RELATIVE \((p, q)-\varphi\) ORDER ORIENTED SOME GROWTH PROPERTIES OF \(p\)-ADIC ENTIRE FUNCTIONS

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ABSTRACT. Let \(K\) be a complete ultrametric algebraically closed field and \(A(K)\) be the \(K\)-algebra of entire function on \(K\). For any \(p\) adic entire functions \(f \in A(K)\) and \(r > 0\), we denote by \(|f|_r\) the number \(\sup \{ |f(x)| : |x| = r \}\) where \(|.\)| is a multiplicative norm on \(A(K)\). In this paper we introduce the concept of relative \((p, q)-\varphi\) order where \(p, q\) are any two positive integers and \(\varphi(r) : [0, +\infty) \to (0, +\infty)\) is a non-decreasing unbounded function of \(r\). Then we study some growth properties of \(p\)-adic entire functions on the basis of their relative \((p, q)-\varphi\) order.

1. Definitions

Let \(K\) be an algebraically closed field of characteristic 0, complete with respect to a \(p\)-adic absolute value \(|.\|\) (example \(C_p\)). For any \(\alpha \in K\) and \(R \in [0, +\infty]\), the closed disk \(\{ x \in K : |x - \alpha| \leq R \}\) and the open disk \(\{ x \in K : |x - \alpha| < R \}\) are denoted by \(d(\alpha, R)\) and \(d(\alpha, R^-)\) respectively. Also \(C(\alpha, r)\) denotes the circle \(\{ x \in K : |x - \alpha| = r \}\). Moreover \(A(K)\) represents the \(K\)-algebra of analytic functions in \(K\) i.e. the set of power series with an infinite radius of convergence. For the most comprehensive study of analytic functions inside a disk or in the whole field \(K\), we refer the reader to the books \([10, 11, 15, 17]\). During the last several years the ideas of \(p\)-adic analysis have been studied from different aspects and many important results were gained (see \([8]\) to \([13, 12, 14]\)).

Let \(f \in A(K)\) and \(r > 0\), then we denote by \(|f|_r\) the number \(\sup \{ |f(x)| : |x| = r \}\) where \(|.\)| is a multiplicative norm on \(A(K)\). Moreover, if \(f\) is not a constant, the \(|f|_r\) is strictly increasing function of \(r\) and tends to \(+\infty\) with \(r\) therefore there exists its inverse function \(|\tilde{f}| : (|f(0)|, +\infty) \to (0, +\infty)\) with \(\lim_{s \to +\infty} |\tilde{f}|(s) = \infty\).

Further \(f \in A(K)\) and \(g \in A(K)\) are said to be asymptotically equivalent if there exists \(l\), \(0 < l < \infty\) such that \(\frac{|f(l)|}{|g(r)|} \to l\) as \(r \to \infty\) and in this case we write \(f \sim g\). If \(f \sim g\) then clearly \(g \sim f\).

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For $x \in [0, \infty)$ and $k \in \mathbb{N}$, we define $\log^k x = \log \left( \log^{k-1} x \right)$ and $\exp^k x = \exp \left( \exp^{k-1} x \right)$ where $\mathbb{N}$ is the set of all positive integers. We also denote $\log^0 x = x$ and $\exp^0 x = x$. Throughout the paper, $\log$ denotes the Neperian logarithm. Further we assume that throughout the present paper $p, q$ and $m$ always denote positive integers. Taking this into account the $(p, q)$-th order and $(p, q)$-th lower order of an entire function $f \in \mathcal{A}(\mathbb{K})$ are defined as follows:

**Definition 1.** [3] Let $f \in \mathcal{A}(\mathbb{K})$. Then the $(p, q)$-th order and $(p, q)$-th lower order of $f$ are respectively defined as:

$$
\rho^{(p,q)}(f) = \limsup_{r \to \infty} \frac{\log^{|p|} |f| (r)}{\log^{|q|} r} \quad \text{and} \quad \lambda^{(p,q)}(f) = \liminf_{r \to \infty} \frac{\log^{|p|} |f| (r)}{\log^{|q|} r}.
$$

Definition 1 avoids the restriction $p \geq q$ of the original definition of $(p, q)$-th order (respectively $(p, q)$-th lower order) of entire functions introduced by Juneja et al. [16] in complex context.

