SOME PROPERTIES OF VARIABLE-ORDER FRACTIONAL CALCULUS

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ABSTRACT. The topic of fractional calculus (derivatives and integrals of arbitrary orders) is enjoying growing interest not only in Mathematics, but also in Physics, Engineering and Mathematical Biology (see [?] for example). In this paper we are concerned with the fractional-order differential equations describe Games with non-uniform interaction rate [?] and Asymmetric games [?]. Existence, uniqueness and stability of the solutions of these systems are studied.

1. INTRODUCTION

Recently, operators and differential equations of variable-order have been considered, see [2]-[29]. In these works, authors considered the applications of derivative of variable-order in various science such as anomalous diffusion modeling, mechanical applications, multi-fractional Gaussian noises. Among of these, there have many works dealing with numerical methods for some class of variable-order fractional differential equations, for instance, [8], [10], [16], [17], [18], [19], [22], [23], [26]. Moreover, a physical experimental study of variable-order operators has been considered in [15]. A comparative study of constant-order and variable-order models have been considered in [13]. The fractional differential equations are the generalization of differential equations of integer order. The fractional operators (fractional derivatives and integrals) are the generalization of integer-order differential and integral operators. It is well known that the motivation for those works arises from both the development of the theory of fractional calculus itself and the applications of such constructions in various sciences such as physics, chemistry, aerodynamics, electro-dynamics of complex medium, polymer rheology, and so on.

2010 Mathematics Subject Classification. 26A33, 34B15.
Key words and phrases. variable-order fractional integral, variable-order fractional derivative, semigroup property, Laplace transform.
This research was funded by the National Natural Science Foundation of China (11671181).
In view of the Laplace transform, semigroup property of the constant order fractional integration operators and some other properties of fractional derivatives and integrals, the fractional differential equations can be transformed into their equivalent integral equations. However, we notice that there has few results for the variable-order fractional integrals and derivatives. Therefore, the main contribution of this paper is to show that the variable-order fractional integrals do not have semigroup properties, and that there is no explicit connection between the Laplace transforms of function \( x(t) \) and variable-order integrals and derivatives to \( x(t) \).

2. Preliminaries

We give several definitions of variable-order fractional integrals and derivatives, which can be found in [10]. We let \(-\infty < a < b < \infty\).

**Definition 2.1** ([10]). Let \( p : [a, b] \times \mathbb{R} \rightarrow (0, +\infty) \), the left Riemann-Liouville fractional integral of order \( p(t, x(t)) \) for function \( x(t) \) are defined as the following two types

\[
I^{p(t,x(t))}_{a+} x(t) = \int_a^t \frac{(t-s)^{p(t,x(t))-1}}{\Gamma(p(t,x(t))} x(s)ds, \ t > a, \tag{2.1}
\]

\[
I^{p(t,x(t))}_{a+} x(t) = \int_a^t \frac{(t-s)^{p(s,x(s))-1}}{\Gamma(p(s,x(s))} x(s)ds, \ t > a. \tag{2.2}
\]

**Definition 2.2** ([10]). Let \( p : [a, b] \times \mathbb{R} \rightarrow (0, +\infty) \), the left Riemann-Liouville fractional derivative of order \( p(t, x(t)) \) for function \( x(t) \) are defined as the following two types

\[
D^{p(t,x(t))}_{a+} x(t) = \left( \frac{d}{dt} \right)^n \int_a^t \frac{(t-s)^{n-p(t,x(t))-1}}{\Gamma(n-p(t,x(t))} x(s)ds, \ t > a, \tag{2.3}
\]

\[
D^{p(t,x(t))}_{a+} x(t) = \left( \frac{d}{dt} \right)^n \int_a^t \frac{(t-s)^{n-p(s,x(s))-1}}{\Gamma(n-p(s,x(s))} x(s)ds, \ t > a, \tag{2.4}
\]

where \( n \in \mathbb{N} \) satisfies \( n-1 < p(t,x(t)) \leq n \) for all \((t, x) \in [a, b] \times \mathbb{R}\).

Obviously, the Riemann-Liouville fractional integral \( I^\delta_{a+} \) and derivative \( D^\delta_{a+} \) of function \( x(t) \) are particular cases of (2.1), (2.2) and (2.3), (2.4) (see [1]), i.e.

\[
I^\delta_{a+} x(t) = \left( \frac{d}{dt} \right)^n \int_a^t \frac{(t-s)^{n-\delta}}{\Gamma(n-\delta)} x(s)ds, \ t > a,
\]

\[
D^\delta_{a+} x(t) = \left( \frac{d}{dt} \right)^n \int_a^t \frac{(t-s)^{n-\delta}}{\Gamma(n-\delta)} x(s)ds, \ t > a,
\]

where \( n \in \mathbb{N} \) satisfies \( n-1 < p(t,x(t)) \leq n \) for all \((t, x) \in (0, +\infty) \times \mathbb{R}\).

