BOUNDS ASSOCIATED TO HADAMARD INEQUALITY VIA GENERALIZED INTEGRAL OPERATORS AND APPLICATIONS FOR CONFORMABLE AND FRACTIONAL INTEGRALS

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Abstract. Error bounds of Hadamard type inequalities have been studied extensively in the literature. This work is dedicated to a chain of Hadamard inequalities for various kinds of integral operators. In this study error bound for a version of the Hadamard inequality for a generalized integral operator is established. Some particular cases are discussed which have connection with already known results.

1. Fractional and Conformable Integrals

The study of fractional order derivatives and integrals received more attention after the formulation of electrochemical problems. In recent years the subject is studied extensively due to its applications in different areas of natural sciences such as: quantum mechanical calculations, chemical analysis of aqueous solutions, design of heat flux meters, transmission line theory etc. see [19]. For the detailed mathematical study of fractional integral and derivative operators, see [10, 11].

The classical fractional integral operator known as Riemann-Liouville fractional integral is defined as follows [10, 11]:

**Definition 1.** Let $f \in L_1[c, d]$. Then Riemann-Liouville fractional integral operators of order $\beta > 0$ with $c \geq 0$ are defined as follows:

\[
\begin{align*}
\beta J_{c^+} f(x) &= \frac{1}{\Gamma(\beta)} \int_c^x (x-\tau)^{\beta-1} f(\tau) d\tau, \quad x > c, \\
\beta J_{d^-} f(x) &= \frac{1}{\Gamma(\beta)} \int_x^d (\tau-x)^{\beta-1} f(\tau) d\tau, \quad x < d.
\end{align*}
\]

Mubeen et al. [18] gave the $k$-analogue of Riemann-Liouville integrals.

**Definition 2.** Let $f \in L_1[c, d]$. Then the $k$-fractional integrals of order $\beta, k > 0$
with $c \geq 0$ are defined as follows:
\[
\beta J^k_{c+} f(x) = \frac{1}{k! k(\beta)} \int_c^x (x-\tau)^{\frac{\beta}{k}-1} f(\tau) d\tau, \quad x > c, \tag{3}
\]
\[
\beta J^k_{d-} f(x) = \frac{1}{k! k(\beta)} \int_x^d (\tau-x)^{\frac{\beta}{k}-1} f(\tau) d\tau, \quad x < d. \tag{4}
\]

Sarikaya et al. \textsuperscript{22} introduced the notion of $(k,s)$-Riemann-Liouville fractional integrals as follows:

**Definition 3.** Let $f \in L^1_{[c,d]}$. Then $(k,s)$-Riemann-Liouville fractional integral operators of order $\beta > 0$ with $c \geq 0$ are defined by:
\[
\beta J^k_{c+} f(x) = (s+1)^{1-\frac{\beta}{k}} \frac{\Gamma(k)}{k! k(\beta)} \int_c^x (x^{s+1} - \tau^{s+1})^{\frac{\beta}{k}-1} \tau^s f(\tau) d\tau, \quad x > c, \tag{5}
\]
\[
\beta J^k_{d-} f(x) = (s+1)^{1-\frac{\beta}{k}} \frac{\Gamma(k)}{k! k(\beta)} \int_x^d (\tau^{s+1} - x^{s+1})^{\frac{\beta}{k}-1} \tau^s f(\tau) d\tau, \quad x < d, \tag{6}
\]
where $k > 0$, $s \in \mathbb{R} - \{0\}$.

Khalil et al. \textsuperscript{14} gave conformable fractional integrals as follows:

**Definition 4.** Let $\beta \in (0,1)$. A function $f : [c,d] \to \mathbb{R}$ is $\beta$-fractional integrable on $[c,d]$ if the integral
\[
I^\beta_{c+} (f)(x) = \int_c^x f(\tau) d\tau = \int_c^x f(\tau) \tau^{\beta-1} d\tau, \quad x > c, \tag{7}
\]
exists and is finite. $L^\beta_{c+} ([c,d])$ is the class of $\beta$-fractional integrable functions on $[c,d]$.

Recently Khan et al. \textsuperscript{15} defined a generalized conformable integral operator as follows:

**Definition 5.** Let $f$ be a conformable integrable function on the interval $[c,d] \subseteq [0,\infty)$. The left and right-sided generalized conformable fractional integrals of order $\beta > 0$ with $r \in \mathbb{R}$, $\gamma \in (0,1)$, $r + \gamma \neq 0$ are defined by
\[
\beta J^\gamma_{c+} f(x) = \frac{(r+\gamma)^{1-\beta}}{\Gamma(\beta)} \int_c^x (x^{r+\gamma} - \tau^{r+\gamma})^{\beta-1} \tau^r f(\tau) d\tau, \quad x > c, \tag{8}
\]
\[
\beta J^\gamma_{d-} f(x) = \frac{(r+\gamma)^{1-\beta}}{\Gamma(\beta)} \int_x^d (\tau^{r+\gamma} - x^{r+\gamma})^{\beta-1} \tau^r f(\tau) d\tau, \quad x < d. \tag{9}
\]
A compact form of aforementioned fractional integral operator is defined as follows \textsuperscript{10, 11}:

**Definition 6.** Let $f : [c,d] \to \mathbb{R}$ be an integrable function. Also let $g$ be an increasing and positive function on $(c,d)$, having continuous derivative $g'$ on $(c,d)$. The left and right-sided fractional integrals of a function $f$ with respect to another function $g$ on $[c,d]$ of order $\beta > 0$ are defined as:
\[
\beta g J^\gamma_{c+} f(x) = \frac{1}{\Gamma(\beta)} \int_c^x [g(x) - g(\tau)]^{\beta-1} g'(\tau) f(\tau) d\tau, \quad x > c, \tag{10}
\]
\[
\beta g J^\gamma_{d-} f(x) = \frac{1}{\Gamma(\beta)} \int_x^d [g(\tau) - g(x)]^{\beta-1} g'(\tau) f(\tau) d\tau, \quad x < d. \tag{11}
\]
Kwun \textsuperscript{13} et al. gave the $k$-fractional analogue of integrals \textsuperscript{10} and \textsuperscript{11} as follows:
Definition 7. Let $g : [c, d] \rightarrow \mathbb{R}$ be an increasing and positive monotone function on $[c, d]$, having a continuous derivative $g'$ on $(c, d)$. The left and right-sided fractional integrals of a function $f$ with respect to another function $g$ on $[c, d]$ of order $\beta, k > 0$ are defined by:

$$
\frac{\beta}{g} J^k_c f(x) = \frac{1}{k \Gamma_k(\beta)} \int_c^x [g(x) - g(\tau)]^{\beta-1} g'(\tau) f(\tau) d\tau, \quad x > c, \quad (12)
$$

$$
\frac{\beta}{g} J^k_d f(x) = \frac{1}{k \Gamma_k(\beta)} \int_x^d [g(\tau) - g(x)]^{\beta-1} g'(\tau) f(\tau) d\tau, \quad x < d, \quad (13)
$$

where $\Gamma_k(.)$ is the $k$-gamma function.

Raina [20], gave the following fractional integral operator by using special functions as follows:

Definition 8. Let $f \in L_1 [c, d]$ . The left and right-sided integrals with special functions are denoted and defined by

$$
\frac{\sigma}{\rho} \zeta_{\mu,c}^+ f(x) = \int_c^x \frac{F_{\rho,\mu}^\sigma (w(x - \tau)^\rho)}{(x - \tau)^{1-\mu}} f(\tau) d\tau, \quad x > c, \quad (14)
$$

$$
\frac{\sigma}{\rho} \zeta_{\mu,d}^- f(x) = \int_x^d \frac{F_{\rho,\mu}^\sigma (w(x - \tau)^\rho)}{(x - \tau)^{1-\mu}} f(\tau) d\tau, \quad x < d, \quad (15)
$$

where $\rho, \mu > 0$, coefficients $\sigma(k)$ generate a bounded sequence of positive real numbers and

$$
F_{\rho,\mu}^\sigma (x) = \sum_{m=0}^{\infty} \frac{\sigma(m)}{(\rho m + \mu)^m} x^m, \quad |x| < R, \text{ with } R > 0. \quad (16)
$$

Tunc et al. [25], generalize the operator of Raina as follows:

Definition 9. For $k > 0$, let $g : [c, d] \rightarrow \mathbb{R}$ be an increasing and positive monotone function having a continuous derivative $g'$ on $(c, d)$. The left and right sided generalized $k$-fractional integrals of $f$ with respect to the function $g$ on $[c, d]$ are respectively defined as follows:

$$
\frac{\sigma}{\rho} \zeta^{k,g}_{\mu,c} f(x) = \int_c^x \frac{F_{\rho,\mu}^{\sigma,k} (w (g(x) - g(\tau))^\rho)}{(g(x) - g(\tau))^{1-\mu}} g'(\tau) f(\tau) d\tau, \quad x > c, \quad (17)
$$

$$
\frac{\sigma}{\rho} \zeta^{k,g}_{\mu,d} f(x) = \int_x^d \frac{F_{\rho,\mu}^{\sigma,k} (w (g(\tau) - g(x))^\rho)}{(g(\tau) - g(x))^{1-\mu}} g'(\tau) f(\tau) d\tau, \quad x < d, \quad (18)
$$

with the coefficients $\sigma(n)$ ($n \in \mathbb{N} \cup \{0\}$) form a bounded sequence of positive real numbers and

$$
F_{\rho,\mu}^{\sigma,k} (x) := \sum_{n=0}^{\infty} \frac{\sigma(n)}{k \Gamma_k(\rho kn + \mu)} x^n, \quad (\rho, \mu > 0; |x| < R) \text{ with } R > 0. \quad (19)
$$

Recently Farid [3] introduced a generalized integral operator as follows:

Definition 10. Let $f, g : [c, d] \rightarrow \mathbb{R}, 0 < c < d$, be the functions such that $f \in L_1[c, d]$ be positive and $g$ be differentiable and increasing. Also let $\frac{g'}{g}$ be increasing function on $[0, \infty)$. Then for $x \in [c, d]$, the left and right integral operators are defined by

$$
F_{\rho,\mu}^{\sigma,g} f(x) = \int_c^x \frac{\varphi (g(x) - g(\tau))}{g(x) - g(\tau)} g'(\tau) f(\tau) d\tau, \quad x > c, \quad (20)
$$
defined by Dragomir in [1] are obtained.

operators with order \( \mu > 0 \) will be obtained.

