ON CERTAIN INTEGRAL TRANSFORM INVOLVING GENERALIZED BESSEL-MAITLAND FUNCTION AND APPLICATIONS

MUSHARRAF ALI, WASEEM A. KHAN AND IDREES A. KHAN

Abstract. In this article, we establish a new integral formula involving the generalized Bessel-Maitland function defined by Khan et al. [9], which is expressed in terms of generalized (Wright) hypergeometric function. Some interesting and special cases of our main result are also considered.

1. Introduction

In recent years, many authors (see, e.g., [1-4]) have developed numerous integral formulas involving a variety of special functions. Also many integral formulas associated with the Bessel functions of several kinds have been presented (see, e.g., [6-9]). Those integrals involving Bessel-Maitland functions are not only of great interest to the pure mathematics, but they are often of extreme importance in many branches of theoretical and applied physics and engineering (see [20]). Several methods for evaluating infinite or finite integrals involving Bessel-Maitland functions have been known (see, e.g., [4] and [19]). However, these methods usually work on a case-by-case basis.

Currently, Ghayasuddin and Khan [4], Khan et al. [6-9] gave certain interesting new class of integral formulas involving the generalized Bessel-Maitland function, which are expressed in terms of the generalized (Wright) hypergeometric function. In the present sequel to the aforementioned investigations, we present two generalized integral formulas involving generalized Bessel-Maitland functions, which are expressed in terms of the generalized (Wright) hypergeometric function. Some special cases and the (potential) usefulness of our main results are also considered and remarked, respectively.

2010 Mathematics Subject Classification. 33C45, 33C60, 33E12.
Key words and phrases. Generalized Bessel-Maitland function, Wright hypergeometric function and integrals, Mittag-Leffler function.
Submitted December 09, 2018.
The Bessel-Maitland function $J_{\nu}^{\mu}(z)$ [12;Eq.(8.3)] defined by the following series representation:

$$J_{\nu}^{\mu}(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n! \Gamma(\mu n + \nu + 1)} = \phi(\mu, \nu + 1; -z). \quad (1.1)$$

Singh et al. [19] introduced the following generalization of Bessel-Maitland function as:

$$J_{\nu,\gamma}^{\mu}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}(-z)^n}{\Gamma(\mu n + \nu + 1)n!}, \quad (\gamma)_0 = 1, (\gamma)_{qn} = \frac{\Gamma(\gamma + qn)}{\Gamma(\gamma)} \text{ denotes the generalized Pochhammer symbol.} \quad (1.2)$$

In particular Khan et al. [9] introduced and investigated a new extension of Bessel-Maitland function as follows:

$$J_{\alpha,\beta,\mu,\nu,\gamma,\sigma,\delta}^{\mu,\nu,\gamma,\delta}(z) = \sum_{n=0}^{\infty} \frac{(\mu)_{pn}(\gamma)_{qn}(-z)^n}{\Gamma(n \beta + \alpha + 1)(\delta)_{pn}(\nu)_{n\sigma}}, \quad (1.3)$$

where $\alpha, \beta, \mu, \nu, \gamma, \sigma, \delta \in \mathbb{C}, \Re(\mu) \geq 0, \Re(\nu) \geq -1, \Re(\gamma) \geq 0, \Re(\delta) \geq 0; p, q > 0$ and $q < \Re(\alpha) + p$.

Relation with Mittag-Leffler functions:

(i) On replacing $\alpha$ by $\alpha - 1$ in (1.4), we get the following interesting relation:

$$J_{\alpha-1,\beta,\mu,\nu,\gamma,\sigma,\delta}^{\mu,\nu,\gamma,\delta}(-z) = E_{\alpha,\beta,\nu,\sigma,\delta}^{\mu,\nu,\gamma,\delta}(z), \quad (1.5)$$

where $E_{\alpha,\beta,\nu,\sigma,\delta}^{\mu,\nu,\gamma,\delta}(z)$ is the Mittag-Leffler function defined by Khan and Ahmed [10].

(ii) On setting $\mu = \nu = \sigma = \rho = 1$ and replacing $\alpha$ by $\alpha - 1$ in (1.4), we get

$$J_{\alpha-1,1,1,\delta}^{1,1,\gamma,\delta}(-z) = E_{\alpha,1,1,\delta}^{\gamma,\delta}(z), \quad (1.6)$$

where $E_{\alpha,1,1,\delta}^{\gamma,\delta}(z)$ is the Mittag-Leffler function defined by Salim and Faraz [17].