The following are some properties for the Riemann-Liouville fractional integral and derivative, which can be founded in [1].

**Proposition 2.1** ([1]). The Riemann-Liouville fractional integral defined for function \( x(t) \in L(a,b) \) exists almost everywhere.

**Proposition 2.2** ([1]). The equality \( I^\gamma_{a+} I^\delta_{a+} f(t) = I^\gamma_{a+} I^\delta_{a+} f(t) = I^{\gamma+\delta}_{a+} f(t) \), \( \gamma, \delta > 0 \) holds for \( f \in L(a,b) \).

**Proposition 2.3** ([1]). The equality \( D^\gamma_{a+} I^\delta_{a+} f(t) = f(t), n-1 < \gamma \leq n(n \in \mathbb{N}^+) \) holds for \( f \in L(a,b) \).
Proposition 2.4 (II). Let $n - 1 < \alpha \leq n (n \in \mathbb{N}^+)\), then the differential equation
\[ D_{a+}^\alpha u = 0 \]
has a unique solution
\[ u(t) = c_1(t - a)^{\alpha - 1} + c_2(t - a)^{\alpha - 2} + \cdots + c_n(t - a)^{\alpha - n}, c_i \in \mathbb{R}. \]

Proposition 2.5 (II). Let $n - 1 < \alpha \leq n (n \in \mathbb{N}^+)\), $u \in L(a, b)$, $D_{a+}^\alpha u \in L(a, b)$. Then the following equality holds
\[ I_{a+}^\alpha D_{a+}^\alpha u(t) = u(t) + c_1(t - a)^{\alpha - 1} + c_2(t - a)^{\alpha - 2} + \cdots + c_n(t - a)^{\alpha - n}, c_i \in \mathbb{R}. \]

Proposition 2.6 (II). Assume the Riemann-Liouville fractional integral $I_{0+}^{\delta}$ ($\delta > 0$) and Laplace transform of function $f(t)$ exist, then the Laplace transform of the Riemann-Liouville fractional integral $I_{0+}^{\delta} f(t)$ is
\[ L\{I_{0+}^{\delta} f; s\} = s^{-\delta} F(s), \]
where $F(s)$ is Laplace transform of function $f(t)$.

Proposition 2.7 (II). Let the Riemann-Liouville fractional derivative $D_{0+}^{\alpha}$ ($n - 1 < \alpha \leq n, n \in \mathbb{N}^+$) and Laplace transform of function $f(t)$ exist, then the Laplace transform of the Riemann-Liouville fractional derivative $D_{0+}^{\delta} f(t)$ is
\[ L\{D_{0+}^{\delta} f; s\} = s^{\delta} F(s) - \sum_{k=0}^{n-1} s^{n-k-1} D^k (I_{0+}^{-\delta} f(0+)), \]
where $F(s)$ is Laplace transform of function $f(t)$.

3. SOME FACTS FOR VARIABLE-ORDER FRACTIONAL CALCULUS

We first consider existence result for variable-order fractional integrals defined by (2.1) and (2.2).

Lemma 3.1. Let $p : [a, b] \times \mathbb{R} \to [p^*, q^*]$ ($0 < p^* < q^* < +\infty$) be a real function. Then for $x \in L(a, b)$, variable-order fractional integrals (2.1) and (2.2) exist almost everywhere, provided that function $\frac{1}{\Gamma(p(t, x)))}$ is bounded.

Proof. We only verify the result for variable-order fractional integral (2.2). The proof for variable-order fractional integral (2.1) is similar to it.

By the bounded assumption of $\frac{1}{\Gamma(p(t, x)))}$, letting $L_p = \sup_{a \leq t \leq b} |\frac{1}{\Gamma(p(t, x)))}|$. For $a \leq s \leq t \leq b$, when $p^* \leq p(s, x(s)) \leq q^*$, we have known that
\[ (t - s)^{p(s, x(s)) - 1} \leq (t - s)^{p^* - 1}; \tag{3.1} \]
\[ (t - s)^{p(s, x(s)) - 1} \leq (t - s)^{q^* - 1}. \tag{3.2} \]

Thus, from (3.1), (3.2), for $a \leq s \leq t \leq b$, we have
\[ (t - s)^{p(s, x(s)) - 1} \leq \max\{(t - s)^{p^* - 1}, (t - s)^{q^* - 1}\} = (t - s)^{\alpha - 1}, \tag{3.3} \]
where $\alpha$ denotes $p^*$ or $q^*$.
Thus, for $x \in L(a, b)$, by the definition of variable-order fractional integral (2.2), using (3.3), we have

$$|I^{p(t,x(t))}_{a+}x(t)| \leq \left| \int_a^t \frac{(t - s)^{p(s,x(s))-1}}{\Gamma(p(s,x(s)))}x(s)ds \right| \leq L_p \int_a^t (t - s)^{p(s,x(s))-1}\left| x(s) \right| ds \leq L_p \int_a^t (t - s)^{\alpha - 1}\left| x(s) \right| ds.$$  

It follows from Proposition 2.1 that

$$\int_a^t (t - s)^{\alpha - 1}\left| x(s) \right| ds = \Gamma(\alpha)I^{\alpha}_{a+}|x(t)|,$$

which implies that variable-order fractional integral (2.2) exists almost everywhere for $x \in L(a, b)$. Thus we complete the proof.

