INITIAL BOUNDS FOR A CLASS OF bi–UNIVALENT FUNCTIONS OF COMPLEX ORDER ASSOCIATED WITH CHEBYSHEV POLYNOMIALS

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Abstract. In this paper, we obtain initial coefficient bounds for functions belong to a subclass of bi–univalent functions by using the Salagean differential operator and Chebyshev polynomials and also we find Fekete-Szego inequalities for functions in this class.

1. Introduction

Let $S$ be the class of analytic and univalent functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, z \in \mathbb{U} = \{z : z \in \mathbb{C} : |z| < 1 \}. \quad (1)$$

For $f$ and $g$ analytic in $\mathbb{U}$, we say that the function $f$ is subordinate to $g$ in $\mathbb{U}$ ($f(z) \prec g(z)$) if there exists an analytic Schwarz function $w(z)$ in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| < 1 \ (z \in \mathbb{U})$, such that $f(z) = g(w(z))$ (see [19]).

Indeed, it is known that

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Rightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}),$$

and if $g$ is univalent in $\mathbb{U}$, then

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \iff f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

It is well known (see Duren [13]) that every function $f \in S$ has an inverse map $f^{-1}$, defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U}),$$

and

$$f(f^{-1}(w)) = w \quad (|w| < r_o(f); \ r_o(f) \geq \frac{1}{4}).$$

In fact, the inverse function $g = f^{-1}$ is given by

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \ldots \quad (2)$$
A function $f \in S$ is said to be bi-univalent function in $U$ if both $f$ and $f^{-1}$ are univalent in $U$. Denote by $\Delta$ the class of bi-univalent functions in $U$. For a history and examples of functions which are (or which are not) in the class $\Delta$, together with various other properties one can refer to [1, 9, 11, 15, 17, 21, 22, 24, 27].

The Chebyshev polynomials of the first and second kinds are well known and defined by (see [2, 10, 12, 14, 16, 18])

$$T_k(t) = \cos kt \quad \text{and} \quad U_k(t) = \frac{\sin(k+1)t}{\sin \theta} \quad (-1 < t < 1),$$

where the degree of the polynomial is $k$ and $t = \cos \theta$.

Consider the function

$$H(z, t) = \frac{1}{1 - 2zt + z^2}.$$  

Note that if $t = \cos \alpha$, $\alpha \in \left(\frac{-\pi}{3}, \frac{\pi}{3}\right)$, then for all $z \in U$

$$H(z, t) = 1 + \sum_{k=1}^{\infty} \frac{\sin(k+1)\alpha}{\sin \alpha} z^k$$

$$= 1 + 2 \cos \alpha + (3 \cos^2 \alpha - \sin^2 \alpha) z^2 + \ldots. \quad (3)$$

Thus, we have [26]

$$H(z, t) = 1 + U_1(t)z + U_2(t)z^2 + \ldots \quad (z \in U, \; t \in (-1, 1)), \quad (4)$$

where $U_{k-1} = \frac{\sin(k \arccos t)}{\sqrt{1 - t^2}}$, for $k \in \mathbb{N} = \{1, 2, \ldots\}$, are the second kind of the Chebyshev polynomials. Also, it is known that

$$U_k(t) = 2tU_{k-1}(t) - U_{k-2}(t), \quad (5)$$

and

$$U_1(t) = 2t, \quad U_2(t) = 4t^2 - 1, \quad U_3(t) = 8t^3 - 4t, \quad U_4(t) = 16t^4 - 12t^2 + 1, \ldots \quad (6)$$

The Chebyshev polynomials $T_k(t)$, $t \in [-1, 1]$, of the first kind have the generating function of the form

$$\sum_{k=0}^{\infty} T_k(t) z^k = \frac{1 - tz}{1 - 2tz + z^2} \quad (z \in U). \quad (7)$$

The first kind of Chebyshev polynomial $T_k(t)$ and second kind of Chebyshev polynomial $U_k(t)$ are connected by:

$$\frac{dT_k(t)}{dt} = kU_{k-1}(t); \quad T_k(t) = U_k(t) - tU_{k-1}(t); \quad 2T_k(t) = U_k(t) - U_{k-2}(t). \quad (8)$$