(iii) On setting $\mu = \nu = \sigma = \rho = \delta = p = 1$ and replacing $\alpha$ by $\alpha - 1$ in (1.4), we get

$$J_{\alpha-1,1,1,1,1}^{1,1,\gamma,\delta}(z) = E_{\alpha,1,1,1,1}^{\gamma,\delta}(z), \quad (1.7)$$

where $E_{\alpha,1,1,1,1}^{\gamma,\delta}(z)$ is the Mittag-Leffler function defined by Shukla and Prajapati [18].
(iv) On setting $\mu = \nu = \sigma = \rho = \delta = p = q = 1$ and replacing $\alpha$ by $\alpha - 1$ in (1.4), we get
\[
J_{\alpha-1,\beta,1,1,1,1}^{1,1,1,1,1}(-z) = E_{\alpha,\beta}(z),
\]
where $E_{\alpha,\beta}(z)$ is the Mittag-Leffler function defined by Prabahkar [14].

(v) On setting $\mu = \nu = \sigma = \rho = \delta = \gamma = p = q = 1$ and replacing $\alpha$ by $\alpha - 1$ in (1.4), we get
\[
J_{\alpha-1,\beta,1,1,1,1}^{1,1,1,1,1}(-z) = E_{\alpha,\beta}(z),
\]
where $E_{\alpha,\beta}(z)$ is the Mittag-Leffler function defined by Wiman [21].

(vi) On setting $\mu = \nu = \sigma = \rho = \delta = \gamma = p = q = 1$, $\alpha = 0$ and replacing $\alpha$ by $\alpha - 1$ in (1.4), we get
\[
J_{0,\beta,1,1,1,1}^{1,1,1,1,1}(-z) = E_{\beta}(z),
\]
where $E_{\beta}(z)$ is the Mittag-Leffler function defined by Ghosta Mittag-Leffler [11].

The generalization of the generalized hypergeometric series $pF_q$ is due to Fox [5] and Wright ([21], [22], [23]) who studied the asymptotic expansion of the generalized (Wright) hypergeometric function defined by (see [17, p.21]; see also [15]):
\[
p\Psi^q_p \left[ \begin{array}{c}
(\alpha_1, A_1), \ldots, (\alpha_p, A_p); \\
(\beta_1, B_1), \ldots, (\beta_q, B_q);
\end{array} \right] z = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(\alpha_j + A_j k) z^k}{\prod_{j=1}^{q} \Gamma(\beta_j + B_j k) k!},
\]
where the coefficients $A_1, \ldots, A_p$ and $B_1, \ldots, B_q$ are positive real numbers such as
\[
(i) \ 1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j > 0 \text{ and } 0 < |z| < \infty; \ z \neq 0.
\]
(ii) $1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j = 0$ and $0 < |z| < A_1^{-A_1} \cdots A_p^{-A_p} B_1 B_2 \cdots B_q B_q$.

A special case of (1.11) is
\[
p\Psi^q_p \left[ \begin{array}{c}
(\alpha_1, 1), \ldots, (\alpha_p, 1); \\
(\beta_1, 1), \ldots, (\beta_q, 1);
\end{array} \right] z = \prod_{j=1}^{p} \frac{\Gamma(\alpha_j)}{\prod_{j=1}^{q} \Gamma(\beta_j)} pF_q \left[ \begin{array}{c}
\alpha_1, \ldots, \alpha_p; \\
\beta_1, \ldots, \beta_q;
\end{array} \right] z,
\]
where $pF_q$ is the generalized hypergeometric series defined by [15]
\[
pF_q \left[ \begin{array}{c}
\alpha_1, \ldots, \alpha_p; \\
\beta_1, \ldots, \beta_q;
\end{array} \right] z = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n z^n}{(\beta_1)_n \cdots (\beta_q)_n n!}
\]
\[
= pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z),
\]
where $(\lambda)_n$ is the Pochhammer’s symbol (see [15]).

For our present investigation the following interesting and useful result due to Obhettinger [13] will be required:
\[
\int_0^\infty x^{\mu-1}(x + a + \sqrt{x^2 + 2ax})^{-\lambda} \, dx = 2\lambda a^{-\lambda} \left(\frac{a}{2}\right)^\mu \frac{\Gamma(2\mu)\Gamma(\lambda - \mu)}{\Gamma(1 + \lambda + \mu)}, \quad (1.16)
\]
provided \(0 < \Re(\mu) < \Re(\lambda)\).