When $q = 1$, we get the definitions of generalized order and generalized lower order of an entire function $f \in \mathcal{A}(\mathbb{K})$ which symbolize as $\rho^{(p)}(f)$ and $\lambda^{(p)}(f)$ respectively. If $p = 2$ and $q = 1$ then we write $\rho^{(2,1)}(f) = \rho(f)$ and $\lambda^{(2,1)}(f) = \lambda(f)$ where $\rho(f)$ and $\lambda(f)$ are respectively known as order and lower order of $f \in \mathcal{A}(\mathbb{K})$ introduced by Boussaf et al. [8].

In this connection we just introduce the following definition:

**Definition 2.** An entire function $f \in \mathcal{A}(\mathbb{K})$ is said to have index-pair $(p, q)$ if $b < \rho^{(p,q)}(f) < \infty$ and $\rho^{(p-1,q-1)}(f)$ is not a nonzero finite number, where $b = 1$ if $p = q$ and $b = 0$ otherwise. Moreover if $0 < \rho^{(p,q)}(f) < \infty$, then

$$
\rho^{(p-n,q)}(f) = \infty \quad \text{for} \quad n < p,
$$

$$
\rho^{(p,q-n)}(f) = 0 \quad \text{for} \quad n < q,
$$

$$
\rho^{(p+n,q+n)}(f) = 1 \quad \text{for} \quad n = 1, 2, \ldots.
$$

Similarly for $0 < \lambda^{(p,q)}(f) < \infty$:

$$
\lambda^{(p-n,q)}(f) = \infty \quad \text{for} \quad n < p,
$$

$$
\lambda^{(p,q-n)}(f) = 0 \quad \text{for} \quad n < q,
$$

$$
\lambda^{(p+n,q+n)}(f) = 1 \quad \text{for} \quad n = 1, 2, \ldots.
$$

An entire function $f \in \mathcal{A}(\mathbb{K})$ of index-pair $(p, q)$ is said to be of regular $(p, q)$-th growth if its $(p, q)$-th order coincides with its $(p, q)$-th lower order, otherwise $f$ is said to be of irregular $(p, q)$-th growth.

The concepts of $(p, q)$-$\varphi$ order and $(p, q)$-$\varphi$ lower order of entire functions in complex context were introduced by Shen et al. [18] where $p \geq q \geq 1$ and $\varphi : [0, +\infty) \to (0, +\infty)$ is a non-decreasing unbounded function. For details about $(p, q)$-$\varphi$ order and $(p, q)$-$\varphi$ lower order, one may see [18]. Considering the ideas developed by Shen et al. [18], one can define the $(p, q)$-$\varphi$ order and $(p, q)$-$\varphi$ lower order of an entire function $f \in \mathcal{A}(\mathbb{K})$ respectively in the following way:

**Definition 3.** Let $f \in \mathcal{A}(\mathbb{K})$. Also let $\varphi(r) : [0, +\infty) \to (0, +\infty)$ be a non-decreasing unbounded function of $r$. The $(p, q)$-$\varphi$ order $\rho^{(p,q)}(f, \varphi)$ and $(p, q)$-$\varphi$ lower order $\lambda^{(p,q)}(f, \varphi)$ of $f$ are respectively defined as:

$$
\rho^{(p,q)}(f, \varphi) = \limsup_{r \to \infty} \frac{\log^{|p|} |f| (r)}{\log^{|q|} \varphi(r)} \quad \text{and} \quad \lambda^{(p,q)}(f, \varphi) = \liminf_{r \to \infty} \frac{\log^{|p|} |f| (r)}{\log^{|q|} \varphi(r)}.
$$
If \( \varphi(r) = r \), then Definition 1 is a special case of Definition 3.

Extending the notion of index-pair \((p, q)\), one may also introduce the definition of index-pair \((p, q), \varphi\) in the following manner:

**Definition 4.** An entire function \( f \in \mathcal{A}(\mathbb{K}) \) is said to have index-pair \((p, q), \varphi\) if \( b < \rho^{(p,q)}(f, \varphi) < \infty \) and \( \rho^{(p-1,q-1)}(f, \varphi) \) is not a nonzero finite number, where \( b = 1 \) if \( p = q \) and \( b = 0 \) otherwise. Moreover if \( 0 < \rho^{(p,q)}(f, \varphi) < \infty \), then

\[
\begin{align*}
\rho^{(p-n,q)}(f, \varphi) &= \infty \quad \text{for} \quad n < p, \\
\rho^{(p,q-n)}(f, \varphi) &= 0 \quad \text{for} \quad n < q, \\
\rho^{(p+n,q+n)}(f, \varphi) &= 1 \quad \text{for} \quad n = 1, 2, \cdots.
\end{align*}
\]