For $f(z) \in S$, the Salagean operator is defined by (see [23] and [3, 4, 5, 6, 7, 8])

$$D^1 f(z) = Df(z) = zf'(z),$$

$$\vdots$$

$$D^n f(z) = D(D^{n-1} f(z)) = z(D^{n-1} f(z))',$$

$$= z + \sum_{k=2}^{\infty} k^n a_k z^k \quad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1, 2, \ldots\}). \quad (9)$$
By using the Salagean differential operator for $g$ of the form (2), Vijaya et al. [25] (also see [20]) defined $D^n g(w)$ as follows:

$$D^n g(w) = w - a_2 2^n w^2 + (2a_2^2 - a_3)3^n w^3 + \ldots . \quad (10)$$

**Definition 1.** For $\alpha \geq 0, b \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and $t \in (-1, 1)$, a function $f \in \Delta$ of form (1) is said to be in the class $R^n_\Delta (b, \alpha, t)$ if the following subordinations hold:

$$1 + \frac{1}{b} \left[ \frac{z(D^n f(z))'' + \alpha z^2 (D^n f(z))'''}{(1 - \alpha)D^n f(z) + \alpha z(D^n f(z))''} - 1 \right] \prec H(z, t) = \frac{1}{1 - 2tz + z^2}, \quad (11)$$

and

$$1 + \frac{1}{b} \left[ \frac{z(D^n g(w))' + \alpha z^2 (D^n g(w))''}{(1 - \alpha)D^n g(w) + \alpha z(D^n g(w))'} - 1 \right] \prec H(w, t) = \frac{1}{1 - 2tw + w^2}, \quad (12)$$

where $z, w \in \mathbb{U}$ and $g$ is given by (2).

For suitable choices of $n, \alpha$ and $b$, we obtain:

(i) $R^n_\Delta (b, \alpha, t) = R^n_\Delta (b, \alpha, t)$, where

$$1 + \frac{1}{b} \left[ \frac{zf'(z) + \alpha z^2 f''(z)}{(1 - \alpha)f(z) + \alpha zf'(z)} - 1 \right] \prec H(z, t) = \frac{1}{1 - 2tz + z^2},$$

and

$$1 + \frac{1}{b} \left[ \frac{zg'(w) + \alpha z^2 g''(w)}{(1 - \alpha)g(w) + \alpha zg'(w)} - 1 \right] \prec H(w, t) = \frac{1}{1 - 2tw + w^2},$$

(ii) $R^n_\Delta (0, 0, t) = R^n_\Delta (b, t)$, if

$$1 + \frac{1}{b} \left[ \frac{z(D^n f(z))'}{D^n f(z)} - 1 \right] \prec H(z, t) = \frac{1}{1 - 2tz + z^2};$$

and

$$1 + \frac{1}{b} \left[ \frac{z(D^n g(w))'}{D^n g(w)} - 1 \right] \prec H(w, t) = \frac{1}{1 - 2tw + w^2},$$

(iii) $R^n_\Delta (b, 1, t) = R^n_\Delta (b, t)$, if

$$1 + \frac{1}{b} \left[ \frac{z(D^n f(z))'}{(1 - \alpha)f(z) + \alpha zf'(z)} - 1 \right] \prec H(z, t) = \frac{1}{1 - 2tz + z^2},$$

and

$$1 + \frac{1}{b} \left[ \frac{z(D^n g(w))'}{(1 - \alpha)g(w) + \alpha zg'(w)} - 1 \right] \prec H(w, t) = \frac{1}{1 - 2tw + w^2},$$

(iv) $R^n_\Delta (1, \alpha, t) = R^n_\Delta (\alpha, t)$, if

$$\frac{zf'(z) + \alpha z^2 f''(z)}{(1 - \alpha)f(z) + \alpha zf'(z)} \prec H(z, t) = \frac{1}{1 - 2tz + z^2},$$

and

$$\frac{zg'(w) + \alpha z^2 g''(w)}{(1 - \alpha)g(w) + \alpha zg'(w)} \prec H(w, t) = \frac{1}{1 - 2tw + w^2},$$

(v) $R^n_\Delta (1, 0, t) = R^n_\Delta (t)$, if

$$\frac{z(D^n f(z))'}{D^n f(z)} \prec H(z, t) = \frac{1}{1 - 2tz + z^2}, \prec H(z, t) = \frac{1}{1 - 2tz + z^2},$$

$$\frac{z(D^n f(z))'}{D^n f(z)} \prec H(z, t) = \frac{1}{1 - 2tz + z^2}, \prec H(z, t) = \frac{1}{1 - 2tz + z^2},$$

$$\frac{z(D^n f(z))'}{D^n f(z)} \prec H(z, t) = \frac{1}{1 - 2tz + z^2}, \prec H(z, t) = \frac{1}{1 - 2tz + z^2},$$

$$\frac{z(D^n f(z))'}{D^n f(z)} \prec H(z, t) = \frac{1}{1 - 2tz + z^2}, \prec H(z, t) = \frac{1}{1 - 2tz + z^2},$$
and \[ \frac{z(D^n g(w))'}{D^n g(w)} < H(w, t) = \frac{1}{1 - 2tw + w^2}. \]