2. Main Result

**Theorem 2.1.** If \(\alpha, \beta, \gamma, \mu, \eta, \nu, \sigma, \rho, \delta, \lambda \in \mathbb{C}\), \(\sigma + \eta + p - \rho - q > 0\), \(\Re(\alpha) > 0\), \(\Re(\beta) > 0\), \(\Re(\gamma) > 0\), \(\Re(\mu) > 0\), \(\Re(\eta) \geq -1\), \(\Re(\nu) > 0\), \(\Re(\sigma) > 0\), \(\Re(\rho) > 0\), \(\Re(\delta) > 0\), \(\Re(\eta) > 0\), \(p, q > 0\) and \(q < \Re(\alpha) + p\), then

\[
\int_0^\infty x^{\alpha-1}(x + a + \sqrt{x^2 + 2ax})^{-\beta} J_{\eta,\lambda,\nu,\sigma,\delta,p}^{\mu,\rho,\gamma,q} \left(\frac{y}{x + a + \sqrt{x^2 + 2ax}}\right) \, dx
= 2^{1-\alpha} a^{\alpha-\beta} \frac{\Gamma(2\alpha)\Gamma(\nu)\Gamma(\delta)}{\Gamma(\mu)\Gamma(\gamma)} \Psi_5 \left[ \begin{array}{c} (\mu, \rho), (\gamma, q), (\beta - \alpha, 1), (\beta + 1, 1), (1, 1); \\ (\nu, \sigma), (\lambda + 1, \eta), (\beta, 1), (1 + \beta + \alpha, 1), (\delta, p) \end{array} \right].
\]

**Proof.** In order to establish our main integral (2.1), we denote the left hand side of (2.1) by I and then by using (1.4), to get:

\[
I = \int_0^\infty x^{\alpha-1}(x + a + \sqrt{x^2 + 2ax})^{-\beta} \times \sum_{n=0}^\infty \frac{(\mu)_{\rho n}(\gamma)_{\eta n}}{\Gamma(\eta + \lambda + 1)(\nu)_{\sigma n}(\delta)_{\mu n}} \left(\frac{-y}{x + a + \sqrt{x^2 + 2ax}}\right)^n \, dx.
\]

Evaluating the above integral with the help of (1.16) and after little simplification, we found:

\[
I = 2^{1-\alpha} a^{\alpha-\beta} \frac{\Gamma(2\alpha)\Gamma(\nu)\Gamma(\delta)}{\Gamma(\mu)\Gamma(\gamma)} \times \sum_{n=0}^\infty \frac{\Gamma(\mu + \rho n)\Gamma(\gamma + q n)\Gamma(\beta + n + \alpha)\Gamma(\beta + n + 1)\Gamma(\nu + \sigma n)\Gamma(\eta + \lambda + 1)\Gamma(\beta + n + \alpha)\Gamma(\beta + n + 1)\Gamma(1 + \beta + n + \alpha)\Gamma(\beta + n + 1)\Gamma(1 + \beta + n + \alpha)}{n!(\delta + pn)!} \left(\frac{-y}{a}\right)^n.
\]

Which upon using (1.11) yields (2.1). This completes the proof of our main result.

**Variation of (2.1):** It the conditions of our main result be satisfied, then the following integral formula holds true:

\[
\int_0^\infty x^{\alpha-1}(x + a + \sqrt{x^2 + 2ax})^{-\beta} J_{\eta,\lambda,\nu,\sigma,\delta,p}^{\mu,\rho,\gamma,q} \left(\frac{y}{x + a + \sqrt{x^2 + 2ax}}\right) \, dx
= 2^{1-\alpha} a^{\alpha-\beta} \frac{\Gamma(2\alpha)\Gamma(\beta + 1)\Gamma(\beta - \alpha)}{\Gamma(\beta)\Gamma(\lambda + 1)\Gamma(1 + \beta + \alpha)} \times \rho^{\rho + q} \Delta(\rho, \mu), \Delta(q, \gamma), \beta + 1, \beta - \alpha, 1; \quad \Delta(\sigma, \nu), \Delta(p, \delta), \Delta(\eta, \lambda + 1), \beta, 1 + \beta + \alpha; \quad \frac{-\rho^\rho q^q y^{\rho + q}}{\sigma^\sigma \eta^\eta \rho^\rho a^a},
\]

(2.4)
where $\Delta(m; l)$ abbreviates the array of $m$ parameters $\frac{l}{m}$, $\frac{l+1}{m}$, ..., $\frac{l+m-1}{m}$, $m \geq 1$.