Similarly for \( 0 < \lambda^{(p,q)}(f, \varphi) < \infty \),

\[
\begin{align*}
\lambda^{(p-n,q)}(f, \varphi) &= \infty \quad \text{for} \quad n < p, \\
\lambda^{(p,q-n)}(f, \varphi) &= 0 \quad \text{for} \quad n < q, \\
\lambda^{(p+n,q+n)}(f, \varphi) &= 1 \quad \text{for} \quad n = 1, 2, \cdots.
\end{align*}
\]

An entire function \( f \in \mathcal{A}(\mathbb{K}) \) of index-pair \((p, q), \varphi\) is said to be of regular \((p, q)-\varphi\) growth if its \((p, q)-\varphi\) order coincides with its \((p, q)-\varphi\) lower order, otherwise \( f \) is said to be of irregular \((p, q)-\varphi\) growth.

However the notion of relative order was first introduced by Bernal [1]. In order to make some progress in the study of \((p, q)-\varphi\) growth, one may introduce the definition of \((p, q)-\varphi\) growth.

Further the function \( f \in \mathcal{A}(\mathbb{K}) \), for which relative order and relative lower order with respect to another function \( g \in \mathcal{A}(\mathbb{K}) \) are the same is called a function of regular relative growth with respect to \( g \). Otherwise, \( f \) is said to be irregular relative growth with respect to \( g \).

In the case of relative order, it therefore seems reasonable to define suitably the \((p, q)\)-th relative order of entire function belonging to \mathcal{A}(\mathbb{K}). With this in view one may introduce the definition of \((p, q)\)-th relative order \( \rho_{g}^{(p,q)}(f) \) and \((p, q)\)-th relative lower order \( \lambda_{g}^{(p,q)}(f) \) of an entire function \( f \in \mathcal{A}(\mathbb{K}) \) with respect to another entire function \( g \in \mathcal{A}(\mathbb{K}) \), in the light of index-pair which are as follows:

**Definition 5.** [3] Let \( f, g \in \mathcal{A}(\mathbb{K}) \). Also let the index-pairs of \( f \) and \( g \) are \((m, q)\) and \((m, p)\), respectively. Then the \((p, q)\)-th relative order \( \rho_{g}^{(p,q)}(f) \) and \((p, q)\)-th relative lower order \( \lambda_{g}^{(p,q)}(f) \) of \( f \) with respect to \( g \) are defined as

\[
\begin{align*}
\rho_{g}^{(p,q)}(f) &= \limsup_{r \to \infty} \frac{\log^{[p]} g \left( \|f\| \left( r \right) \right)}{\log r} = \limsup_{r \to \infty} \frac{\log^{[p]} g \left( \|f\| \left( r \right) \right)}{\log r}, \\
\lambda_{g}^{(p,q)}(f) &= \liminf_{r \to \infty} \frac{\log^{[p]} g \left( \|f\| \left( r \right) \right)}{\log^{[q]} r} = \liminf_{r \to \infty} \frac{\log^{[p]} g \left( \|f\| \left( r \right) \right)}{\log^{[q]} r}.
\end{align*}
\]

Now in order to make some progress in the study of relative order, one may introduce the definitions of relative \((p, q)-\varphi\) order and relative \((p, q)-\varphi\) lower order of entire functions belonging to \mathcal{A}(\mathbb{K}) and to investigate some of its properties, which
we attempt in this paper. With this in view one may introduce the definition of relative \((p, q)\)-\(\varphi\) order \(\rho_{g}^{(p,q)}(f, \varphi)\) and relative \((p, q)\)-\(\varphi\) lower order \(\lambda_{g}^{(p,q)}(f, \varphi)\) of an entire function \(f \in A(\mathbb{K})\) with respect to another entire function \(g \in A(\mathbb{K})\) which are as follows:

**Definition 6.** Let \(f, g \in A(\mathbb{K})\) and \(\varphi(r) : [0, +\infty) \to (0, +\infty)\) be a non-decreasing unbounded function of \(r\). Also let the index-pairs of \(f\) and \(g\) are \((m, q)\)-\(\varphi\) and \((m, p)\), respectively. The relative \((p, q)\)-\(\varphi\) order denoted as \(\rho_{g}^{(p,q)}(f, \varphi)\) and relative \((p, q)\)-\(\varphi\) lower order denoted by \(\lambda_{g}^{(p,q)}(f, \varphi)\) of \(f\) with respect to \(g\) are defined as

\[
\rho_{g}^{(p,q)}(f, \varphi) = \limsup_{r \to \infty} \frac{\log[p] \left\lfloor g(r) \right\rfloor}{\log[q] \varphi(r)} = \limsup_{r \to \infty} \frac{\log[p] \left\lfloor g(r) \right\rfloor}{\log[q] \varphi(r)},
\]

and

\[
\lambda_{g}^{(p,q)}(f, \varphi) = \liminf_{r \to \infty} \frac{\log[p] \left\lfloor g(r) \right\rfloor}{\log[q] \varphi(r)} = \liminf_{r \to \infty} \frac{\log[p] \left\lfloor g(r) \right\rfloor}{\log[q] \varphi(r)}.
\]

If \(\varphi(r) = r\), then Definition 5 is a special case of Definition 6. Further if relative \((p, q)\)-\(\varphi\) order and the relative \((p, q)\)-\(\varphi\) lower order of \(f\) with respect to \(g\) are the same, then \(f\) is called a function of regular relative \((p, q)\)-\(\varphi\) growth with respect to \(g\). Otherwise, \(f\) is said to be irregular relative \((p, q)\)-\(\varphi\) growth with respect to \(g\).

The main aim of this paper is to establish some results related to the growth rates of \(p\)-adic entire functions on the basis of relative \((p, q)\)-\(\varphi\) order and relative \((p, q)\)-\(\varphi\) lower order.

## 2. Results

First of all, we recall one related known property which will be needed in order to prove our results, as we see in the following lemma.

**Lemma 1.** \([6]\) Let \(f \in A(\mathbb{K})\) and \(\alpha > 1\), \(0 < \beta < \alpha\), then for all large \(r\),

\[
\beta |f| (r) \leq |f| (\alpha r).
\]

**Theorem 1.** Let \(f, g, h \in A(\mathbb{K})\). Also let \(0 < \lambda_{h}^{(m,q)}(f, \varphi) \leq \rho_{h}^{(m,q)}(f, \varphi) < \infty\) and \(0 < \lambda_{h}^{(m,p)}(g) \leq \rho_{h}^{(m,p)}(g) < \infty\). Then

\[
\lambda_{h}^{(m,q)}(f, \varphi) \leq \rho_{h}^{(m,q)}(f, \varphi) \leq \min \left\{ \lambda_{h}^{(m,q)}(f, \varphi), \rho_{h}^{(m,q)}(f, \varphi) \lambda_{h}^{(m,p)}(g), \rho_{h}^{(m,p)}(g) \right\} \leq \max \left\{ \lambda_{h}^{(m,q)}(f, \varphi), \rho_{h}^{(m,q)}(f, \varphi) \right\} \leq \lambda_{h}^{(m,q)}(f, \varphi) \leq \rho_{h}^{(m,q)}(f, \varphi) \leq \frac{\lambda_{h}^{(m,q)}(f, \varphi)}{\rho_{h}^{(m,p)}(g)}.
\]

**Proof.** From the definitions of \(\rho_{g}^{(p,q)}(f, \varphi)\) and \(\lambda_{g}^{(p,q)}(f, \varphi)\) we get that

\[
\log \rho_{g}^{(p,q)}(f) = \limsup_{r \to \infty} \left( \log[p+1] \left\lfloor g(r) \right\rfloor - \log[q+1] \varphi \left( \left\lfloor f(r) \right\rfloor \right) \right),
\]

\[
\log \lambda_{g}^{(p,q)}(f) = \liminf_{r \to \infty} \left( \log[p+1] \left\lfloor g(r) \right\rfloor - \log[q+1] \varphi \left( \left\lfloor f(r) \right\rfloor \right) \right).
\]
Now from the definitions of $\rho_{h}^{(m,q)}(f, \varphi)$ and $\lambda_{h}^{(m,q)}(f, \varphi)$, it follows that

$$\log \rho_{h}^{(m,q)}(f) = \limsup_{r \to \infty} \left( \log^{[m+1]} \hat{h}(r) - \log^{[q+1]} \varphi \left( \hat{f}(r) \right) \right), \quad (3)$$

$$\log \lambda_{h}^{(m,q)}(f) = \liminf_{r \to \infty} \left( \log^{[m+1]} \hat{h}(r) - \log^{[q+1]} \varphi \left( \hat{f}(r) \right) \right). \quad (4)$$