(vi) \[ R_{A}^n(1, 1, t) = R_{A}^n(t), \]

if

\[ \frac{z(D^n f(z))''}{(D^n f(z))'} < H(z, t) = \frac{1}{1 - 2tz + z^2}. \]

and

\[ \frac{z(D^n g(w))''}{(D^n g(w))'} < H(w, t) = \frac{1}{1 - 2tw + w^2}. \]

(vii) \[ R_{A}^n((1 - \lambda)e^{-i\theta} \cos \theta, \alpha, t) = R_{A}^n(\lambda, \theta, \alpha, t) \quad (0 \leq \lambda < 1, |\theta| < \frac{\pi}{2}), \]

if

\[ e^{i\theta} \left[ \frac{z(D^n f(z))' + az^2(D^n f(z))''}{(1 - \alpha)D^n f(z) + az(D^n f(z))'} \right] < H(z, t)(1 - \lambda) \cos \theta + \lambda \cos \theta + i \sin \theta, \]

and

\[ e^{i\theta} \left[ \frac{z(D^n g(w))' + az^2(D^n g(w))''}{(1 - \alpha)D^n g(w) + az(D^n g(w))'} \right] < H(w, t)(1 - \lambda) \cos \theta + \lambda \cos \theta + i \sin \theta. \]

In this paper, we obtain the initial coefficients bounds and Fekete-Szego problem for functions in the class \( R_{A}^n(b, \alpha, t). \)

2. COEFFICIENT BOUNDS FOR THE FUNCTION CLASS \( R_{A}^n(b, \alpha, t) \)

Unless indicated, we assume that \( \alpha \geq 0, t \in (-1, 1), f \) given by (1.1) and \( b \in \mathbb{C}. \)

**Theorem 1.** Let \( f \in R_{A}^n(b, \alpha, t). \) Then

\[ |a_2| \leq \frac{|b|2t\sqrt{\pi}}{\sqrt{\{4t^2\{b(2+4\alpha)^{3n}-(1+\alpha)^22^{2n}\}-(1+\alpha)^22^{2n}\}+\{1+3(1+\alpha)^22^{2n}\}}} \quad (13) \]

and

\[ |a_3| \leq \frac{4t^2|b|^2}{(1+\alpha)^22^{2n}} + \frac{2t|b|}{(2+4\alpha)^3n}. \quad (14) \]

**Proof.** Let \( f \in R_{A}^n(b, \alpha, t) \) and \( g = f^{-1}. \) From (11) and (12), we have

\[ 1 + \frac{1}{b} \left[ \frac{z(D^n f(z))'}{(1 - \alpha)D^n f(z) + az(D^n f(z))'} \right] = 1 + U_1(t)p(z) + U_2(t)p^2(z) + \ldots \quad (15) \]

and

\[ 1 + \frac{1}{b} \left[ \frac{z(D^n g(w))'}{(1 - \alpha)D^n g(w) + az(D^n g(w))'} \right] = 1 + U_1(t)q(w) + U_2(t)q^2(w) + \ldots \quad (16) \]

for some analytic functions

\[ p(z) = c_1z + c_2z^2 + c_3z^3 + \ldots \quad (z \in \mathbb{U}), \]

\[ q(w) = d_1w + d_2w^2 + d_3w^3 + \ldots \quad (w \in \mathbb{U}), \]

such that \( p(0) = q(0) = 0, \) \( |p(z)| < 1 \quad (z \in \mathbb{U}) \) and \( |q(w)| < 1 \quad (w \in \mathbb{U}). \) It is well known that if \( |p(z)| < 1 \) and \( |q(w)| < 1, \) then

