HANKEL DETERMINANT FOR BI-BAZELEVIC FUNCTION INVOLVING ERROR AND SIGMOID FUNCTION DEFINED BY DERIVATIVE CALCULUS VIA CHEBYSHEV POLYNOMIALS

I. T. AWOLERE

Abstract. In this investigation, a new subclass of bi-univalent function was introduced by means of differential operator involving error function and modified activation function via Chebyshev polynomials. The coefficients bounds and second Hankel determinant of this class were established using subordination principle.

1. Introduction

Let $H$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which are analytic in the open unit disk $E = \{z : z \in \mathbb{C}, |z| < 1\}$, which satisfying normalized condition $f(0) = f'(0) - 1 = 0$.

A function $f$ is said to be univalent in a domain $E$ if it is injective in $E$ and it is call a starlike domain with respect to origin denoted by $f \in S^*$. A function $f \in S$ convex function is one which map unit disk conformally onto a convex domain denoted by $K$. For a given $0 \leq \gamma < 1$, a function $f \in A$ is call a starlike of order $\gamma$, class of such function denoted by $S^*(\gamma)$ if and only if $\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \gamma$ and $z \in E$.

Furthermore, for $0 \leq \gamma < 1$, a function $f \in A$ is called convex function of order $\gamma$, class of such function denoted by $K(\gamma)$ if $\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \gamma$.

Fractional calculus is considered one of the important branches of mathematical analysis and enormous progress has taken place recently in the study of fractional integral and differential operators because various operators provides tools of considerable importance in geometric function theory. Different types of fractional integral and differential operators were introduced by different researchers such as Salagean operator, Ruscheweyh operator, Strivastava operator, Abdunaby el al.

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operator, e.t.c [see (3, 9, 12, 30, 37, 39)]. For details about the fractional calculus operator see [2, 22, 47].

Also, in recent time, several authors studies family of analytic function involving special functions such as activation, error function, Bessel function defined by $A$ to find different conditions such that the member of such functions have certain geometric properties like univalency, starlikeness and convexity. But concern of this paper is that of error function which was by normalized by Ramanchandran [38] as

$$\text{Erf}(z) = \frac{\sqrt{\pi}}{2} \text{erf}(\sqrt{z}) = z + \sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{(2k-1)(k-1)!} z^k$$

(2)

Originally before normalization the error function is given by [4] as

$$\text{erf}(z) = 2 \sqrt{\frac{\pi}{2}} \int_0^z e^{-t^2} dt = 2 \sqrt{\frac{\pi}{2}} \sum_{k=2}^{\infty} \frac{(-1)^{k-1} z^{2k+1}}{(2k+1)k!}$$

(3)

The function (3) is an important in estimate the probability of observing a particle in a specified region [3]. Error function also present in which is a part of transport phenomena, significance in many discipline of physics, chemistry, biology, thermomechanics and mass flow [4]. See [13] and [5] for properties and inequalities of error function. It is well-known that every function $f \in S$ has an inverse $f^{-1}$ defined by

$$f^{-1}(f(z)) = z, \quad (z \in E)$$

$$f(f^{-1}(w)) = w, \quad \left(|w| < r_0(f), \quad r_0(f) \geq \frac{1}{4}\right)$$

It is easily seen from above that where

$$f^{-1}(f(z))^\alpha = z^\alpha, \quad (z \in E)$$

$$f(f^{-1}(w))^\alpha = w^\alpha, \quad \left(|w| < r_0(f), \quad r_0(f) \geq \frac{1}{4}\right)$$

Let $\sum$ denote the class of bi-univalent in $E$ by the relation (1). The idea of bi-univalent was conceptualized by Lewin [26] in 1967 and he confirmed that the second coefficient $|a_2| < 1.51$. However, Netanyahu [27] later proved that $\max_{f} |a_2| = \frac{4}{3}$. The coefficient problem for each of coefficients $|a_n|$ is still an open problem see ([11], [27], [28]) for details. For more work on bi-univalent one can see [8, 17, 21, 30, 31, 32, 41, 43, 45, 46, 48].