(17) and (18) will be re-captured.

operators will be obtained defined in [1] as follows:

\[
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\]

\[
\frac{\varphi (g(\tau) - g(x))}{g(\tau) - g(x)} g'(\tau) f(\tau) d\tau, \ x < d. \tag{21}
\]

Following remark gives the summary of conformable and fractional integral operators which can be deduced from the last definition by different settings of \( \varphi \) and \( g \).

**Remark 1.**

(i) If \( \varphi(x) = x^k \mathcal{F}_{c}^{\sigma,k}(w(x)^\rho) \) in (20) and (21), then fractional integral operators (17) and (18) will be re-captured.

(ii) If \( g(x) = x \) in (17) and (18), then fractional integral operators (14) and (15) will be obtained.

(iii) If \( \varphi(x) = x^k \mathcal{F}_{c}^{\sigma} w(x)^\rho \) in (20) and (21), then fractional integral operators defined by Tunc et al in [25] will be re-captured.

(iv) If \( \varphi(x) = x^k \mathcal{F}_{c}^{\sigma,k}(w(x)^\rho) \) in (20) and (21) and \( g(x) = x \), then \( k \)-analogue of fractional integral operators (14) and (15) defined by Tunc et al in [25] will be re-captured.

(v) If \( \varphi(x) = x^k \mathcal{F}_{c}^{\sigma,k}(w(x)^\rho) \), then (20) and (21) will give the generalized Hadamard \( k \)-fractional integral operators defined in [25], subject to the condition that \( g(x) = \ln x \).

(vi) If \( \varphi(x) = x^k \mathcal{F}_{c}^{\sigma,k}(w(x)^\rho) \) and \( g(x) = \frac{x^{s+1}}{(s+1)} \), \( s \in \mathbb{R} - \{ -1 \} \), then (20) and (21) reduced to the \((k,s)\)-fractional integral operators with special functions defined in [25].

(vii) If \( \varphi(x) = \frac{1}{\Gamma(\beta)} x^\beta \), and \( g(x) = -x^{-1} \), then generalized fractional integral operators with exponential kernel will obtained defined in [1] as follows:

\[
\frac{\beta}{\Gamma(\beta)} E_{c+} f(x) = \frac{1}{\beta} \int_{c}^{x} \exp \left( -\frac{1}{\beta} (g(x) - g(\tau)) \right) f(\tau) d\tau, \ x > c, \tag{22}
\]

\[
\frac{\beta}{\Gamma(\beta)} E_{d-} f(x) = \frac{1}{\beta} \int_{x}^{d} \exp \left( -\frac{1}{\beta} (g(\tau) - g(x)) \right) f(\tau) d\tau, \ x < d. \tag{23}
\]

(viii) If \( \varphi(x) = \frac{1}{\Gamma(\beta)} x^\beta \), and \( g(x) = -x^{-1} \), then Harmonic fractional integral operators will be obtained defined in [1] as follows:

\[
\frac{\beta R_{c+}}{\Gamma(\beta)} f(x) = \frac{x^{1-\beta}}{\Gamma(\beta)} \int_{c}^{x} (x - \tau)^{\beta-1} f(\tau) d\tau, \ x > c, \tag{24}
\]

\[
\frac{\beta R_{d-}}{\Gamma(\beta)} f(x) = \frac{x^{1-\beta}}{\Gamma(\beta)} \int_{x}^{d} (\tau - x)^{\beta-1} f(\tau) d\tau, \ x < d. \tag{25}
\]

(ix) If \( \varphi(x) = \frac{1}{\Gamma(\mu)} x^\mu \), and \( g(x) = \exp(\beta x) \), then \( \beta \)-Exponential fractional integral operators with order \( \mu > 0 \) will be obtained [1]:

\[
\frac{\beta}{\Gamma(\mu)} Z f(x) = \frac{1}{\Gamma(\mu)} \int_{c}^{x} (\exp(\beta x) - \exp(\beta \tau))^{\mu-1} \exp(\beta \tau) f(\tau) d\tau, \ x > c, \tag{26}
\]
\[
\beta \mathfrak{D}_{d^{-}} f(x) = \frac{\beta}{\Gamma(\mu)} \int_{x}^{d} (\exp(\beta \tau) - \exp(\beta x))^{\mu-1} \exp(\beta \tau) f(\tau) d\tau, \ x < d. \quad (27)
\]

(x) If \( \varphi(x) = x^\beta \ln x \), then left and right-sided logarithmic fractional integrals which were introduced in [11] will be obtained:

(xi) If \( \varphi(x) = \frac{1}{\Gamma(\beta)} x^\beta \) and \( g(x) = \frac{1}{1+s} x^{1+s}, 1 + s \neq 0 \), then (5) and (5) will be obtained.