\[ |c_j| \leq 1 \quad \text{and} \quad |d_j| \leq 1 \quad \text{for all} \quad j \in \mathbb{N}. \quad (19) \]
From (15) — (18), we have
\[
\frac{1}{b} \left[ \frac{z(D^n f(z))' + \alpha z^2(D^n f(z))''}{(1 - \alpha)D^n f(z) + \alpha z(D^n f(z))'} - 1 \right] = \frac{1}{b} \left\{ (1 + \alpha) 2^n a_2 z + \left[ (2 + 4\alpha) 3^n a_3 - (1 + \alpha)^2 2^{2n} a_2^2 \right] z^2 + ... \right\} = U_1(t)c_1 z + \left[ U_1(t)c_2 + U_2(t)c_1^2 \right] z^2 + ...
\quad \text{(20)}
\]
and
\[
\frac{1}{b} \left[ \frac{z(D^n g(w))' + \alpha z^2(D^n g(w))''}{(1 - \alpha)D^n g(w) + \alpha z(D^n g(w))'} - 1 \right] = \frac{1}{b} \left\{ -(1 + \alpha) 2^n a_2 w + \right\} \left[ (2 + 4\alpha) 3^n a_3 - (1 + \alpha)^2 2^{2n} a_2^2 \right] w^2 + ...ight\} = U_1(t)d_1 w + \left[ U_1(t)d_2 + U_2(t)d_1^2 \right] w^2 + ...
\quad \text{. (21)}
\]
Equating the coefficients in (20) and (21) we get
\[
\frac{1}{b} (1 + \alpha) 2^n a_2 = U_1(t)c_1, \quad \text{(22)}
\]
\[
\frac{1}{b} \left[ (2 + 4\alpha) 3^n a_3 - (1 + \alpha)^2 2^{2n} a_2^2 \right] = U_1(t)c_2 + U_2(t)c_1^2, \quad \text{(23)}
\]
and
\[
\frac{1}{b} \left\{ (4 + 8\alpha) 3^n - (1 + \alpha)^2 2^{2n} a_2^2 \right\} = U_1(t)d_2 + U_2(t)d_1. \quad \text{(25)}
\]
From (22) and (24) we obtain
\[
c_1 = -d_1 \quad \text{(26)}
\]
and
\[
\frac{1}{b^2} (1 + \alpha)^2 2^{2n+1} a_2^2 = U_1^2(t) \left( c_1^2 + d_1^2 \right). \quad \text{(27)}
\]
Also, (23) and (25) yield
\[
\frac{1}{b} \left[ (4 + 8\alpha) 3^n - (1 + \alpha)^2 2^{2n+1} \right] a_2^2 = U_1(t) \left( c_2 + d_2 \right) + U_2(t) \left( c_1^2 + d_1^2 \right), \quad \text{(28)}
\]
which by (??), leads to
\[
\left[ (4 + 8\alpha) 3^n - (1 + \alpha)^2 2^{2n+1} - \frac{U_2(t)2^{2n+1}}{bU_1^2(t)} (1 + \alpha)^2 \right] a_2^2 = bU_1(t) \left( c_2 + d_2 \right). \quad \text{(29)}
\]
From (6), (19) and (29), we have (13).
Next, by subtracting (??) from (23), we have
\[
\frac{2}{b} (2 + 4\alpha) 3^n (a_3 - a_2^2) = U_1(t) \left( c_2 - d_2 \right) + U_2(t) \left( c_1^2 - d_1^2 \right). \quad \text{(30)}
\]
Further, in view of (26), we obtain
\[
a_3 = a_2^2 + \frac{bU_1(t)}{2(2 + 4\alpha) 3^n} (c_2 - d_2). \quad \text{(30)}
\]
Hence using (27) and applying (6), we get (14).
This completes the proof of Theorem 1.

Taking \( n = 0 \) in Theorem 1, we get the following consequence.

**Corollary 1.** Let \( f \in \Delta \) be in the class \( R_\Delta (b, \alpha, t) \). Then

\[
|a_2| \leq \frac{|b| 2|\pi|}{\sqrt{|\{4t^2 \{b[(2+4\alpha)^2-(1+\alpha)^2]-[(1+\alpha)^2+(1+\alpha)^2]\}\}|},
\]

and

\[
|a_3| \leq \frac{4t^2 |b|^2}{(1+\alpha)^2} + \frac{2t |b|}{(2+4\alpha)}.
\]

Taking \( \alpha = 1 \) in Corollary 1, we get the following consequence.

**Corollary 2.** Let \( f \in \Delta \) be in the class \( R_\Delta (b, t) \). Then

\[
|a_2| \leq \frac{|b| 2|\pi|}{\sqrt{|t^2[2(2+4\alpha)+4\alpha]-1|}},
\]

and

\[
|a_3| \leq t^2 |b|^2 + \frac{t |b|}{3}.
\]

Taking \( b = e^{-i\theta}(1-\lambda) \cos \theta \) \((0 \leq \lambda < 1, |\theta| < \frac{\pi}{2})\) in Corollary 2, we get the following consequence.

**Corollary 3.** Let \( f \in \Delta \) be in the class \( R_\Delta (\lambda, \theta, t) \). Then

\[
|a_2| \leq \frac{t|\pi| (1-\lambda) \cos \theta}{\sqrt{|t^2[2-\cos \theta]-4|}},
\]

and

\[
|a_3| \leq t^2 (1-\lambda)^2 \cos^2 \theta + \frac{t(1-\lambda) \cos \theta}{3}.
\]

Taking \( \lambda = 0 \) in Corollary 3, we get the following consequence.