Now, Noorman and Thomas [29] stated the $q^{th}$ Hankel determinant for $q \geq 1$ and $n \geq 1$

$$H_2(2) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q+1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q+2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q-2} & \cdots & a_{n+2q-2} \end{vmatrix}$$

(4)

where $a_m^{\prime}$s are coefficient of various power of $z$ in $f(z)$. Later, this determinant has been repeatedly investigated by several authors and researchers [15, 19, 23, 37, 38] among others. In particular, Fekete and Szego consider the Hankel determinant of $f \in A$ for $q = 2$ and $n = 1$

$$H_2(2) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix}$$

(5)
Then they later generalized the estimate as \( |a_3 - \mu a_2^2 | \) where \( \mu \) is real and \( f \in S \).
Recently, Fadipe et al.[19] considered a function involving modified Sigmoid function

\[
f_\gamma(z) = z + \sum_{k=2}^{\infty} a_k \gamma(s) z^k
\]  
(6)

where \( \gamma(s) = \frac{2}{1 + e^{-s}}, \ s \geq 0 \) and \( s \) is real. Furthermore, if \( f \) and \( g \) are analytic in \( E \), and \( f \) is as defined (1) and \( g(z) = z + \sum_{k=2}^{\infty} b_k z^k \) then

\[
(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z).
\]  
(7)

From (1) and (6) using (7) we define

\[
 Erf_\gamma(z) = z + \sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{(2k-1)(k-1)!} \gamma(s) a_k z^k
\]
(8)

where all parameters are as earlier defined.

From (8) we have

\[
 Erf_\gamma(z)^\alpha = \left( z + \sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{(2k-1)(k-1)!} \gamma(s) a_k z^k \right)^\alpha
\]
(9)

Expand (9) binomially we get

\[
 Erf_\gamma(z)^\alpha = z^\alpha - \frac{\alpha}{3}(\gamma(s)a_2) z^{\alpha + 1} + \left[ \frac{\alpha}{10} \gamma(s)a_3 + \frac{\alpha(\alpha - 1)}{18} \gamma^2(s)a_2^2 \right] z^{\alpha + 2}
\]

\[
\left[ - \frac{\alpha}{42} \gamma(s)a_4 - \frac{\alpha(\alpha - 1)}{30} \gamma^2(s)a_2a_3 - \frac{\alpha(\alpha - 1)(\alpha - 2)}{162} \gamma^3(s)a_2^2 \right] z^{\alpha + 3}
\]  
(10)

\[
 Erg_\gamma(w)^\alpha = w^\alpha + \frac{\alpha}{3}(\gamma(s)a_2) w^{\alpha + 1} + \left[ \frac{\alpha(\alpha + 1)}{18} \gamma^2(s)a_2^2 - \frac{\alpha}{10} \gamma(s)a_3 \right] w^{\alpha + 2}
\]

\[
\left[ \frac{\alpha}{42} \gamma(s)a_4 - \frac{\alpha(\alpha + 1)}{30} \gamma^2(s)a_2a_3 + \frac{\alpha(\alpha + 1)(\alpha + 2)}{162} \gamma^3(s)a_2^2 \right] w^{\alpha + 3}
\]  
(11)

Applying Salagean derivative operator [41] on (10) and (11) we get

\[
\frac{D^n}{\alpha^n z^\alpha} Erf_\gamma(z)^\alpha = z^\alpha - \frac{\alpha}{3} \left( \frac{\alpha + 1}{\alpha} \right)^n \gamma(s)a_2 z^{\alpha + 1} + \left[ \frac{\alpha}{10} \gamma(s)a_3 + \frac{\alpha(\alpha - 1)}{18} \gamma^2(s)a_2^2 \right] \left( \frac{\alpha + 2}{\alpha} \right)^n z^{\alpha + 2}
\]

\[
\left[ - \frac{\alpha}{42} \gamma(s)a_4 - \frac{\alpha(\alpha - 1)}{30} \gamma^2(s)a_2a_3 - \frac{\alpha(\alpha - 1)(\alpha - 2)}{162} \gamma^3(s)a_2^2 \right] \left( \frac{\alpha + 3}{\alpha} \right)^n z^{\alpha + 3}
\]  
(12)

\[
\frac{D^n}{\alpha^n w^\alpha} Erg_\gamma(w)^\alpha = w^\alpha + \frac{\alpha}{3} \left( \frac{\alpha + 1}{\alpha} \right)^n \gamma(s)a_2 w^{\alpha + 1} + \left[ \frac{\alpha(\alpha + 1)}{18} \gamma^2(s)a_2^2 - \frac{\alpha}{10} \gamma(s)a_3 \right] \left( \frac{\alpha + 2}{\alpha} \right)^n w^{\alpha + 2}
\]

\[
\left[ \frac{\alpha}{42} \gamma(s)a_4 - \frac{\alpha(\alpha + 1)}{30} \gamma^2(s)a_2a_3 + \frac{\alpha(\alpha + 1)(\alpha + 2)}{162} \gamma^3(s)a_2^2 \right] \left( \frac{\alpha + 3}{\alpha} \right)^n w^{\alpha + 3}
\]  
(13)

Researchers like [1], [20], [32], [33], [34], [42] and the likes have used (10) to define several classes of analytic functions and obtained interesting results which are too numerous to discuss.

