A NOTE ON ORDINARY HYPERGEOMETRIC SERIES AND BAILEY’S TRANSFORM

LAKSHMI NARAYAN MISHRA, RAVINDRA KUMAR YADAV AND DEEPMALA

Abstract. In this paper, making use of Bailey’s transform and certain known summation formula, we have established certain interesting transformation formula of ordinary hypergeometric series.

1. Introduction, Notations and Definition:

The generalized ordinary hypergeometric series \( \,_{r}F_{s} \) is defined by

\[
\,_{r}F_{s}[a_1, a_2, ..., a_r; b_1, b_2, ..., b_s; z] = \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n...(a_r)_n}{n!(b_1)_n(b_2)_n...(b_s)_n} z^n. \tag{1.1}
\]

In the above series

(i) If \( r \leq s \), then it converges for \(|z| < \infty\).

(ii) If \( r = s + 1 \), then series converges for \(|z| < 1\).

(iii) If \( r > s + 1 \), then series converges only at \( z = 0 \).

Gauss’s hypergeometric series is defined as

\[
\,_{2}F_{1}[a, b; c; z] = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} z^n, \tag{1.2}
\]

where

\[
(a)_n = a(a+1)(a+2)...(a+n-1), n = 1, 2, ...
\]

\[
(1) = 1.
\]

\[
(\frac{a}{b}) = \frac{(-1)^r}{(1-a)\Gamma(c-a)}.
\]

\[
(a)_{m+n} = (a)_m(a+m)_n.
\]

Gauss- summation formula is

\[
\,_{2}F_{1}[a; b; c; 1] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{1.3}
\]

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provided that \( R1(c - a - b) > 0 \). \[ 6; (1.7.6) p.28 \]

Some interesting formula for multi-basic hypergeometric series appear in the work of \([1 - 7]\). Also, many useful summations and transformations for elliptic hypergeometric series have been established by \([8 - 19]\) In 1947, Bailey’s established a remarkable, simple and useful transformation formula, which is given in the following form.

If

\[
\beta_n \sum_{r=0}^{n} \alpha_r u_{n-r} v_{n+r}  \tag{1.4}
\]

and

\[
\gamma_n = \sum_{r=0}^{\infty} \delta_{r+n} u_r v_{r+2n}  \tag{1.5}
\]

then subject to convergence conditions,

\[
\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n.  \tag{1.6}
\]

Where \( \alpha_r, \delta_r, u_r \) and \( v_r \) are functions of \( r \) alone.

In this paper, we shall also use the following summations.

\[
\begin{align*}
3F2[a, b, -n; (1 + a - b), (1 + a + n); 1] &= \frac{(1 + a)n(a + \frac{n}{2} - b)_n}{(1 + \frac{n}{2})n(1 + a - b)_n}  \tag{1.7} \\
2F1[a, b; (1 + a + b); 1] &= \frac{(1 + a)_n(1 + b)_n}{n!(1 + a + b)_n}  \tag{1.8} \\
5F4[a, (1 + \frac{a}{2}), b, c; \frac{a}{2}, (1 + a - b), (1 + a - c), (1 + a - d); 1] &= \frac{(1 + a)_n(1 + b)_n(1 + c)_n(1 + d)_n}{n!(1 + a - b)_n(1 + a - c)_n(1 + a - d)_n}  \tag{1.9}
\end{align*}
\]

provided that, \( a = b + c + d \)

\[
3F2[a, b, c; d; (a + b + c - d); 1] = \frac{(1 + a)_n(1 + b)_n(1 + c)_n}{n!(d)_n(a + b + c - d)_n}  \tag{1.10} \\
\]

\[
A+1F_A[a_0, a_1, ..., a_A; (1 + b_1), (1 + b_2), ..., (1 + b_A); 1] = \frac{(1 + a_0)_N(1 + a_1)_N(1 + a_2)_N... (1 + a_A)_N}{N!(1 + b_1)_N(1 + b_2)_N(1 + b_3)_N... (1 + b_A)_N}  \tag{1.11} \\
\]

Under the condition

\[
\begin{align*}
\sum_{i=0}^{A} a_i &= b_1 + b_2 + b_3 + ... + b_A, \\
\sum_{i=0}^{A} a_i a_j &= b_1 b_2 + b_2 b_3 + ... + b_A b_{A-1} \\
\end{align*}
\]


\[a_0a_1a_2\ldots a_A = b_1b_2b_3\ldots b_A,\]

2. Main Results:

In this section, we shall establish our main results.

