \frac{K_c^\delta f(z)}{z} - \frac{1}{1 - \beta} \right| < \frac{1}{1 - \beta}, \quad z \in E
\]

which implies that

\[
\text{Re} \left\{ \frac{K_c^\delta f(z)}{z} \right\} > 0, \quad z \in E.
\]

We complete the proof of the theorem. \(\square\)

By setting \(\beta = 0\) in Theorem 2.3, we readily obtain the following

**Corollary 2.4.** If \(f \in A\) satisfies

\[
\text{Re} \left\{ \frac{z (K_c^\delta f(z))'}{K_c^\delta f(z)} \right\} > \frac{1}{2}, \quad z \in E
\]

then

\[
\frac{z}{K_c^\delta f(z)} < \frac{1 + z}{1 - z}, \quad z \in E
\]

and

\[
\left| \frac{K_c^\delta f(z)}{z} - 1 \right| < 1, \quad z \in E.
\]

**Theorem 2.5.** If \(f \in A\) satisfies

\[
\text{Re} \left\{ \frac{z (K_c^\delta f(z))'}{K_c^\delta f(z)} \right\} < \frac{\beta (2 - \gamma) - (2 + \gamma)}{2(\beta - 1)}, \quad z \in E
\]

for some \(\beta (-1 < \beta \leq 0)\) and \(0 < \beta \leq 1\) then

\[
\left( \frac{K_c^\delta f(z)}{z} \right)^\frac{1}{\gamma} < \frac{1 + \beta z}{1 - z}, \quad z \in E.
\]

Then implies that

\[
\left( \frac{K_c^\delta f(z)}{z} \right)^\frac{1}{\gamma} > \frac{1 - \beta}{2}, \quad z \in E.
\]

**Proof.** Let us define the function \(\omega(z)\) by

\[
\frac{K_c^\delta f(z)}{z} = \left( \frac{1 - \beta \omega(z)}{1 - \omega(z)} \right)^\gamma, \quad \omega(z) \neq 1.
\]

Clearly, \(\omega(z)\) is analytic in \(E\) and \(\omega(0) = 0\). We want to prove that \(|\omega(z)| < 1\) in \(E\). Since

\[
\frac{z (K_c^\delta f(z))'}{K_c^\delta f(z)} = \gamma \left( \frac{z \omega'(z)}{1 - \omega(z)} - \frac{\beta z \omega'(z)}{1 - \beta \omega(z)} \right) + 1.
\]
We see that
\[
\text{Re}\left\{ \frac{z(K^\delta f(z))'}{K^\delta f(z)} \right\} = \text{Re}\left\{ \gamma \left( \frac{z\omega'(z)}{1 - \omega(z)} - \frac{\beta z\omega'(z)}{1 - \beta \omega(z)} \right) + 1 \right\} < \frac{\beta(2 - \gamma) - (2 + \gamma)}{2(\beta - 1)}, \quad z \in E,
\]
for \(-1 < \beta \leq 0\) and \(0 < \gamma \leq 1\). If there exists a point \((z_0 \in E)\) such that
\[
\max_{|z|<|z_0|} |\omega(z)| = |\omega(z_0)| = 1
\]
then by Lemma 1.2, gives us that \(\omega(z_0) = e^{i\theta}\) and \(z_0\omega'(z_0) = k\omega(z_0), \ k \geq 1\).

Thus we have
\[
z_0 \left( \frac{K^\delta f(z_0)}{K^\delta f(z_0)} \right)' = \gamma \left( \frac{z_0\omega'(z_0)}{1 - \omega(z_0)} - \frac{\beta z_0\omega'(z_0)}{1 - \beta \omega(z_0)} \right) + 1
= 1 + \frac{k}{1 - e^{i\theta}} - \frac{k}{1 - \beta e^{i\theta}}.
\]

Therefore, we have
\[
\text{Re}\left\{ \frac{z_0 \left( K^\delta f(z_0) \right)'}{K^\delta f(z_0)} \right\} = 1 + \frac{\gamma k(1 - \beta^2)}{2(1 + \beta^2 - 2\beta \cos \theta)}.
\]

Thus implies that, for \(-1 < \beta \leq 0\) and \(0 < \gamma \leq 1\)
\[
\text{Re}\left\{ \frac{z_0 \left( K^\delta f(z_0) \right)'}{K^\delta f(z_0)} \right\} \geq \frac{\beta(2 - \gamma) - (2 + \gamma)}{2(\beta - 1)}.
\]

This contradicts the condition in the theorem.

Hence, there is no \(z_0 \in E\) such that \(|\omega(z_0)| = 1\) for all \(z \in E\), that is
\[\left( \frac{K^\delta f(z)}{z} \right)^{\frac{1}{2}} < \frac{1 - \beta z}{1 - z}, \quad z \in E.
\]
Furthermore, since
\[\omega(z) = \frac{\left( \frac{K^\delta f(z)}{z} \right)^{\frac{1}{2}} - 1}{\left( \frac{K^\delta f(z)}{z} \right)^{\frac{1}{2}} - \beta}\]
and \(|\omega(z)| < 1, \ (z \in E)\), we conclude that
\[\left( \frac{K^\delta f(z)}{z} \right)^{\frac{1}{2}} > \frac{1 - \beta}{2}, \quad z \in E,
\]
we complete the proof of the theorem.

\[\square\]

References

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