(xii) If \( \varphi(x) = \frac{1}{\Gamma(\beta)} x^\beta \) and \( g(x) = \ln x \), then Hadamard fractional integral operators will be obtained [10]. In recent past the researchers have utilized various kinds of integral operators especially fractional and conformable integral operators to establish the well known Hadamard inequality for example see [2], [4], [5], [6], [7], [8], [10], [17], [23], [24], [25] and the references therein. Recently Hadamard inequality via generalized integral operators (20) and (21) is established in [21]. The aim of this paper is to study the error bounds of the Hadamard inequality for integral operators (20) and (21). These error bounds have interesting consequences for estimation of Hadamard inequalities for conformable and fractional integral operators.

The paper is organized as follows. In Section 2 an identity is established by using integral operators (20) and (21). By using this identity error bounds of the Hadamard inequality for integral operators (20) and (21) are established. In Section 3, by considering appropriate settings of functions several error bounds for corresponding Hadamard inequalities for fractional and conformable integral operators are obtained.

2. Main Results

In this paper \( \varphi \) and \( g \) are same as defined in Definition [1]. Following notations will be use frequently in this study:

\[
\tilde{u}(x) := u(c + d - x), \quad U(x) := u(x) + \tilde{u}(x),
\]

\[
\Delta_t^0(\varphi, g) = \int_{0}^{t} \frac{\varphi(g(sd + (1-s)c) - g(c))}{g(sd + (1-s)c) - g(c)} g'(sd + (1-s)c) ds,
\]

and

\[
\nabla_t^0(\varphi, g) = \int_{0}^{t} \frac{\varphi(g(d) - g(sc + (1-s)d))}{g(d) - g(sc + (1-s)d)} g'(sc + (1-s)d) ds.
\]

The following lemma is useful to establish the bounds of the Hadamard inequality for integral operators (20) and (21).

**Lemma 1.** Let \( u : [c, d] \to \mathbb{R} \) be a differentiable mapping on \((c, d)\) with \( c < d \). If
u' \in L^1_c, d]$, then the following equalities for integral operators (20) and (21) hold:

\[
\frac{u(c) + u(d)}{2} - \frac{1}{2(d - c)} \left( \Delta^t_0(\varphi, g) + \nabla^t_0(\varphi, g) \right) \left[ F^x_g U(b) + F^y_g U(a) \right] = \left\{ \begin{array}{ll}
\frac{d - c}{2} \int_0^1 \Omega_1(t)u'(tc + (1 - t)d) dt \\
\frac{d - c}{2} \int_0^1 \Omega_2(t)U'(td + (1 - t)c) dt,
\end{array} \right.
\]

where

\[
\Omega_1(t) = \left( \Delta^{1-t}_0(\varphi, g) - \Delta^t_0(\varphi, g) \right) + \left( \nabla^{1-t}_0(\varphi, g) - \nabla^t_0(\varphi, g) \right),
\]

\[
\Omega_2(t) = \Delta^t_0(\varphi, g) + \nabla^t_0(\varphi, g).
\]

**Proof.** It is easy to see that,

\[
\int_0^1 \left[ \Delta^{1-t}_0(\varphi, g) - \Delta^t_0(\varphi, g) \right] u'(tc + (1 - t)d) dt
\]

\[
= \int_0^1 \Delta^t_0(\varphi, g)[u'(td + (1 - t)c) - u'(tc + (1 - t)d)] dt
\]

and

\[
\int_0^1 \left[ \nabla^{1-t}_0(\varphi, g) - \nabla^t_0(\varphi, g) \right] u'(tc + (1 - t)d) dt
\]

\[
= \int_0^1 \nabla^t_0(\varphi, g)[u'(td + (1 - t)c) - u'(tc + (1 - t)d)] dt.
\]

Clearly,

\[
\int_0^1 \Omega_2(t)U'(td + (1 - t)c) dt = \int_0^1 \Delta^t_0(\varphi, g)U'(td + (1 - t)c) dt
\]

\[
+ \int_0^1 \nabla^t_0(\varphi, g)U'(td + (1 - t)c) dt.
\]

Let

\[
I_c = \int_0^1 \Delta^t_0(\varphi, g)U'(td + (1 - t)c) dt
\]

and

\[
I_d = \int_0^1 \nabla^t_0(\varphi, g)U'(td + (1 - t)c) dt.
\]