**Corollary 4.** Let \( f \in \Delta \) be in the class \( R_\Delta (\theta, t) \). Then

\[
|a_2| \leq \frac{t|\pi| \cos \theta}{\sqrt{|t^2[2 \cos \theta]-4|}},
\]

and

\[
|a_3| \leq t^2 \cos^2 \theta + \frac{t \cos \theta}{3}.
\]

3. FEKETE- SZEGO INEQUALITIES FOR THE CLASS \( R_\Delta^w (b, \alpha, t) \)

**Theorem 2.** If \( f \in R_\Delta^w (b, \alpha, t) \) and \( \xi \in \mathbb{R} \), then

\[
|a_3 - \xi a_2^2| \leq \begin{cases} 
\frac{2b|t|}{(2+4\alpha)^3 \pi}, & |\xi - 1| \leq k \\
\frac{8|b|^2 |\xi - 1|^3}{|\{4t^2 \{b[(2+4\alpha)^3-2(1+\alpha)]^2-2(1+\alpha)^2 \}+(1+\alpha)^2 \}^2|}, & |\xi - 1| \geq k,
\end{cases}
\]

where \( k = \frac{|\{4t^2 \{b[(2+4\alpha)^3-(1+\alpha)^2]^2-2(1+\alpha)^2 \}+(1+\alpha)^2 \}^2|}{4t^2 |b[(2+4\alpha)^3]-1|} \).

**Proof.** From (29) and (30)

\[
(a_3 - \xi a_2^2) = (1-\xi) \left[ \frac{b^2 U_1^2(t)(c_2 - d_2)}{M_1^2(t)(4+8a^3)/3 -(1+\alpha)^2 2^{n+1}} - U_2(t)(1+\alpha)^2 2^{n+1}c_2 + \frac{b U_1^2(t)}{2(2+4\alpha)^3} \right] + \frac{b U_1^2(t)}{2(2+4\alpha)^3} (c_2 - d_2)
\]

\[
= b U_1^2(t) \left[ \left( h(\xi) + \frac{1}{2(2+4\alpha)^3} \right) c_2 + \left( h(\xi) - \frac{1}{2(2+4\alpha)^3} \right) d_2 \right],
\]

where

\[
h(\xi) = \frac{b(1-\xi)U_1^2(t)}{b U_1^2(t)(4+8\alpha)^3 - (1+\alpha)^2 2^{n+1} - U_2(t)(1+\alpha)^2 2^{n+1}}.
\]
Then, by taking the modulus of (32) and considering (6), we have

\[
|a_3 - \xi a_2^2| \leq \begin{cases} 
\frac{2|b|}{(2+4\alpha)^3}, & |h(\xi)| \leq \frac{1}{(2+4\alpha)^3} \\
\frac{4|b| |h(\xi)| t,} & |h(\xi)| \geq \frac{1}{(2+4\alpha)^3}.
\end{cases}
\]

This completes the proof of Theorem 2.

Taking \( \xi = 1 \) in Theorem 2, we get the following consequence.

**Corollary 5.** Let the function \( f \in \Delta \) given by (1) be in the class \( R^\alpha_\Delta(b, \alpha, t) \). Then

\[
|a_3 - a_2^2| \leq \frac{2|b| t}{(2 + 4\alpha)^3}.
\]

Taking \( \alpha = 1 \) and \( n = 0 \) in Corollary 5, we get the following consequence.

**Corollary 6.** For \( t \in (-1, 1) \), let the function \( f \in \Delta \) given by (1) be in the class \( R_\Delta(b, t) \). Then

\[
|a_3 - a_2^2| \leq \frac{t|b|}{3}.
\]

Taking \( b = e^{-it}(1 - \lambda) \cos \theta \) \((0 \leq \lambda < 1, |\theta| < \frac{\pi}{2})\) in Corollary 6, we get the following consequence.

**Corollary 7.** For \( t \in (-1, 1) \), let the function \( f \in \Delta \) given by (1) be in the class \( R_\Delta(\lambda, \theta, t) \). Then

\[
|a_3 - a_2^2| \leq \frac{t(1 - \lambda) \cos \theta}{3}.
\]

Taking \( \lambda = 0 \) in Corollary 7, we get the following consequence.

**Corollary 8.** For \( t \in (-1, 1) \), let the function \( f \in \Delta \) given by (1) be in the class \( R_\Delta(\theta, t) \). Then

\[
|a_3 - a_2^2| \leq \frac{t \cos \theta}{3}.
\]

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**References**


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