**Proof:** By writing the right hand side of (2.4) in the original summation and using the result

$$\Gamma(\alpha + n) = \Gamma(\alpha)(\alpha)_{n}$$

and

$$(l)_{kn} = k^{kn}(\frac{l}{k})_{n}(\frac{l+1}{k})_{n}(\frac{l+k-1}{k})_{n},$$

(Gauss multiplication theorem) in (2.3) and assuming up the given series with the help of (1.15), we easily arrive at our required result (2.4).

3. **Special Cases**

(i) On replacing $\lambda$ by $\lambda - 1$ in (2.1) and then by using (1.5), we obtain:

$$\int_{0}^{\infty} x^{\alpha-1}(x + a + \sqrt{x^{2} + 2ax})^{-\beta} E_{\eta, \lambda, \nu, \sigma, \delta, \rho}^{\mu, \nu, q, y} \left( \frac{y}{x + a + \sqrt{x^{2} + 2ax}} \right) dx$$

$$= 2^{1-\alpha} a^{\alpha-\beta} \frac{\Gamma(2\alpha)\Gamma(\nu)}{\Gamma(\mu)\Gamma(\gamma)} \left[ \begin{array}{c} (\mu, \rho), (\gamma, q), (\beta - \alpha, 1), (\beta + 1, 1), (1, 1); \\ (\nu, \sigma), (\lambda, \eta), (\beta, 1), (1 + \beta + \alpha, 1), (\delta, p); \\ \frac{y}{a} \end{array} \right],$$

(3.1)

where $\alpha, \beta, \gamma, \mu, \eta, \nu, \sigma, \rho, \delta, \lambda \in \mathbb{C}$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(\mu) > 0$, $\Re(\nu) > 0$, $\Re(\lambda) > 0$, $\Re(\delta) > 0$, $\Re(\sigma) > 0$, $\Re(\rho) > 0$, $\Re(\eta) > 0$, $p, q > 0$ and $q < \Re(\alpha) + p$.

(ii) On replacing $\lambda$ by $\lambda - 1$ in (2.4) and then by using (1.5), we attain:

$$\int_{0}^{\infty} x^{\alpha-1}(x + a + \sqrt{x^{2} + 2ax})^{-\beta} E_{\eta, \lambda, \nu, \sigma, \delta, \rho}^{\mu, \nu, q, y} \left( \frac{y}{x + a + \sqrt{x^{2} + 2ax}} \right) dx$$

$$= 2^{1-\alpha} a^{\alpha-\beta} \frac{\Gamma(2\alpha)\Gamma(\beta + 1)}{\Gamma(\beta + 1)\Gamma(1 + \beta + \alpha)} \times_{\rho+q+3} F_{\gamma+\eta+p+2} \left[ \begin{array}{c} \Delta(\rho, \mu), \Delta(q, \gamma), \beta + 1, \beta - \alpha, 1; \\ \Delta(\sigma, \nu), \Delta(p, \delta), \Delta(\eta, \lambda), \beta, 1 + \beta + \alpha; \\ \frac{\rho^{q}q^{y}}{\sigma^{\gamma}\eta^{p}a} \end{array} \right],$$

(3.2)

where $\alpha, \beta, \gamma, \mu, \eta, \nu, \sigma, \rho, \delta, \lambda \in \mathbb{C}$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(\mu) > 0$, $\Re(\nu) > 0$, $\Re(\lambda) > 0$, $\Re(\delta) > 0$, $\Re(\sigma) > 0$, $\Re(\rho) > 0$, $\Re(\eta) > 0$, $p, q > 0$ and $q < \Re(\alpha) + p$.

(iii) On setting $\mu = \nu = \rho = \sigma = 1$ and replacing $\lambda$ by $\lambda - 1$ in (2.1) and then by using (1.6), we find:
\[ \int_0^\infty x^{\alpha-1} (x + a + \sqrt{x^2 + 2ax})^{-\beta} E^{\gamma,q}_{\eta,\lambda,p} \left( \frac{y}{x + a + \sqrt{x^2 + 2ax}} \right) dx \]

\[ = 2^{1-\alpha} a^{-\beta} \frac{\Gamma(2\alpha)\Gamma(\delta)}{\Gamma(\gamma)} 4\Psi_4 \left[ \begin{array}{c} (\gamma, q), (\beta - \alpha, 1), (\beta + 1, 1), (1, 1); \\ (\lambda, \eta), (\beta, 1), (1 + \beta + \alpha, 1), (\delta, p); \end{array} \right], \quad (3.3) \]

where \( \alpha, \beta, \gamma, \eta, \delta, \lambda \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\lambda) > 0, \Re(\delta) > 0, \Re(\eta) > 0, p, q > 0 \) and \( q < \Re(\alpha) + p \).