Then one can have
\[(d-c)I_c = \Delta_0^1(\varphi, g)U(d) - \int_0^1 \frac{\varphi(g(td + (1-t)c) - g(c))}{(g(td + (1-t)c) - g(c))} g'(td + (1-t)c)U(td + (1-t)c)dt.\]

By suitable change of variables and applying definition \(20\) and \(21\) of generalized integral operators, we have
\[(d-c)I_c = \Delta_0^1(\varphi, g)[u(c) + u(d)] - \frac{1}{(d-c)} F_{d-c}^{\varphi, g} U(c). \quad (33)\]

Similarly,
\[(d-c)I_d = \nabla_0^1(\varphi, g)[u(c) + u(d)] - \frac{1}{(d-c)} F_{c}^{\varphi, g} U(d), \quad (34)\]

From \(28\), \(29\), \(30\), \(33\) and \(34\), the required equalities can be achieved.

**Remark 2.** The aforementioned lemma holds for all kinds of integral operators comprises in Remark \(1\). In particular one can obtain \([2, \text{Lemma 2.1}], [6, \text{Lemma 2.4}], [5, \text{Lemma 2.3}], [16, \text{Theorem 4.1}], [23, \text{Lemma 5}], [24, \text{Lemma 2}], [25, \text{Lemma 1}]\) etc. Furthermore, some new equalities can be obtain for operators \([5], [6], [8], [9], [12], [13], [22], [23], [24], [25], [26], [27]\) by using appropriate settings of \(\varphi\) and \(g\) as given in Remark \(1\).

**Theorem 1.** Let \(g : [c, d] \rightarrow \mathbb{R}\) be a positive monotone increasing function on \((c, d]\), having continuous derivatives \(g'\) on \((c, d]\). Let \(u : [c, d] \rightarrow \mathbb{R}\) be a differentiable mapping on \((c, d]\). If \(|u'|\) is convex on \([c, d]\), then the following inequality for generalized fractional integrals \(20\) and \(21\) hold:
\[
\left| \frac{u(c) + u(d)}{2} - \frac{1}{2(d-c)} \left( \frac{F_{c}^{\varphi, g} U(d) + F_{d-c}^{\varphi, g} U(c)}{\Delta_0^1(\varphi, g) + \nabla_0^1(\varphi, g)} \right) \right| \leq \frac{d-c}{\Delta_0^1(\varphi, g) + \nabla_0^1(\varphi, g)} \int_0^1 t |\Omega_1(t)| |u'(tc) + (1-t)d| dt, \quad (35)\]

where \(\Omega_1(t)\) is same as defined in Lemma \(2\).

**Proof.** By Lemma \(2\) and property of modulus, we have
\[
\left| \frac{u(c) + u(d)}{2} - \frac{1}{2(d-c)} \left( \frac{F_{c}^{\varphi, g} U(d) + F_{d-c}^{\varphi, g} U(c)}{\Delta_0^1(\varphi, g) + \nabla_0^1(\varphi, g)} \right) \right| \leq \frac{d-c}{2(\Delta_0^1(\varphi, g) + \nabla_0^1(\varphi, g))} \int_0^1 |\Omega_1(t)| |u'(tc + (1-t)d)| dt.
\]

By convexity of \(|u'|\), we get
where, 

\[
\frac{u(c) + u(d)}{2} - \frac{1}{2(d-c)} \left( \Delta^1_0 \phi, g \right) + \nabla^l_0 (\phi, g) \left[ F^\phi \xi U(d) + F^\phi \eta U(c) \right] \\
\leq \frac{d - c}{2 \left( \Delta^1_0 (\phi, g) + \nabla^l_0 (\phi, g) \right)} \left[ |u'(c)| \int_0^1 t |\Omega_1(t)| dt + |u'(d)| \int_0^1 (1 - t) |\Omega_1(t)| dt \right] \\
= \frac{d - c}{2 \left( \Delta^1_0 (\phi, g) + \nabla^l_0 (\phi, g) \right)} \left[ |u'(c)| \int_0^1 t |\Omega_1(t)| dt + |u'(d)| \int_0^1 t |\Omega_1(1 - t)| dt \right]. 
\]

Note that 

\[
\Omega_1 (1 - t) = \Delta^1_0 (\phi, g) - \Delta^1_0 (\phi, g) + \nabla^l_0 (\phi, g) - \nabla^l_0 (\phi, g) \\
= - \Omega_1 (t). 
\]

Using value of \( \Omega_1 (1 - t) \) in above inequality, we get required inequality \(^{[35]}\).

**Remark 3.** The aforementioned inequality gives error bounds of Hadamard inequalities of all kinds of integral operators comprises in Remark \(^{[1]}\). In particular by using suitable settings of \( \phi \) and \( g \) as given in Remark \(^{[1]}\), one can obtain \(^{[2]}\) Theorem 2.2, \(^{[6]}\) Theorem 2.5, \(^{[5]}\) Theorem 2.4, \(^{[16]}\) Theorem 4.1, \(^{[23]}\) Theorem 6, \(^{[23]}\) Corollary 5, \(^{[24]}\) Theorem 3 and \(^{[25]}\) Theorem 2.