(i) Choosing \(\alpha_r = \frac{(a_r)(b_r)}{r!(1+a+b)_r}\) and \(u_r = \frac{1}{r!}, v_r = \frac{1}{(1+a)_r}\) and \(\delta_r = (\alpha_r)(\beta_r)\) in (1.7), then using (1.4) and (1.5), we get

\[
\beta_n = \frac{(1 + \frac{a}{2} - b)_n}{n!(1 + \frac{a}{2})_n(1 + a - b)_n}
\]

and

\[
\gamma_n = \frac{(\alpha)_n(\beta_n)}{(1 + a - \alpha)_n(1 + a - \beta)_n} \times \frac{\Gamma(1 + a)\Gamma(1 + a - \alpha - \beta)}{\Gamma(1 + a - \alpha)\Gamma(1 + a - \beta)}.
\]

Putting the value of \(\alpha_n, \beta_n, \gamma_n\) and \(\delta_n\) in (1.6), we get

\[
_{4}F_{3}[\alpha, \beta, a, b; (1 + a - b), (1 + a - \alpha), (1 + a - \beta); -1] = \\
\frac{\Gamma(1 + a)\Gamma(1 + a - \alpha - \beta)}{\Gamma(1 + a - \alpha)\Gamma(1 + a - \beta)} \times _{3}F_{2}[\alpha, \beta, (1 + \frac{a}{2} - b); (1 + a - b), (1 + \frac{a}{2}); 1].
\]

(ii) Choosing

\[
\alpha_r = \frac{(a_r)(b_r)}{r!(1+a+b)_r},
\]

and \(u_r = v_r = 1,\) and \(\delta_r = z^r\)
in (1.8), then using (1.4) and (1.5), we get

\[
\beta_n = \frac{(1+a)_n(1+b)_n}{n!(1+a+b)_n} \text{ and } \gamma_n = \frac{z^n}{(1-z)}.
\]

Putting the value of \(\alpha_n, \beta_n, \gamma_n\) and \(\delta_n\) in (1.6), we get

\[
_{2}F_{1}[a, b; (1 + a + b); z] = (1 - z) \times _{2}F_{1}[(1 + a), (1 + b); (1 + a + b); z].
\]

(iii) Again by choosing

\[
\alpha_r = \frac{(a_r)(b_r)}{r!(1+a+b)_r},
\]

and \(u_r = v_r = 1,\) and \(\delta_r = rz^r\)
in (1.8), then using (1.4) and (1.5), we get

\[
\beta_n = \frac{(1+a)_n(1+b)_n}{n!(1+a+b)_n} \text{ and } \gamma_n = \{z^{n+1} + \frac{n z^n}{(1-z)}\}.
\]

Putting the value of \(\alpha_n, \beta_n, \gamma_n\) and \(\delta_n\) in (1.6), we get

\[
\frac{z}{(1-z)} \times _{2}F_{1}[a, b; (1+a+b); z] + \frac{abz}{(1+a+b)(1-z)} \times _{2}F_{1}[(1+a), (1+b); (2+a+b); z]
\]

\[
= \frac{(1+a)(1+b)z}{(1+a+b)} \times _{2}F_{1}[(2+a), (2+b); (2+a+b); z]
\]

(iv) Choosing

\[
\alpha_r = \frac{(a_r)(1+a-b)_r(1+a-c)_r(1+a-d)_r}{r!(1+a-b)_r(1+a-c)_r(1+a-d)_r},
\]

and \(u_r = v_r = 1,\) and \(\delta_r = z^r\)
in (1.9), then using (1.4) and (1.5), we get

\[
\beta_n = \frac{(1+a)_n(1+b)_n(1+c)_n(1+d)_n}{n!(1+a-b)_n(1+a-c)_n(1+a-d)_n} \text{ and } \gamma_n = \frac{z^n}{(1-z)}.
\]
Putting the value of \( \alpha_n, \beta_n, \gamma_n \) and \( \delta_n \) in (1.6), we get

\[
\begin{align*}
5F_4[a, (1 + \frac{a}{2}), b, c, (a - b - c); \frac{a}{2}, (1 + a - b, (1 + a - c), (1 + b + c); z] &= \\
z(1 - z) \times 4F_3[(1 + a), (1 + b), (1 + c), (1 + a - b - c); (1 + c - b), (1 + a - c), (1 + b + c); z]
\end{align*}
\]

(2.4)

(v) Again choosing

\[
\alpha_r = \frac{(a), (1 + \frac{r}{2}), (b), (c), (d), (e)}{\Gamma(1 + a - b), (1 + a - c), (1 + a - d), (1 + a - e)},
\]

and \( u_r = v_r = 1, \) and \( \delta_r = rz^r \) in (1.9), then using (1.4) and (1.5), we get

\[
\beta_n = \frac{(1 + a)n_{(1 + b)n(1 + c)n(1 + d)n}}{n!(1 + a - b)n(1 + a - c)n(1 + a - d)n} \quad \text{and} \quad \gamma_n = \{z^{n+1} + \frac{n}{1 - z}\}.
\]