The Chebyshev polynomials are a sequence of orthogonal polynomials which are
practically related to De Moivres formular and which are defined recursively. Chebyshev polynomials play an important role in numerical analysis. For details about Chebyshev polynomials see ([6],[7],[10],[19]). There are four kinds of Chebyshev polynomials, we shall only concern ourself with Chebyshev polynomials of second kind. Chebyshev polynomials of second is given as

$$U_k(t) = \frac{\sin(k + 1)\alpha}{\sin\alpha} \quad t \in [-1, 1],$$

where $k$ denotes the degree of the polynomial and $t = \cos\alpha$. The Chebyshev polynomials of the second kind $U_k(t), \ t \in [-1, 1]$ have the generating function of the form

$$H(z, t) = \frac{1}{1 - 2t z + z^2} = 1 + \sum_{k=1}^{\infty} \frac{\sin(k + 1)\alpha}{\sin\alpha} z^n.$$

Note that if $t = \cos\alpha, \ \alpha \in \left(\frac{\pi}{3}, \frac{\pi}{2}\right)$.

Thus

$$H(z, t) = \frac{1}{1 - 2\cos\alpha z + z^2} = 1 + \sum_{k=1}^{\infty} \frac{\sin(k + 1)\alpha}{\sin\alpha} z^n.$$

Using Fadipe et al [19], we state that

$$H(z, t) = 1 + U_1(t)z + U_2(t)z^2 + \ldots \quad (z \in E, \ t \in [-1, 1]),$$

where

$$U_{k-1} = \frac{\sin(k\arccos t)}{\sqrt{1 - t^2}}, \quad k \in \mathbb{N}$$

are Chebyshev polynomial of the second kind.

It well known that

$$U_k(t) = 2tU_{k-1}(t) - U_{k-2}(t)$$

so that

$$U_1(t) = 2t, \quad U_2(t) = 4t^2 - 1, \quad U_3(t) = 8t^3 - 4t.$$  

2. Lemma and Definitions

**Lemma 1** [14]: If $w(z) = b_1 z + b_2 z^2 + \ldots, b_1 \neq 0$ is analytic and satisfies $|w(z)| < 1$ in the unit disk $E$, then for each $0 < r < 1, |w'(z)| < 1$ and $|w(re^{i\theta})|$ unless $w(z) = e^{i\theta}z$ for some number $\theta$.

**Lemma 2** [42]: The power series for $p(z) = 1 + \sum_{k=2}^{\infty} k^n c_k z^k$. function Let the $f \in B_n$ be given by (1) then

$$2c_2 = c_1^3 + x(4 - c_1^2)$$

for some $x, |x| < 1$ and

$$4c_2 = c_1^3 + 2x(4 - c_1^2)c_1 - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z$$

for some $x, |x| < 1$.

**Definition 1**: A function $Erf^\alpha$ given by (5) is said to be in the class $TH^\alpha_{\Sigma}(\alpha, \lambda, \gamma(s)), \ \alpha > 0, \ s \geq 0$ if it satisfies the following

$$(1 - \lambda)\frac{D^n Erf_\lambda(z)^\alpha}{\alpha^n z^\alpha} + \lambda\frac{D^{n+1} Erf_\lambda(z)^\alpha}{\alpha^{n+1} z^\alpha} < H(z, t)$$

and

$$(1 - \lambda)\frac{D^n Erf_\gamma(w)^\alpha}{\alpha^n w^\alpha} + \lambda\frac{D^{n+1} Erf_\gamma(w)^\alpha}{\alpha^{n+1} w^\alpha} < H(w, t)$$
where \( \gamma(s) = \frac{2}{1 + e^{-s}} \)