3. Error bounds associated to Hadamard inequalities via conformable and fractional integrals

In this section we construct error bounds of the Hadamard inequalities for various kinds of fractional and conformable integral operators.

**Theorem 2.** Let \( g : [c, d] \to \mathbb{R} \) be a positive monotone increasing function on \( (c, d) \), having continuous derivatives \( g' \) on \( (c, d) \). Let \( u : [c, d] \to \mathbb{R} \) be a differentiable mapping on \( (c, d) \). If \( |u'| \) is convex on \( [c, d] \), then the following inequality for operators \(^{[12]}\) and \(^{[13]}\) holds:

\[
\left| \frac{u(c) + u(d)}{2} - \frac{\Gamma_k(\beta + k)}{4 [g(d) - g(c)]^\frac{\beta}{\pi}} \left[ g^\beta J^k_+ U(d) + g^\beta J^k_- U(c) \right] \right| \\
\leq \frac{\beta_k A^\beta (c, d)}{4 [g(d) - g(c)]^\frac{\beta}{\pi} (d - c)} \left[ |u'(c)| + |u'(d)| \right], \tag{36}
\]

where,

\[
\beta_k A^\beta (c, d) = \frac{\beta_k}{\pi} \chi^\beta (d, c) + \frac{\beta_k}{\pi} \chi^\beta (c, d) - \frac{\beta_k}{\pi} \chi^\beta (d, c) - \frac{\beta_k}{\pi} \chi^\beta (c, c), \tag{37}
\]

and

\[
\frac{\beta_k}{\pi} \chi^\beta (x, y) := \int_c^x |x - t| [g(y) - g(t)]^\frac{\beta}{\pi} dt - \int_c^y |x - t| [g(y) - g(t)]^\frac{\beta}{\pi} dt, \tag{38}
\]
for all \( x, y \in [c, d] \).

**Proof.** Let us define the function \( \varphi \) by \( \varphi(t) = \frac{t^\beta}{\Gamma_0(\beta)} \). Then we have

\[
\Omega_1(t) = \frac{1}{(d-c)\Gamma_k(\beta + k)}[(g(tc + (1-t)d) - g(c))^{\frac{\beta}{\pi}} - (g(tc + (1-t)d) - g(c))^{\frac{\beta}{\pi}} \\
+ (g(tc + (1-t)d) - g(c))^{\frac{\beta}{\pi}} - (g(tc + (1-t)d) - g(c))^{\frac{\beta}{\pi}}]
\]

and

\[
\Delta^1_0(\varphi, g) + \nabla^1_0(\varphi, g) = \frac{2}{(d-c)\Gamma_k(\beta + k)}[(g(d) - g(c))^{\frac{\beta}{\pi}}] \tag{39}
\]

Also by change of variables we have

\[
\int_0^1 t |\Omega_1(t)| \, dt = \frac{1}{(d-c)^3\Gamma_k(\beta + k)} \int_c^d (d-x) |\psi(x)| \, dt, \tag{40}
\]

where

\[
\psi(x) = (g(x) - g(c))^{\frac{\beta}{\pi}} - (g(c + d - x) - g(c))^{\frac{\beta}{\pi}} - (g(d) - g(c + d - x))^{\frac{\beta}{\pi}} - (g(d) - g(x))^{\frac{\beta}{\pi}}.
\]

Observe that \( \psi \) is a non-decreasing function on \([c, d]\). We have indeed,

\[
\psi(c) = 2 (g(c))^{\beta} - 2 (g(d))^{\beta} < 0,
\]

\[
\psi\left(\frac{c+d}{2}\right) = 0
\]

and

\[
\psi(d) = 2 (g(d))^{\beta} - 2 (g(c))^{\beta} > 0.
\]

Hence we have,

\[
\int_c^d (d-x) |\psi(x)| \, dx = I_1 + I_2 + I_3 + I_4, \tag{41}
\]

where

\[
I_1 = \int_c^{c+d} (d-x)[g(c + d - x) - g(c)]^{\frac{\beta}{\pi}} \, dx - \int_{c+d}^d (d-x)[g(c + d - x) - g(c)]^{\frac{\beta}{\pi}} \, dx,
\]

\[
I_2 = \int_c^{c+d} (d-x)[g(d) - g(x)]^{\frac{\beta}{\pi}} \, dx - \int_{c+d}^d (d-x)[g(d) - g(x)]^{\frac{\beta}{\pi}} \, dx,
\]

\[
I_3 = - \int_c^{c+d} (d-x)[g(d) - g(c + d - x)]^{\frac{\beta}{\pi}} \, dx + \int_{c+d}^d (d-x)[g(d) - g(c + d - x)]^{\frac{\beta}{\pi}} \, dx,
\]

\[
I_4 = - \int_c^{c+d} (d-x)[g(x) - g(c)]^{\frac{\beta}{\pi}} \, dx + \int_{c+d}^d (d-x)[g(x) - g(c)]^{\frac{\beta}{\pi}}.
\]
By (38),
\[ I_2 = \frac{\beta}{k} \chi^\beta(b, b), \quad I_4 = \frac{\beta}{k} \chi^\beta(b, a), \] (42)
and by suitable change of variables,
\[ I_1 = \frac{\beta}{k} \chi^\beta(c, c), \quad I_3 = \frac{\beta}{k} \chi^\beta(c, d), \] (43)
Using (37), (41), (42) and (43) in (40), one have
\[ \int_0^1 t |\Omega_1(t)| \, dt = \frac{\beta}{k} A^\beta(a, b) \]
(44)
Thus inequality (36) along with (39) and (44) reduced to the required inequality (36).