Putting the value of \( \alpha_n, \beta_n, \gamma_n \) and \( \delta_n \) in (1.6), we get

\[
\begin{align*}
\frac{z}{(1 - z)^2} \times 5F_4[a, (1 + \frac{a}{2}), b, c, (a - b - c); \frac{a}{2}, (1 + a - b, (1 + a - c), (1 + b + c); z] &= \\
z[a(1 + \frac{a}{2})bc(a - b - c)] \\
\times \frac{(1 - z)\frac{a}{2}(1 + a - b)(1 + a - c)(1 + b + c)}{(1 + a - c)(1 + a - b)(1 + b + c)}
\end{align*}
\]

(5F_4[(1 + a), (2 + \frac{a}{2}), (1 + b), (1 + c), (a - b - c - 1); (1 + \frac{a}{2}), (2 + a - b), (2 + a - c), (2 + b + c); z] \quad (2.5)

(vi) Choosing

\[
\alpha_r = \frac{(a), (b), (c), (d)}{\Gamma(1 + d)}, \quad u_r = v_r = 1, \quad \text{and} \quad \delta_r = rz^r
\]

in (1.10), then using (1.4) and (1.5), we get

\[
\beta_n = \frac{(1 + a)n(1 + b)n(1 + c)n}{n!(1 + a - b)n(1 + a - c)n(1 + a - d)n} \quad \text{and} \quad \gamma_n = \frac{z^{n+1}}{(1 - z)^2}.
\]

Putting the value of \( \alpha_n, \beta_n, \gamma_n \) and \( \delta_n \) in (1.6), we get

\[
3F_2[a, b, c; d, (a + b + c - d); z] = (1 - z)3F_2[(1 + a), (1 + b), (1 + c); d, (a + b + c - d); z] \quad (2.6)
\]

(vii) Again choosing

\[
\alpha_r = \frac{(a), (b), (c), (d)}{\Gamma(1 + d)}, \quad u_r = v_r = 1, \quad \text{and} \quad \delta_r = rz^r
\]

in (1.10), then using (1.4) and (1.5), we get

\[
\beta_n = \frac{(1 + a)n(1 + b)n(1 + c)n}{n!(1 + a - b)n(1 + a - c)n(1 + a - d)n} \quad \text{and} \quad \gamma_n = \frac{z^{n+1}}{(1 - z)^2}.
\]

Putting the value of \( \alpha_n, \beta_n, \gamma_n \) and \( \delta_n \) in (1.6), we get

\[
\begin{align*}
\frac{z}{(1 - z)^2} \times 3F_2[a, b, c; d, (a + b + c - d); z] + \frac{abcz}{(1 - z)d(a + b + c - d)} \times \\
\times 3F_2[(1 + a), (1 + b), (1 + c); (1 + d), (1 + a + b + c - d); z] &= \\
\frac{(1 + a)(1 + b)(1 + c)z}{d(a + b + c - d)} \times 3F_2[(2 + a), (2 + b), (2 + c); (1 + d), (1 + a + b + c - d); z]
\end{align*}
\]

(2.7)

(viii) Choosing
\[
\alpha_r = \frac{(a_0),(a_1),(a_2),...,(a_A)}{N!(1+b_1)x_1(1+b_2)x_2...} \\
u_r = v_r = 1, \text{ and } \delta_r = z^r \text{ in (1.11), then using (1.4) and (1.5), we get} \\
\beta_n = \frac{(1+na_1)x(1+na_2)x...}{N!(1+b_1)x_1(1+b_2)x_2...} \text{ and } \gamma_n = \frac{z^n}{(1-z)^n}.
\]

Putting the value of \( \alpha_n, \beta_n, \gamma_n \) and \( \delta_n \) in (1.6), we get

\[
A+1 F_A[a_0, a_1, ..., a_A; (1+b_1), (1+b_2), ... (1+b_A); z] = (1-z)A+1 F_A[(1+a_0), (1+a_1), ..., (1+a_A); (1+b_1), (1+b_2), ... (1+b_A); z]. \tag{2.8}
\]

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References


Lakshmi Narayan Mishra
Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology (VIT) University, Vellore 632 014, Tamil Nadu, India
E-mail address: lakshminarayann Mishra04@gmail.com, lakshminarayan.mishra@vit.ac.in

Ravindra Kumar Yadav
Department of Mathematics, GLA University, Mathura, Uttar Pradesh, India-281406
E-mail address: rkyadav81@yahoo.com, ravindra.yadav@glu.ac.in

Deepmala
Mathematics Discipline, PDPM Indian Institute of Information Technology, Design and Manufacturing, Jabalpur, Dumna Airport Road, P.O.: Khamaria, Jabalpur 482 005, Madhya Pradesh, India.
E-mail address: dmrai23@gmail.com, deepmala@iiitdmj.ac.in