**Remark 3.1.** If $p : [a,b] \to (0,1)$ is a real continuous function. Then for $x \in C_r[a,b] = \{x(t) \in C[a,b], \int_a^t x(s)ds \in C[a,b]\}$ (0 < $r < \min_{a \leq t \leq b} |p(t)|$), then variable-order fractional integrals (2.1) and (2.2) exist for any points on $[a,b]$.

In fact, it follows from the continuity of function $\Gamma(p(t))$ that $L_p = \max_{a \leq t \leq b} \left| \frac{1}{\Gamma(p(t))} \right|$ exists.

Let $\beta = \min_{a \leq t \leq b} p(t)$, then, for $a \leq s \leq t \leq b$, when $\beta \leq p(s,x(s)) < 1$, we have known that

if $0 \leq t - s \leq 1$, then $(t - s)^{p(s)-1} \leq (t - s)^{\beta - 1};$ \hspace{1cm} (3.4)

if $1 < t - s < +\infty$, then $(t - s)^{p(s)-1} \leq 1.$ \hspace{1cm} (3.5)

Thus, from (3.4), (3.5), for $a \leq s \leq t \leq b$, we have

$$(t - s)^{p(s)-1} \leq \max\{(t - s)^{\beta - 1}, 1\}. \hspace{1cm} (3.6)$$

For $x \in C_r[a,b]$, by the definition of variable-order fractional integral (2.2), using (3.6), we have

$$|I^{p(t)}_{a+}x(t)| \leq L_p \int_a^t (t - s)^{p(s)-1}\left| x(s) \right| ds \leq L_p \int_a^t (t - s)^{\beta - 1}s^{-r}\left| x(s) \right| ds \leq L_p \max_{0 \leq t \leq b} t^r \left| x(t) \right| \int_a^t (t - s)^{\beta - 1}s^{-r}ds \leq L_p \max_{0 \leq t \leq b} t^r \left| x(t) \right| \frac{(t - a)^{\beta - r}\Gamma(\beta)\Gamma(1 - r)}{\Gamma(1 - r + \beta)}.$$
or

$$|I_{a+}^{p(t)}x(t)| \leq L_p \int_a^t (t-s)^{p(s)-1}|x(s)| ds$$

$$= L_p \int_a^t (t-s)^{\beta-1}s^{-r}|x(s)| ds$$

$$\leq L_p \max_{0 \leq t \leq b} t^r |x(t)| \int_a^t s^{-r} ds$$

$$= \frac{L_p \max_{0 \leq t \leq b} t^r |x(t)|(t-a)^{1-r}}{1-r}$$

$$\leq \frac{L_p \max_{0 \leq t \leq b} t^r |x(t)|(b-a)^{1-r}}{1-r},$$

both the two estimations imply that variable-order fractional integral (2.2) exists everywhere for $x \in C_r[a,b]$. The proof for variable-order fractional integral (2.1) is similar to that.

**Example 3.1** Let $x(t) = t^\frac{2}{3}$, $p(t, x(t)) = \frac{1}{3} + (x(t))^\frac{2}{3} + 0.5 \sin(\pi t) = \frac{1}{3} + t + 0.5 \sin(\pi t)$, $0 \leq t \leq 1$, we will calculate the variable order integral $I_{0+}^{p(t,x(t))}x(t)$ defined in (2.1).

By (2.1), for $0 \leq t \leq 1$, we obtain that

$$I_{0+}^{p(t,x(t))}x(t) = \left. \frac{1}{\Gamma(\frac{1}{3} + t + 0.5 \sin(\pi t))} \int_0^t (t-s)^{t+0.5 \sin(\pi t)-\frac{2}{3}}s^{\frac{2}{3}} ds \right|_{s=0} = \frac{\Gamma(\frac{2}{3})t^{1+t+0.5 \sin(\pi t)}}{\Gamma(2+t+0.5 \sin(\pi t))},$$

obviously, $\Gamma(\frac{2}{3})t^{1+t+0.5 \sin(\pi t)} \Gamma(2+t+0.5 \sin(\pi t))^{-1}$ is bounded for $t \in [0,1]$.

**Example 3.2** Let $x(t) = t^\frac{2}{3}$, $p(t, x(t)) = \frac{1}{3} + (x(t))^\frac{2}{3} + 0.5 \sin(\pi t) = \frac{1}{3} + t + 0.5 \sin(\pi t)$, $0 \leq t \leq 1$, we will calculate the variable order integral $I_{0+}^{p(t,x(t))}x(t)$ defined in (2.2).