**Corollary 1.** Let \( u : [c, d] \to \mathbb{R} \) be a differentiable mapping on \((c, d)\). If \(|u'|\) is convex on \([c, d]\), then the following inequality for operators (5) and (6) hold:
\[
\left| \frac{u(c) + u(d)}{2} - \frac{(r + \alpha)^\beta \Gamma(\beta + 1)}{4 [d^{r+\alpha} - c^{r+\alpha}]} \left[ r J^\alpha_{c^+} U(d) + r J^\alpha_{d^-} U(c) \right] \right| \\
\leq \frac{\beta}{r} B^\alpha(c, d) \\
\leq \frac{\beta}{4 [d^{r+\alpha} - c^{r+\alpha}]} (|u'(c)| + |u'(d)|),
\] (45)
where,
\[ \frac{\beta}{r} B^\alpha(c, d) = \frac{\beta}{r} \varrho^\alpha(d, d) + \frac{\beta}{r} \varrho^\alpha(c, d) - \frac{\beta}{r} \varrho^\alpha(d, c) - \frac{\beta}{r} \varrho^\alpha(c, c), \]
and
\[ \frac{\beta}{r} \varrho^\alpha(x, y) := \int_c^d |x-t| |y^{r+\alpha} - t^{r+\alpha}|^\beta dt - \int \frac{d}{x} |x-t| |y^{r+\alpha} - t^{r+\alpha}|^\beta dt, \]
for all \( x, y \in [c, d] \).

**Proof.** Considering \( g(t) = \frac{t^{r+\alpha}}{r^{r+\alpha}}, \) \( r + \alpha \neq 0, \alpha \in (0, 1) \) and \( k = 1, \) in inequality (36), one gets the required result.

**Corollary 2.** Let \( u : [c, d] \to \mathbb{R} \) be a differentiable mapping on \((c, d)\). If \(|u'|\) is convex on \([c, d]\), then the following inequality for operators (5) and (6) hold:
\[
\left| \frac{u(c) + u(d)}{2} - \frac{(1 + s)^\beta \Gamma(\beta + 1)}{4 [d^{1+s} - c^{1+s}]} \left[ s J^k_{c^+} U(d) + s J^k_{d^-} U(c) \right] \right| \\
\leq \frac{\beta}{s} C^k(c, d) \\
\leq \frac{\beta}{4 [d^{1+s} - c^{1+s}]} (|u'(c)| + |u'(d)|),
\] (46)
where,
\[ \frac{\beta}{s} C^k(c, d) = \frac{\beta}{s} \omega^k(d, d) + \frac{\beta}{s} \omega^k(c, d) - \frac{\beta}{s} \omega^k(d, c) - \frac{\beta}{s} \omega^k(c, c), \]
and
\[ \frac{\beta}{s} \omega^k(x, y) := \int_c^{c+d} |x-t| |y^{1+s} - t^{1+s}|^\beta dt - \int \frac{d}{x} |x-t| |y^{1+s} - t^{1+s}|^\beta dt, \]
for all $x, y \in [c, d]$.

**Proof.** By using $g(t) = t^{k-1} + s$, $s \in \mathbb{R} - \{0\}$, in inequality (36), one gets the required inequality.

**Corollary 3.** Let $u : [c, d] \rightarrow \mathbb{R}$ be a differentiable mapping on $(c, d)$. If $|u'|$ is convex on $[c, d]$, then the following inequality for $\beta$-exponential fractional integrals (26) and (27) of order $\mu$ hold:

$$
\left| \frac{u(c) + u(d)}{2} - \frac{\Gamma(\mu + 1)}{4(\exp(\beta d) - \exp(\beta c))^\mu} \left[ \mu \mathcal{S}_{c}^{+} U(d) + \mu \mathcal{S}_{d}^{-} U(c) \right] \right| 
\leq \frac{\beta \mu N(c, d)}{4(\exp(\beta d) - \exp(\beta c))^\mu (d - c)} (|u'(c)| + |u'(d)|),
$$

where

$$
\beta \mu N(c, d) = \beta \mu \Lambda(d, d) + \beta \mu \Lambda(c, d) - \beta \mu \Lambda(d, c) - \beta \mu \Lambda(c, c),
$$

and

$$
\beta \mu \Lambda(x, y) := \frac{c + d}{2} \int_{c}^{d} \frac{|x - t|}{|y - t|} \exp(\beta t) dt - \frac{d}{4(\exp(\beta d) - \exp(\beta c))^\mu} \int_{c}^{d} \frac{|x - t|}{|y - t|} \exp(\beta t) dt,
$$

for all $x, y \in [c, d]$.