**Theorem 1:** Let \( f \in TH^e_{\alpha}(\lambda, \gamma(s)) \). Then

\[
|a_2| \leq \frac{6t\sqrt{2t}}{\sqrt{[2a\alpha^2 \alpha]^{\alpha}[\alpha + 2\lambda] - (4t^2 - 1) ([\alpha + 2\lambda] \gamma)^2(s)}} \tag{19}
\]

\[
|a_3| \leq \frac{20t^2}{\sum_{\alpha} [\alpha + \lambda] \gamma(s)} + \frac{20t}{\sum_{\alpha} [\alpha + 2\lambda] \gamma(s)} \tag{20}
\]

\[
|a_4| \leq \frac{168t^3}{\sum_{\alpha} [\alpha + \lambda] \gamma(s)^2} + \frac{168t^3}{\sum_{\alpha} [\alpha + \lambda] \gamma(s)} - \frac{28t^3(\alpha + 4)}{28(4\alpha^2 - 1)} - \frac{28t^3(\alpha + 4)}{28(4\alpha^2 - 1)} \tag{21}
\]

**Proof:** It is evident from (17) and (18) that

\[
-\frac{1}{3} \left( \frac{\alpha + 1}{\alpha} \right)^n [\alpha + \lambda] \gamma(s) a_2 = U_1(t)c_1 \tag{22}
\]

\[
\frac{1}{10} \left( \frac{\alpha + 2}{\alpha} \right)^n [\alpha + 2\lambda] a_2 \gamma(s) + \frac{1}{18} \left( \frac{\alpha + 2}{\alpha} \right)^n [\alpha + 2\lambda] \gamma^2(s) a_2^2 = c_2 U_1(t) + c_2^2 U_2(t) \tag{23}
\]

\[
-\frac{1}{42} \left( \frac{\alpha + 3}{\alpha} \right)^n [\alpha + 2\lambda] a_2 \gamma(s) - \frac{1}{30} \left( \frac{\alpha + 3}{\alpha} \right)^n [\alpha + 3\lambda] \gamma^2(s) a_2^2 \tag{24}
\]

\[
\frac{1}{18} \left( \frac{\alpha + 2}{\alpha} \right)^n [\alpha + 2\lambda] \gamma^2(s) a_2^2 - \frac{1}{10} \left( \frac{\alpha + 2}{\alpha} \right)^n [\alpha + 2\lambda] \gamma(s) a_3 = d_2 U_1(t) + d_2^2 U_2(t) \tag{26}
\]

and

\[
\frac{1}{42} \left( \frac{\alpha + 3}{\alpha} \right)^n [\alpha + 2\lambda] \gamma(s) a_4 - \frac{\alpha(\alpha + 1)}{30} \left( \frac{\alpha + 3}{\alpha} \right)^n [\alpha + 3\lambda] \gamma^2(s) a_2^2 \tag{27}
\]

From (22) and (25)

\[
c_1 = -d_1, \quad a_2 = \frac{3U_1(t)c_1}{\sum_{\alpha} [\alpha + \lambda] \gamma(s)} \tag{28}
\]

and

\[
\frac{2}{9} \left( \frac{\alpha + 1}{\alpha} \right)^n [\alpha + \lambda] \gamma^2(s) a_2^2 = U_1^2(t) [c_1^2 + d_1^2] \tag{29}
\]
It is evident from (23), (26), (28) and (29) that

\[
a_2^2 = \frac{9U_1^2(t) |c_2 + d_2|}{\alpha U_1^2(t) \left( \frac{\alpha + 2}{\alpha} \right)^n [\alpha + 2\lambda - 2U_2(t) \left( \frac{\alpha + 1}{\alpha} \right)^n n[\alpha + \lambda]^2] \gamma^2(s)}
\]

(30)

It is fairly well known [14] that if \(|w(z)| < 1 \text{ and } |v(z)| < 1\)

\[|c_i| \leq 1, \text{ and } |d_i| \leq 1, \text{ for all } i \in N\]

(31)

Applying Lemma 1 in conjunction with (31) and (16) on (30) readily yields

\[|a_2| \leq \frac{6t\sqrt{2t}}{\sqrt{2\alpha t^2 \left( \frac{\alpha + 2}{\alpha} \right)^n [\alpha + 2\lambda - (4t^2 - 1) \left( \frac{\alpha + 2}{\alpha} \right)^n [\alpha + \lambda]^2] \gamma^2(s)}}
\]