(iv). On setting \( \mu = \nu = \rho = \sigma = 1 \) and replacing \( \lambda \) by \( \lambda - 1 \) in (2.4) and then by using (1.6), we get:

\[ \int_0^\infty x^{\alpha-1} (x + a + \sqrt{x^2 + 2ax})^{-\beta} E^{\gamma,q}_{\eta,\lambda,p} \left( \frac{y}{x + a + \sqrt{x^2 + 2ax}} \right) dx \]

\[ = 2^{1-\alpha} a^{-\beta} \frac{\Gamma(2\alpha)\Gamma(\beta + 1)\Gamma(\beta - \alpha)}{\Gamma(\beta)\Gamma(\lambda)\Gamma(1 + \beta + \alpha)} \]

\[ \times q + 3 F_{\eta+p+2} \left[ \begin{array}{c} \Delta(q, \gamma), \beta + 1, \beta - \alpha, 1; \\ \Delta(p, \delta), \Delta(\eta, \lambda), \beta, 1 + \beta + \alpha; \end{array} \right], \quad (3.4) \]

where \( \alpha, \beta, \gamma, \eta, \delta, \lambda \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\lambda) > 0, \Re(\delta) > 0, \Re(\eta) > 0, p, q > 0 \) and \( q < \Re(\alpha) + p \).

(v). On setting \( \mu = \nu = \rho = \sigma = \delta = p = 1 \) and replacing \( \lambda \) by \( \lambda - 1 \) in (2.1) and then by using (1.7), we find:

\[ \int_0^\infty x^{\alpha-1} (x + a + \sqrt{x^2 + 2ax})^{-\beta} E^{\gamma,q}_{\eta,\lambda} \left( \frac{y}{x + a + \sqrt{x^2 + 2ax}} \right) dx \]

\[ = 2^{1-\alpha} a^{-\beta} \frac{\Gamma(2\alpha)\Gamma(\beta + 1)\Gamma(\beta - \alpha)}{\Gamma(\beta)\Gamma(\lambda)\Gamma(1 + \beta + \alpha)} \]

\[ \times q + 3 F_{\eta+2} \left[ \begin{array}{c} \Delta(q; \gamma), \beta + 1, \beta - \alpha, 1; \\ \Delta(\eta, \lambda), \beta + 1 + \beta + \alpha; \end{array} \right], \quad (3.5) \]

where \( \alpha, \beta, \gamma, \eta, \lambda \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\lambda) > 0, \Re(\eta) > 0, q \in (0, 1) \cup \mathbb{N} \).

(vi). On setting \( \mu = \nu = \rho = \sigma = \delta = p = 1 \) and replacing \( \lambda \) by \( \lambda - 1 \) in (2.4) and then by using (1.7), we acquire:

\[ \int_0^\infty x^{\alpha-1} (x + a + \sqrt{x^2 + 2ax})^{-\beta} E^{\gamma,q}_{\eta,\lambda} \left( \frac{y}{x + a + \sqrt{x^2 + 2ax}} \right) dx \]

\[ = 2^{1-\alpha} a^{-\beta} \frac{\Gamma(2\alpha)\Gamma(\beta + 1)\Gamma(\beta - \alpha)}{\Gamma(\beta)\Gamma(\lambda)\Gamma(1 + \beta + \alpha)} \]

\[ \times q + 3 F_{\eta+3} \left[ \begin{array}{c} \Delta(q; \gamma), \beta + 1, \beta - \alpha, 1; \\ \Delta(\eta, \lambda), \beta + 1 + \beta + \alpha; \end{array} \right], \quad (3.6) \]
where \( \alpha, \beta, \gamma, \eta, \lambda \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\lambda) > 0, \Re(\eta) > 0, \)
\( q \in (0, 1) \cup \mathbb{N} \).