By (2.2), we obtain that

$$I_{0+}^{p(t,x(t))}x(t) = \int_0^t \frac{(t-s)^{s+0.5 \sin(\pi s)-\frac{2}{3}}}{\Gamma(\frac{1}{3} + s + 0.5 \sin(\pi s))} s^{\frac{2}{3}} ds, 0 \leq t \leq 1.$$  

It follows from the continuity of Gamma function, we know that $\frac{1}{\Gamma(\frac{1}{3} + s + 0.5 \sin(\pi s))}$ is bounded for $s \in [0,1]$, denoted by $M_1$. Moreover, for $0 \leq s \leq t \leq 1$, $(t-s)^{s+0.5 \sin(\pi s)-\frac{2}{3}} \leq (t-s)^{-\frac{2}{3}}$, as a result, we have that

$$\left|I_{0+}^{p(t,x(t))}x(t)\right| \leq M_1 \int_0^t (t-s)^{s+0.5 \sin(\pi s)-\frac{2}{3}}s^{\frac{2}{3}} ds$$

$$\leq M_1 \int_0^t (t-s)^{-\frac{2}{3}}s^{\frac{2}{3}} ds = M_1\Gamma(\frac{5}{3}) \Gamma(\frac{5}{3}) t \leq M_1\Gamma(\frac{1}{3}) \Gamma(\frac{2}{3}),$$

which implies that $I_{0+}^{p(t,x(t))}x(t)$ is bounded for $t \in [0,1]$.

The following are curves for variable-order integral $I_{0+}^{p(t,x(t))}x(t) = \frac{\Gamma(\frac{2}{3})t^{1+t+0.5 \sin(\pi t)}}{\Gamma(2+t+0.5 \sin(\pi t))},$
0 \leq t \leq 1 \text{ defined in 2.1, and variable-order integral } I_{0+}^{p(t,x(t))} x(t) = \int_0^t \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))} \frac{(s-\tau)^{q(s)-1}}{\Gamma(q(s))} f(\tau) d\tau ds, 0 \leq t \leq 1 \text{ defined in 2.2.}

Now, let’s take an example to consider results of Proposition 2.2 for variable-order fractional integrals 2.1 and 2.2.

Let \( p(t, x(t)) = p(t), q(t, x(t)) = q(t) \) be not constant functions, for \( p, q \in C[0, b], 0 < b < +\infty, f(t) = 1 \), \( t \in [0, T] \), we calculate \( I_{0+}^{p(t)} I_{0+}^{q(t)} f(t) \) and \( I_{0+}^{p(t)+q(t)} f(t) \). First, according to (2.1), we have

\[
I_{0+}^{p(t)} I_{0+}^{q(t)} f(t) = \int_0^t \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))} \int_0^s \frac{(s-\tau)^{q(s)-1}}{\Gamma(q(s))} f(\tau) d\tau ds
\]

\[
= \int_0^t \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))} \int_0^s \frac{(s-\tau)^{q(s)-1}}{\Gamma(q(s))} d\tau ds
\]

\[
= \int_0^t \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))} \frac{s^{q(s)}}{\Gamma(1+q(s))} ds
\]

\[
= \int_0^1 \frac{t^{p(t)+q(s)} s^{p(t)}}{\Gamma(p(t))\Gamma(1+q(s))} ds, \quad I_{0+}^{p(t)+q(t)} f(t) = \int_0^t \frac{(t-s)^{p(t)+q(t)-1}}{\Gamma(p(t)+q(t))} ds = \frac{t^{p(t)+q(t)}}{\Gamma(1+p(t)+q(t))}.
\]

Thus, for \( f(t) \equiv 1 \), we see that, if

\[
\int_0^1 \frac{t^{p(t)+q(t)} s^{p(t)} (1-s)^{p(t)-1}}{\Gamma(p(t))\Gamma(1+q(ts))} ds = \frac{t^{p(t)+q(t)}}{\Gamma(1+p(t)+q(t))},
\]

then we could obtain

\[
I_{0+}^{p(t)} I_{0+}^{q(t)} f(t) = I_{0+}^{p(t)+q(t)} f(t).
\]

However, we can’t assert (3.7) holds. We only get

\[
\int_0^1 \frac{t^{p(t)+q(t)} s^{p(t)} (1-s)^{p(t)-1}}{\Gamma(p(t))\Gamma(1+q(t))} ds = \frac{t^{p(t)+q(t)}}{\Gamma(1+p(t)+q(t))}.
\]