Again, from (23) and (26) it is evident that

\[
a_3 = \frac{5U_1^2(t)(c_1^2 + d_1^2)}{2 \left( \frac{\alpha + 1}{\alpha} \right)^n n[\alpha + \lambda] \gamma(s)} + \frac{5U_1(t)(c_2 - d_2)}{2 \left( \frac{\alpha + 2}{\alpha} \right)^n [\alpha + 2\lambda] \gamma(s)}
\]

(32)

Applying Lemma 1 for the coefficients \(c_1, \ c_2, \ d_1, \ d_2\) and making uses of (16) and (31) we have

\[|a_3| \leq \frac{20t^2}{\left( \frac{\alpha + 1}{\alpha} \right)^n n[\alpha + \lambda] \gamma(s)} + \frac{20t}{\left( \frac{\alpha + 2}{\alpha} \right)^n [\alpha + 2\lambda] \gamma(s)}
\]

(33)

Next, by (24) and (27) we notice that

\[
a_4 = \frac{21(C_2^2 + d_2^2)U_3^2(t)}{2 \left( \frac{\alpha + 1}{\alpha} \right)^n (\alpha + \lambda)^3 \gamma(s)} + \frac{21(C_2 - d_2)U_2^2(t)}{2 \left( \frac{\alpha + 1}{\alpha} \right)^n (\alpha + \lambda)(\alpha + 2\lambda) \gamma(s)} - \frac{7(\alpha^2 + 4)U_3^3(t)}{(\alpha + 1)^3 n[\alpha + \lambda]^3 \gamma(s)} - \frac{21(c_3 - d_3)U_3(t)}{(\alpha + 1)^n (\alpha + 3\lambda) \gamma(s)} - \frac{21(c_1 c_2 - d_1 d_2)U_3(t)}{(\alpha + 1)^n (\alpha + 3\lambda) \gamma(s)} - \frac{21(c_3^2 - d_3^2)U_3(t)}{(\alpha + 1)^n (\alpha + 3\lambda) \gamma(s)}
\]

(34)

Again, applying Lemma 1 for the coefficients \(c_1, \ c_2, \ d_1, \ d_2, \ c_3, \ d_3\) and making uses of (16) and (31) we have

\[|a_4| \leq \frac{168t^3}{(\alpha + 1)^n [\alpha + \lambda]^2 \gamma(s)} + \frac{168t^3}{(\alpha + 1)^n (\alpha + 2\lambda)^n [\alpha + \lambda] \alpha + 2\lambda \gamma(s)} + \frac{28t^3(\alpha^2 + 4)}{(\alpha + 1)^3 n[\alpha + \lambda]^3 \gamma(s)} - \frac{84t}{(\alpha + 1)^n [\alpha + 3\lambda] \gamma(s)} - \frac{84t}{(\alpha + 1)^n [\alpha + 3\lambda] \gamma(s)} - \frac{42(8t^2 - 4t)}{(\alpha + 1)^n [\alpha + 3\lambda] \gamma(s)}
\]

(35)

as asserted by (21) which complete the proof.

**Corollary 2:** Let \(f \in TH_n^\Sigma(1, \lambda, \gamma(s))\). Then

\[|a_2| \leq \frac{6t\sqrt{2t}}{\sqrt{2\alpha t^2 3^n [1 + 2\lambda - (4t^2 - 1)3^n [1 + \lambda]^2] \gamma^2(s)}}
\]

(36)

\[|a_3| \leq \frac{20t^2}{2^n [1 + \lambda] \gamma(s)} + \frac{20t}{2^n [1 + 2\lambda] \gamma(s)}
\]

(37)

\[|a_4| \leq \frac{168t^3}{2^n [1 + \lambda]^2 \gamma(s)} + \frac{168t^3}{2^n [1 + \lambda]^2 [\alpha + 2\lambda] \gamma(s)} - \frac{28t^3(\alpha^2 + 4)}{2^n [\alpha + \lambda]^3 \gamma(s)} - \frac{84t}{4^n [1 + 3\lambda] \gamma(s)} - \frac{84t}{4^n [1 + 3\lambda] \gamma(s)} - \frac{42(8t^2 - 4t)}{4^n [1 + 3\lambda] \gamma(s)}
\]

(38)
Theorem 2: Let \( f \in TH^\omega_\Sigma_0(\alpha, \lambda, \gamma(s)) \). Then

\[
|a_2a_4 - a_3^2| \leq \theta \gamma^2 + \beta \gamma
\]  