(vii). On setting \( \mu = \nu = \rho = \sigma = \delta = p = q = 1 \) and replacing \( \lambda \) by \( \lambda - 1 \) in (2.1) and then by using (1.8), we obtain:

\[
\int_0^1 x^{-1}(x + a + \sqrt{x^2 + 2ax})^{-\beta} E_{\eta, \lambda}^{(\gamma)} \left( \frac{y}{x + a + \sqrt{x^2 + 2ax}} \right) dx = 2^{1-a} a^{\alpha-\beta} \frac{\Gamma(2(\alpha))}{\Gamma(\gamma)} \Psi_3 \left[ \begin{array}{c} (\gamma, 1), (\beta - \alpha, 1), (\beta + 1, 1), (1, 1); \\ (\beta, 1), (1 + \beta + \alpha, 1), (\lambda, \eta); \\ \frac{y}{a} \end{array} \right], \quad (3.7)
\]

where \( \alpha, \beta, \gamma, \eta, \lambda \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\lambda) > 0, \Re(\eta) > 0. \)

(viii). On setting \( \mu = \nu = \rho = \sigma = \delta = p = q = 1 \) and replacing \( \nu \) by \( \nu - 1 \) in (2.1) and then by using (1.7), we get:

\[
\int_0^1 x^{-1}(x + a + \sqrt{x^2 + 2ax})^{-\beta} E_{\eta, \lambda}^{(\gamma)} \left( \frac{y}{x + a + \sqrt{x^2 + 2ax}} \right) dx = 2^{1-a} a^{\alpha-\beta} \frac{\Gamma(2(\alpha))}{\Gamma(\gamma)} \Gamma(\beta + 1) \Gamma(\beta - \alpha) \times 4 F_{\eta+2} \left[ \begin{array}{c} \gamma, \beta + 1, \beta - \alpha, 1; \\ \Delta(\eta, \lambda), 1 + \beta + \alpha, \beta; \\ \frac{y}{\eta+a} \end{array} \right], \quad (3.8)
\]

where \( \alpha, \beta, \gamma, \eta, \lambda \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\lambda) > 0, \Re(\eta) > 0. \)

(ix). On setting \( \mu = \nu = \rho = \eta = p = q = \delta = \gamma = 1 \) in (2.1) and then by using (1.8), we attain:

\[
\int_0^1 x^{-1}(x + a + \sqrt{x^2 + 2ax})^{-\beta} E_{\eta, \lambda}^{(\gamma)} \left( \frac{y}{x + a + \sqrt{x^2 + 2ax}} \right) dx = 2^{1-a} a^{\alpha-\beta} \Gamma(2(\alpha)) \Psi_3 \left[ \begin{array}{c} (\beta - \alpha, 1), (\beta + 1, 1), (1, 1); \\ (\beta, 1), (1 + \beta + \alpha, 1), (\lambda, \eta); \\ \frac{y}{a} \end{array} \right], \quad (3.9)
\]

where \( \alpha, \beta, \eta, \lambda \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\lambda) > 0, \Re(\eta) > 0. \)

(x). On setting \( \mu = \nu = \rho = \eta = p = q = \delta = \gamma = 1 \) in (2.1) and then by using (1.8), we obtain:

\[
\int_0^1 x^{-1}(x + a + \sqrt{x^2 + 2ax})^{-\beta} E_{\eta, \lambda}^{(\gamma)} \left( \frac{y}{x + a + \sqrt{x^2 + 2ax}} \right) dx = 2^{1-a} a^{\alpha-\beta} \Gamma(2(\alpha)) \Gamma(\beta + 1) \Gamma(\beta - \alpha) \quad \frac{\Gamma(\beta) \Gamma(1 + \beta + \alpha)}{\Gamma(\gamma)},
\]

\[
\]
\[ \times \mathbf{3F}_{\eta+2} \left[ \begin{array}{c} \beta + 1, \beta - \alpha, 1; \\ \Delta(\eta, \lambda), 1 + \beta + \alpha, \beta; \\ \frac{\eta}{\eta + n} \end{array} \right], \quad (3.10) \]

where \( \alpha, \beta, \eta, \lambda \in \mathbb{C} \), \( \Re(\alpha) > 0, \Re(\beta) > 0 \), \( \Re(\lambda) > 0, \Re(\eta) > 0 \).

**References**


**MUSHARRAF ALI**

DEPARTMENT OF MATHEMATICS, G.F. COLLEGE, SHAHJAHANPUR-242001, INDIA

E-mail address: drmusharrafali@gmail.com
Waseem A. Khan  
Department of Mathematics, Faculty of Science, Integral University, Lucknow-226026, India  
E-mail address: waseem08_khan@rediffmail.com

Idrees A. Khan  
Department of Mathematics, Faculty of Science, Integral University, Lucknow-226026, India  
E-mail address: khanidrees077@gmail.com