![Figure 1](image-url)
Second, according to (2.2), we have

\[ I_{0+} ^{p(t)} f^{q(t)}(t) = \int_0^t \frac{(t-s)^{p(s)-1}}{\Gamma(p(s))} \int_0^s \frac{(s-\tau)^{q(\tau)-1}}{\Gamma(q(\tau))} f(\tau) d\tau ds \]

\[ = \int_0^t \frac{(t-s)^{p(s)-1}}{\Gamma(p(s))} \int_0^s \frac{(s-\tau)^{q(\tau)-1}}{\Gamma(q(\tau))} d\tau ds \]

\[ = \int_0^t \frac{(t-s)^{p(s)-1}}{\Gamma(p(s))} \int_0^1 s^{q(s\mu)(1-\mu)^q(s\mu)-1} \frac{d\mu ds}{\Gamma(q(s\mu))} \]

\[ = \int_0^1 \int_0^t \frac{t^{p(tr)}(1-r)^{p(tr)-1} (tr)^{q(tr\mu)(1-\mu)^q(tr\mu)-1}}{\Gamma(p(tr)) \Gamma(q(tr\mu))} dr d\mu \]

\[ = \int_0^1 \int_0^t \frac{t^{p(tr)+q(tr\mu)}(1-r)^{p(tr)-1} r^{q(tr\mu)(1-\mu)^q(tr\mu)-1}}{\Gamma(p(tr)) \Gamma(q(tr\mu))} dr d\mu, \]

\[ I_{0+} ^{p(t)+q(t)} f(t) = \int_0^t \frac{(t-s)^{p(s)+q(s)-1}}{\Gamma(p(s)+q(s))} f(s) ds = \int_0^1 \frac{t^{p(tr)+q(tr\mu)}(1-r)^{p(tr)-1} r^{q(tr\mu)(1-\mu)^q(tr\mu)-1}}{\Gamma(p(tr)) \Gamma(q(tr\mu))} dr. \]

Thus, for \( f(t) \equiv 1 \), we see that, if

\[ \int_0^1 \frac{t^{p(tr)+q(tr\mu)}(1-r)^{p(tr)-1} r^{q(tr\mu)(1-\mu)^q(tr\mu)-1}}{\Gamma(p(tr)) \Gamma(q(tr\mu))} dr = \frac{t^{p(tr)+q(tr)}(1-r)^{p(tr)+q(tr)-1}}{\Gamma(p(tr) + q(tr))}, \]

then we could obtain

\[ I_{0+} ^{p(t)} I_{0+} ^{q(t)} f(t) = I_{0+} ^{p(t)+q(t)} f(t). \]

However, we can’t assert (3.8) holds. We only get

\[ \int_0^1 \frac{t^{p(tr)+q(tr\mu)}(1-r)^{p(tr)-1} r^{q(tr\mu)(1-\mu)^q(tr\mu)-1}}{\Gamma(p(tr)) \Gamma(q(tr\mu))} dr = \frac{t^{p(tr)+q(tr)}(1-r)^{p(tr)+q(tr)-1}}{\Gamma(p(tr) + q(tr))} \]

Therefore, from the above arguments, we could obtain the following result.

**Lemma 3.2.** Let \( x(t), p(t, x(t)), q(t, x(t)) \) be real functions on finite interval \([0, T]\), assume that variable-order fractional integrals \( I_{0+} ^{p(t, x(t))} x(t), I_{0+} ^{q(t, x(t))} x(t) \) and \( I_{0+} ^{p(t, x(t)) + q(t, x(t))} x(t) \) defined by (2.1), (2.2) exist. In general case, we could claim that

\[ I_{0+} ^{p(t, x(t))} I_{0+} ^{q(t, x(t))} x(t) \neq I_{0+} ^{p(t, x(t)) + q(t, x(t))} x(t), \]

for some points in \([0, T]\).

In particular, for general functions \( 0 < p(t, x(t)) < 1 \) and \( x(t) \), we have

\[ I_{0+} ^{p(t, x(t))} I_{0+} ^{1-p(t, x(t))} x(t) \neq I_{0+} ^{p(t, x(t)) + 1-p(t, x(t))} x(t) = I_{0+} ^1 x(t), \]

for some points in \([0, T]\).

In order to demonstrate the claim of Lemma 3.2, we give the following two examples about variable order fractional integrals (2.1) and (2.2).