(39)

where

\[
\theta = A_1 - B_1 - B_2, \quad \gamma = \frac{-2(B_1 + B_2)}{A_1 - B_1 - B_2}, \quad C = \left(\frac{\alpha + 1}{\alpha}\right)^n \left(\frac{\alpha + 1}{\alpha}\right)^{3n} (\alpha + 3\lambda)^3
\]

\[
A_1 = \frac{336t^4 - 168t^4 \alpha^2 A - 126t^4 B - 1008t^3 - 252t - 100t^4 D}{\left(\frac{\alpha + 3}{\alpha}\right)^n \left(\frac{\alpha + 1}{\alpha}\right)^{4n} (\alpha + 3\lambda)^4 (\alpha + 3\lambda)^2 (s)}, \quad \gamma = 4(B_1 + C_1)
\]

\[
B_1 = \frac{1008t^3 + 252t^2 - 252t}{\left(\frac{\alpha + 3}{\alpha}\right)^n \left(\frac{\alpha + 1}{\alpha}\right)^n (\alpha + \lambda)(\alpha + 3\lambda)^2 (s)}, \quad C_1 = \frac{126t^2}{\left(\frac{\alpha + 3}{\alpha}\right)^n \left(\frac{\alpha + 1}{\alpha}\right)^{3n} (\alpha + \lambda)(\alpha + 3\lambda)^2 (s)}
\]

\[
A = \left(\frac{\alpha + 3}{\alpha}\right)^n (\alpha + \lambda)^3 (\alpha + 3\lambda), \quad B = \left(\frac{\alpha + 3}{\alpha}\right)^n (\alpha + \lambda)^3, \quad D = \left(\frac{\alpha + 3}{\alpha}\right)^n (\alpha + 3\lambda)
\]

Proof: It is evident from (28), (32) and (34) that

\[
a_2a_4 - a_3^2 = \frac{21(2 - \alpha^2)U_1^4(t)c_1^4}{2 (\alpha + 1)^n (\alpha + \lambda)^4 \gamma^2 (s)} + \frac{63c_1^2(c_2 - d_2)U_1^2(t)}{2 (\alpha + 1)^n (\alpha + \lambda)^2 (\alpha + 2\lambda)^2 \gamma^2 (s)} - \frac{25c_1^4U_1^4(t)}{2 (\alpha + 1)^n (\alpha + \lambda)^4 \gamma^2 (s)}
\]

\[
- \frac{252U_1^2(t)c_1^2c_2}{(\alpha + 1)^n (\alpha + \lambda)(\alpha + 3\lambda)^2 \gamma^2 (s)} - \frac{126U_1(t)U_3(t)c_1^4}{(\alpha + 1)^n (\alpha + \lambda)(\alpha + 3\lambda)^2 \gamma^2 (s)} - \frac{63c_3 - d_3U_1^2(t)c_1}{(\alpha + 1)^n (\alpha + \lambda)(\alpha + 3\lambda)^2 \gamma^2 (s)}
\]

\[
- \frac{25c_1^2(c_2 - d_2)U_1^2(t)}{2 (\alpha + 1)^n (\alpha + \lambda)^2 (\alpha + 2\lambda)^2 \gamma^2 (s)} - \frac{25U_1^2(t)c_2 - d_2^2U_1^2(t)}{2 (\alpha + 1)^n (\alpha + \lambda)^2 (\alpha + 2\lambda)^2 \gamma^2 (s)}
\]

(40)

According to Lemma 2 we have

\[
2c_2 = c_1^2 + x(4 - c_1^2) \quad \text{and} \quad 2d_2 = d_1^2 + x(4 - d_1^2)
\]

hence we have

\[
c_2 = d_2
\]

\[
4c_3 = c_1^3 + 2x(4 - c_1^2)c_1 - x^2(4 - c_1^2)c_1 + 2(4 - c_1^2)(1 - |z|^2)
\]

\[
4d_3 = d_1^3 + 2x(4 - d_1^2)d_1 - x^2(4 - d_1^2)d_1 + 2(4 - d_1^2)(1 - |z|^2)
\]

and

\[
c_3 - d_3 = \frac{1}{2}c_1^3 + x(4 - c_1^2)c_1 - \frac{1}{2}x^2(4 - c_1^2)c_1.
\]