**Proof.** Replacing $\beta$ by $\mu$ and using $g(t) = \exp(\beta t)$, $\beta > 0$, $k = 1$, in inequality (36), one gets the required inequality.

**Corollary 4.** Let $u : [c, d] \rightarrow \mathbb{R}$ be a differentiable mapping on $(c, d)$. If $|u'|$ is convex on $[c, d]$, then the following inequality for operators (24) and (25) holds:

$$
\left| \frac{u(c) + u(d)}{2} - \frac{(cd)^{\beta} \Gamma(\beta + 1)}{4(d - c)^{\beta}} \left[ \beta \mathcal{R}_{c} U(d) + \beta \mathcal{R}_{d} U(c) \right] \right| 
\leq \frac{\beta \mu L(c, d)}{4(d - c)^{\beta + 1}} (|u'(c)| + |u'(d)|),
$$

where

$$
\beta \mu L(c, d) = c^{\beta} \left[ \beta \Phi(d, d) + \beta \Phi(c, d) \right] - d^{\beta} \left[ \beta \Phi(c, c) - \beta \Phi(d, c) \right],
$$

and

$$
\beta \Phi(x, y) := \frac{c + d}{2} \int_{c}^{d} \frac{|x - t|}{|y - t|} \exp(\beta t) dt - \frac{d}{4(\exp(\beta d) - \exp(\beta c))^\mu} \int_{c}^{d} \frac{|x - t|}{|y - t|} \exp(\beta t) dt,
$$

for all $x, y \in [c, d]$.

**Proof.** Inequality (36), reduced to the required inequality subject to the condition that $g(t) = -t^{-1}$, $t > 0$, $k = 1$.

**Corollary 5.** Let $u : [c, d] \rightarrow \mathbb{R}$ be a differentiable mapping on $(c, d)$. If $|u'|$ is
convex on \([c, d]\), then the following inequality for Riemann integrals hold:

\[
\left| \frac{u(c) + u(d)}{2} - \frac{1}{2(g(d) - g(c))} \int_c^d g'(t)U(t)dt \right| \\
\leq \frac{1}{(d - c)(g(d) - g(c))} \left( \frac{|u'(c)| + |u'(d)|}{2} \right) \left[ \Theta^{g,g}(x) - \Theta^{d,g}(x) \right],
\]

(49)

where

\[
\Theta^{g,g}(x) = \int_c^x |y - x| g(x)dx - \int_x^d |y - x| g(x)dx.
\]

**Proof.** By taking \(\varphi\) as identity function in inequality (35) and using the same lines as adopted in Theorem (3), we come to the desired inequality.

**Theorem 3.** Let \(g : [c, d] \to \mathbb{R}\) be a positive monotone increasing function on \((c, d]\), having continuous derivatives \(g'\) on \((c, d)\). Let \(u : [c, d] \to \mathbb{R}\) be a differentiable mapping on \((c, d)\). If \(|u'|\) is convex on \([c, d]\), then the following inequality for operators (22) and (23) holds:

\[
\left| \frac{u(c) + u(d)}{2} - \frac{1 - \alpha}{4(1 - \exp(-B))} \left[ \beta E_{c^+} U(d) + \beta E_d U(c) \right] \right| \\
\leq \frac{1}{4(1 - \exp(-B))} N(c, d),
\]

(50)

where

\[
N(c, d) = \xi(d, c) - \xi(c, d) + \xi(c, c) - \xi(d, d),
\]

\[
\xi(x, y) = \int_c^x |x - u| \exp(-A(|u| - g(y)))du \\
- \int_x^d |x - u| \exp(-A(|u| - g(y)))du,
\]

\(A = \frac{1 - \beta}{\beta} \) and \(B = A(g(d) - g(c))\). Proof If we use \(\varphi(u) = \frac{u}{\beta} \exp(-Au)\), where \(A = \frac{1 - \beta}{\beta} \) and \(\beta \in (0, 1)\), then by same lines as followed in the proof of Theorem 3 required inequality (50) is obtained.

4. **Concluding remarks**

This study establishes a very general version of an inequality associated to bounds of Hadamard inequalities. The inequality (35) provides bounds of almost all the Hadamard inequalities via conformable and fractional integral operators available in the literature. Inequalities (36), (45), (46), (47), (48) and (50) gives error bounds of generalized fractional and conformable integral operators. More results can be deduce from inequality (35) by different settings of \(\varphi\) and \(g\). We feel that some generalize inequalities may be obtain by using different kinds of convex functions via integral operators (20) and (21). We hope that this work will attract the attention of researchers working in fractional calculus, mathematical analysis.
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