**Example 3.3** Let \( p(t) = t, 0 \leq t \leq 3, q(t) = \frac{t}{3} + \frac{1}{5}, f(t) = 1, 0 \leq t \leq 3 \). Now, we calculate \( I_{0+} ^{p(t)} I_{0+} ^{q(t)} f(t) \) and \( I_{0+} ^{p(t) + q(t)} f(t) \) defined in (2.1).
We see that
\[
I_{0+}^{(p)} I_{0+}^{(q)} f(t) = \int_0^t \int_0^s \frac{(t-s)^{p-1}}{\Gamma(p)} \frac{(s-\tau)^{q-1}}{\Gamma(q)} f(\tau) d\tau ds
\]
\[
= \int_0^t \int_0^s \frac{(t-s)^{p-1}}{\Gamma(p)} \frac{(s-\tau)^{q-1}}{\Gamma(q)} d\tau ds
\]
\[
= \int_0^t (t-s)^{p-1} \frac{1}{\Gamma(p)}ds
\]
\[
= \int_0^t (t-s)^{p-1} \frac{1}{\Gamma(p)}ds + \int_0^t (t-s)^{p-1} \frac{1}{\Gamma(p)}ds.
\]
We set \(M_1 = \max_{2 \leq t \leq 3} \left| \frac{1}{\Gamma(p)} \right| \) and \(M_2 = \max_{2 \leq t \leq 3} \left| \frac{1}{\Gamma(p)} \right| \). For \(2 \leq t \leq 3\), it follows from
\[
\left| \int_2^t (t-s)^{p-1} \frac{1}{\Gamma(p)}ds \right| \leq M_1 M_2 \int_2^t 3^{t-1} \left| \frac{t-s}{3} \right| ds
\]
\[
\leq M_1 M_2 \int_2^t 3^{t-1} \left( \frac{t-s}{3} \right)^2 ds
\]
\[
= M_1 M_2 \int_2^t 3 t^2 ds
\]
that
\[
\left[ \int_2^t (t-s)^{p-1} \frac{1}{\Gamma(p)}ds \right]_{t=2} = 0.
\]
So, we get
\[
I_{0+}^{(p)} I_{0+}^{(q)} f(t)_{t=2} = \int_0^2 \frac{(2-s)^{2-1} \frac{1}{\Gamma(2)}}{\Gamma(\frac{1}{3} + \frac{1}{3})} ds \approx 1.79
\]
and
\[
I_{0+}^{(p+q)} f(t)_{t=2} = \int_0^2 \frac{(2-s)^{2-1} \frac{1}{\Gamma(2)}}{\Gamma(1 + \frac{2}{3} + \frac{2}{3})} ds = \frac{2^{p+q}(2)\Gamma(2)}{\Gamma(1 + p(2) + q(2))} = \frac{4}{3}.
\]
Therefore,
\[
I_{0+}^{(p)} I_{0+}^{(q)} f(t)_{t=2} \neq I_{0+}^{(p+q)} f(t)_{t=2}.
\]

**Example 3.4** Let \(p(t) = t\), \(q(t) = 1\), \(f(t) = 1\), \(0 \leq t \leq 3\). Now, we calculate \(I_{0+}^{(p)} I_{0+}^{(q)} f(t)_{t=1}\) and \(I_{0+}^{(p+q)} f(t)_{t=1}\) defined in (2.2).
\[
I_{0+}^{(p)} I_{0+}^{(q)} f(t)_{t=1} = \int_0^1 \frac{(1-s)^{s-1}}{\Gamma(s)} ds = \int_0^1 \frac{(1-s)^{s-1}}{\Gamma(s)} ds \approx 0.472.
\]
and
\[
I_{0+}^{(p+q)} f(t)_{t=1} = \int_0^1 \frac{(1-s)^{s}}{\Gamma(s+1)} ds = \int_0^1 \frac{(1-s)^{s}}{s\Gamma(s)} ds \approx 0.686.
\]
Therefore,
\[
I_{0+}^{(p)} I_{0+}^{(q)} f(t)_{t=1} \neq I_{0+}^{(p+q)} f(t)_{t=1}.
\]
Remark 3.2. According to Lemma 3.2, we could claim that variable-order calculus of non-constant functions $p(t, x(t))$ for $x(t)$ defined by (2.1), (2.2) don’t have the properties like Propositions 2.3-2.5.

Integral and derivative of variable-order (2.1), (2.2) and (2.3), (2.4), defined on a finite interval $[a, b]$ of the real line $\mathbb{R}$, are naturally extended to the half-axis on $\mathbb{R}^+$. The variable-order integration and differentiation constructions, corresponding to the ones in (2.1), (2.2) and (2.3), (2.4), having the following forms

\[
I_{0+}^{p(t,x(t))} x(t) = \int_0^t \frac{(t-s)^{p(t,x(t))-1}}{\Gamma(p(t,x(t)))} x(s)ds, \quad t > 0, \quad (3.9)
\]

\[
I_{0+}^{p(t,x(t))} x(t) = \int_0^t \frac{(t-s)^{p(s,x(s))-1}}{\Gamma(p(s,x(s)))} x(s)ds, \quad t > 0. \quad (3.10)
\]

\[
D_{0+}^{p(t,x(t))} x(t) = \left( \frac{d}{dt} \right)^n \int_0^t \frac{(t-s)^{n-p(t,x(t))-1}}{\Gamma(n-p(t,x(t)))} x(s)ds, \quad t > 0, \quad (3.11)
\]

\[
D_{0+}^{p(t,x(t))} x(t) = \left( \frac{d}{dt} \right)^n \int_0^t \frac{(t-s)^{n-p(s,x(s))-1}}{\Gamma(n-p(s,x(s)))} x(s)ds, \quad t > 0. \quad (3.12)
\]

Let $L[x(t); s], L[I_{0+}^{p(t,x(t))} x(t); s], L[D_{0+}^{p(t,x(t))} x(t); s]$ denote the Laplace transforms of functions $x(t), I_{0+}^{p(t,x(t))} x(t)$ and $D_{0+}^{p(t,x(t))} x(t)$. We find that there is no explicit connection between $L[x(t); s]$ and $L[I_{0+}^{p(t,x(t))} x(t); s]$, as a result, there is also no explicit connection between $L[x(t); s]$ and $L[D_{0+}^{p(t,x(t))} x(t); s]$.