Thus upon substitution and letting \( c_1 = c \) we get

\[
a_2a_4 - a_3^2 = \left[\frac{336t^4 - 168t^4 \alpha^2 A - 126t^4 B - 1008t^3 - 252t - 100t^4 D}{(\alpha + 3)^n (\alpha + 1)^n (\alpha + 3\lambda)^4 \gamma^2 (s)}\right] c^4
\]

\[
- \left[\frac{1008t^3 + 252t^2 - 252t}{(\alpha + 1)^n (\alpha + \lambda)(\alpha + 3\lambda)^2 \gamma^2 (s)}\right] c^2(4 - c^2)x + \left[\frac{126t^2}{(\alpha + 1)^n (\alpha + \lambda)(\alpha + 3\lambda)^2 \gamma^2 (s)}\right] c^2(4 - c^2)x^2
\]

where

\[
A = \left(\frac{\alpha + 3}{\alpha}\right)^n (\alpha + \lambda)^3 (\alpha + 3\lambda), \quad B = \left(\frac{\alpha + 3}{\alpha}\right)^n (\alpha + \lambda)^3,
\]

\[
C = \left(\frac{\alpha + 1}{\alpha}\right)^n \left(\frac{\alpha + 1}{\alpha}\right)^{3n} (\alpha + 3\lambda)^3, \quad D = \left(\frac{\alpha + 3}{\alpha}\right)^n (\alpha + 3\lambda)
\]
Since $|c| \leq 1$ by using Lemma 2, we may assume without restriction $c \in [0, 1]$. Thus using the triangle inequality, with $\rho = |x|$ we obtain

$$|a_2a_4 - a_3^2| \leq A_1c^4 + B_1c^2(4 - p^2)\rho + C_1c^2(4 - c^2)\rho^2 = F(c, \rho),$$

where

$$A_1 = \frac{[336t^4 - 168t^4\alpha^2]A - 126t^4B - [1008t^3 - 252t] - 100t^4D}{(\frac{\alpha + 3}{\alpha})^n(\frac{\alpha + 1}{\alpha})^{4n}(\alpha + 3\lambda)^4(\alpha + 3\lambda)\gamma^2(s)}$$

$$B_1 = \frac{1008t^3 + 252t^2 - 252t}{(\frac{\alpha + 3}{\alpha})^n(\frac{\alpha + 1}{\alpha})^{n}(\alpha + \lambda)(\alpha + 3\lambda)\gamma^2(s)}$$

$$C_1 = \frac{126t^2}{(\frac{\alpha + 3}{\alpha})^n(\frac{\alpha + 1}{\alpha})^{n}(\alpha + \lambda)(\alpha + 3\lambda)\gamma^2(s)}$$

Then,

$$\frac{\partial F}{\partial \rho} = B_1c^2(4 - c^2) + 2C_1c^2(4 - c^2)\rho$$

It is clear $\frac{\partial F}{\partial \rho} > 0$ which shows that $F(c, \rho)$ is an increasing function on the closed interval $[0, 1]$. This shows that maximum occurs at $\rho = 1$. Therefore $\max F(c, \rho) = F(c, 1) = G(c)$

Now

$$F(c, 1) = G(c) = A_1c^4 + B_1c^2(4 - c^2) + C_1c^2(4 - c^2)$$

$$G(c) = (A_1 - B_1 - C_1)c^3 + 4(B_1 + C_1)c^2$$

$$G'(c) = 4(A_1 - B_1 - C_1)c^3 + 8(B_1 + C_1)c$$

$$G''(c) = 12(A_1 - B_1 - C_1)c^2 + 8(B_1 + C_1) < 0.$$ (42)

For optimum value of $G(c)$, consider $G'(c) = 0$. From (41) we get

$$c^2 = \frac{-2(B_1 + C_1)}{A_1 - B_1 - B_1} = \gamma$$

(43)

Upon substitution for the value of $c^2$ from (43) in (42), it can be shown that

$$G''(c) = 12(A_1 - B_1 - C_1)c^2 + 8(B_1 + C_1)$$

Therefore, by using second derivative test $G(c)$ has the maximum value at $c$, where $c^2$ is given by (43). Substituting the obtained value of $c^2$ in the expression which gives maximum of $G(c)$ as

$$|a_2a_4 - a_3^2| \leq \theta \gamma^2 + \beta \gamma$$

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I. T. AWOLEREDepartment of Mathematical Sciences, Ondo State University of Science and Technology, Okitipupa, Ondo State, Nigeria.
E-mail address: awolereibrahim01@gmail.com