Lemma 3.3. Let $x(t), p(t, x(t))$ be real functions, assume that variable-order fractional integrals $I_{0+}^{p(t,x(t))} x(t)$ defined by (3.3) exists, and the Laplace transforms $L[x(t); s]$ and $L[I_{0+}^{p(t,x(t))} x(t); s]$ exist. There is no explicit connection between $L[x(t); s]$ and $L[I_{0+}^{p(t,x(t))} x(t); s]$.

Proof. By definitions of variable-order fractional integral (3.9) and Laplace transform, we get

\[
L[I_{0+}^{p(t,x(t))} x(t); s] = \int_0^\infty e^{-st} \int_0^t \frac{(t-\tau)^{p(t,x(t))-1}}{\Gamma(p(t,x(t)))} x(\tau)d\tau d\tau
\]

\[
= \int_0^\infty e^{-st} \int_\tau^\infty \frac{(t-\tau)^{p(t,x(t))-1}}{\Gamma(p(t,x(t)))} x(\tau)d\tau d\tau
\]

\[
= \int_0^\infty e^{-s(\tau+r)} \int_0^\tau \frac{r^{p(\tau+r,x(\tau+r))-1}}{\Gamma(p(\tau+r,x(\tau+r)))} x(\tau)dr d\tau
\]

\[
= \int_0^\infty e^{-st} x(\tau) \int_0^\infty e^{-sr} \frac{r^{p(\tau+r,x(\tau+r))-1}}{\Gamma(p(\tau+r,x(\tau+r)))} dr d\tau
\]

\[
= \int_0^\infty e^{-st} x(\tau) L\left[ \frac{r^{p(\tau+r,x(t+r))-1}}{\Gamma(p(r+r),x(r+r))}; s \right] d\tau.
\]

Since we could’t know what $L\left[ \frac{r^{p(\tau+r,x(t+r))-1}}{\Gamma(p(r+r),x(r+r))}; s \right]$ equals to, this implies that we could’t know the explicit connection between $L[x(t); s]$ and $L[I_{0+}^{p(t,x(t))} x(t); s]$. Thus, we complete the proof.
Example 3.5 We consider the Laplace transforms of functions \( t^\lambda, \lambda > -1 \) and

\[
P^{\lambda(t)}_0 = \int_0^t (t-s)^{\lambda-1} \frac{s^\lambda ds}{\Gamma(t^2)} = \frac{\Gamma(1+\lambda)}{\Gamma(1+\lambda+t^2)} t^{2+\lambda}.
\]

We could know

\[
L[t^\lambda; s] = \int_0^\infty e^{-st} t^\lambda dt = \frac{\Gamma(1+\lambda)}{s^{1+\lambda}},
\]

\[
L[I^{p(t)}_0 t^\lambda; s] = \int_0^\infty e^{-st} \frac{\Gamma(1+\lambda)}{\Gamma(1+\lambda+t^2)} t^{2+\lambda} dt,
\]

(3.13)

\[
L[I^{p(t)}_0 t^\lambda; s] = \int_0^\infty e^{-st} \int_0^t \frac{(t-\tau)^{\lambda-1}}{\Gamma(t^2)} \tau^\lambda d\tau dt
\]

\[
= \int_0^\infty e^{-st} \int_0^{\infty} \frac{(t-\tau)^{\lambda-1}}{\Gamma(t^2)} \tau^\lambda d\tau d\tau
\]

\[
= \int_0^\infty e^{-s(\tau+r)} \int_0^{\infty} \frac{\tau^{\lambda} r^{(\tau+r)^2-1}}{\Gamma((\tau+r)^2)} dr d\tau
\]

\[
= \int_0^\infty e^{-s\tau} \tau^\lambda \int_0^{\infty} e^{-sr} \frac{r^{(\tau+r)^2-1}}{\Gamma((\tau+r)^2)} dr d\tau
\]

\[
= \int_0^\infty e^{-s\tau} \tau^\lambda L\left[ \frac{r^{(\tau+r)^2-1}}{\Gamma((\tau+r)^2)} ; s \right] d\tau.
\]

(3.14)

By (3.13) and (3.14), we could assure that there is no explicit connection between \( L[x(t); s] \) and \( L[I^{p(t,x(t))}_0 x(t); s] \).

By the similar arguments, we could get the same result as Lemma 3.2 for variable-order fractional integral defined by (3.10).

Lemma 3.4. Let \( x(t), p(t, x(t)) \) be real functions, assume that variable-order fractional integrals \( I^{p(t,x(t))}_0 x(t) \) defined by (3.4) exists, and the Laplace transform \( L[x(t); s] \) and \( L[I^{p(t,x(t))}_0 x(t); s] \) exist. There is no explicit connection between \( L[x(t); s] \) and \( L[I^{p(t,x(t))}_0 x(t); s] \).
Proof. By definition of variable-order fractional integral (3.10) and Laplace transform, we get that
\[
L[\mathcal{I}_{0+}^{p(t,x(t))}x(t);s] = \int_0^\infty e^{-st} \int_0^t \frac{(t-\tau)^{p(\tau,x(\tau))}-1}{\Gamma(p(\tau,x(\tau)))} x(\tau)d\tau dt
\]
\[
= \int_0^\infty e^{-st} \int_0^\infty \frac{(t-\tau)^{p(\tau,x(\tau))}-1}{\Gamma(p(\tau,x(\tau)))} x(\tau)d\tau d\tau
\]
\[
= \int_0^\infty e^{-s(\tau+r)} \int_0^\infty \frac{\mathcal{I}_{0+}^{p(\tau,x(\tau))}x(\tau)}{\Gamma(p(\tau,x(\tau)))} d\tau dr
\]
\[
= \int_0^\infty \frac{e^{-s\tau}}{\Gamma(p(\tau,x(\tau)))} x(\tau) \int_0^\infty e^{-s\tau} \mathcal{I}_{0+}^{p(\tau,x(\tau))}x(\tau)d\tau dr
\]
\[
= \int_0^\infty \frac{e^{-s\tau}}{\Gamma(p(\tau,x(\tau)))} x(\tau) \int_0^\infty \frac{\mathcal{I}_{0+}^{p(\tau,x(\tau))}x(\tau)}{\mathcal{L}_{\tau}^{p(\tau,x(\tau))}} d\tau
\]
\[
= \int_0^\infty e^{-s\tau} s^{-p(\tau,x(\tau))} x(\tau)d\tau.
\]
These imply that we couldn’t know explicit connection between \(L[x(t);s]\) and \(L[\mathcal{I}_{0+}^{p(t,x(t))}x(t);s]\). Thus, we complete the proof. \(\square\)

Example 3.6 We consider the Laplace transforms of functions \(t^\lambda, \lambda > -1\) and \(\mathcal{I}_{0+}^{p(t)}t^\lambda = \int_0^t \frac{t^\tau-1}{t^\tau} ds\). We could know
\[
L[\mathcal{I}_{0+}^{p(t)}t^\lambda; s] = \int_0^\infty e^{-st} \int_0^t \frac{(t-\tau)^{\lambda-1}}{\Gamma(\lambda)} d\tau dt
\]
\[
= \int_0^\infty e^{-st} \int_0^\infty \frac{(t-\tau)^{\lambda-1}}{\Gamma(\lambda)} d\tau d\tau
\]
\[
= \int_0^\infty e^{-s(\tau+r)} \int_0^\infty \frac{\mathcal{I}_{0+}^{p(\tau,x(\tau))}x(\tau)}{\Gamma(p(\tau,x(\tau)))} d\tau dr
\]
\[
= \int_0^\infty \frac{e^{-s\tau}}{\Gamma(\lambda)} \mathcal{I}_{0+}^{p(\tau,x(\tau))}x(\tau) \int_0^\infty \frac{\mathcal{I}_{0+}^{p(\tau,x(\tau))}x(\tau)}{\mathcal{L}_{\tau}^{p(\tau,x(\tau))}} d\tau
\]
\[
= \int_0^\infty e^{-s\tau} s^{-p(\tau,x(\tau))} x(\tau)d\tau.
\]
By (3.13) and (3.15), we could assure that there is no explicit connection between \(L[x(t);s]\) and \(L[\mathcal{I}_{0+}^{p(t,x(t))}x(t);s]\).

In view of Lemmas 3.2-3.3 and the connection between Laplace transforms of function \(x(t)\) and its derivative \(x'(t)\), we couldn’t obtain the Laplace transform formula for variable-order fractional derivatives (3.11) and (3.12).
Lemma 3.5. Let $x(t), p(t, x(t))$ be real functions, assume that variable-order fractional derivatives $D^{p(t, x(t))}_{0+} x(t)$ defined by (3.11), (3.12) exist, and the Laplace transform $L[x(t); s]$ and $L[D^{p(t, x(t))}_{0+} x(t); s]$ exist. There is no obvious connection between $L[D^{p(t, x(t))}_{0+} x(t); s]$ and $L[x(t); s]$.

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