Similarly, from the definitions of $\rho_{h}^{(m,p)}(g)$ and $\lambda_{h}^{(m,p)}(g)$, we obtain that

$$\log \rho_{h}^{(m,p)}(g) = \limsup_{r \to \infty} \left( \log^{[m+1]} \hat{h}(r) - \log^{[p+1]} \hat{g}(r) \right), \quad (5)$$

$$\log \lambda_{h}^{(m,p)}(g) = \liminf_{r \to \infty} \left( \log^{[m+1]} \hat{h}(r) - \log^{[p+1]} \hat{g}(r) \right). \quad (6)$$

Therefore from (2), (4) and (5), we get that

$$\log \lambda_{y}^{(p,q)}(f, \varphi) = \liminf_{r \to \infty} \left[ \log^{[m+1]} \hat{h}(r) - \log^{[q+1]} \varphi \left( \hat{f}(r) \right) \right. \left. - \left( \log^{[m+1]} \hat{h}(r) - \log^{[p+1]} \hat{g}(r) \right) \right]$$

i.e.,

$$\log \lambda_{y}^{(p,q)}(f, \varphi) \geq \liminf_{r \to \infty} \left( \log^{[m+1]} \hat{h}(r) - \log^{[q+1]} \varphi \left( \hat{f}(r) \right) \right) \quad \left( \log^{[m+1]} \hat{h}(r) - \log^{[p+1]} \hat{g}(r) \right)$$

i.e.,

$$\log \lambda_{y}^{(p,q)}(f, \varphi) \geq \log \lambda_{h}^{(m,q)}(f, \varphi) - \log \rho_{h}^{(m,p)}(g). \quad (7)$$

Similarly, from (1), (3) and (6), it follows that

$$\log \rho_{y}^{(p,q)}(f, \varphi) = \limsup_{r \to \infty} \left[ \log^{[m+1]} \hat{h}(r) - \log^{[q+1]} \varphi \left( \hat{f}(r) \right) \right. \left. - \left( \log^{[m+1]} \hat{h}(r) - \log^{[p+1]} \hat{g}(r) \right) \right]$$

i.e.,

$$\log \rho_{y}^{(p,q)}(f, \varphi) \leq \limsup_{r \to \infty} \left( \log^{[m+1]} \hat{h}(r) - \log^{[q+1]} \varphi \left( \hat{f}(r) \right) \right) \quad \left( \log^{[m+1]} \hat{h}(r) - \log^{[p+1]} \hat{g}(r) \right)$$

i.e.,

$$\log \rho_{y}^{(p,q)}(f, \varphi) \leq \log \rho_{h}^{(m,q)}(f, \varphi) - \log \lambda_{h}^{(m,p)}(g). \quad (8)$$

Again, in view of (2) we obtain that

$$\log \lambda_{y}^{(p,q)}(f, \varphi) = \liminf_{r \to \infty} \left[ \log^{[m+1]} \hat{h}(r) - \log^{[q+1]} \varphi \left( \hat{f}(r) \right) \right. \left. - \left( \log^{[m+1]} \hat{h}(r) - \log^{[p+1]} \hat{g}(r) \right) \right].$$

By taking $A = \left( \log^{[m+1]} \hat{h}(r) - \log^{[q+1]} \varphi \left( \hat{f}(r) \right) \right)$ and $B = \left( \log^{[m+1]} \hat{h}(r) - \log^{[p+1]} \hat{g}(r) \right)$, we get from above that

$$\log \lambda_{y}^{(p,q)}(f, \varphi) \leq \min \left( \liminf_{r \to \infty} A + \limsup_{r \to \infty} - B, \limsup_{r \to \infty} A + \liminf_{r \to \infty} - B \right)$$
From the conclusion of Theorem 1, let $B < \lambda$.

Therefore in view of (3), (4), (5) and (6) we get from above that
\[
\log \lambda_g^{(p,q)} (f, \varphi) \leq \min \left\{ \log \lambda_h^{(m,q)} (f, \varphi) - \log \lambda_h^{(m,p)} (g), \log \rho_h^{(m,q)} (f, \varphi) - \log \rho_h^{(m,p)} (g) \right\}. \tag{9}
\]

Further from (1) it follows that
\[
\log \rho_g^{(p,q)} (f, \varphi) = \limsup_{r \to \infty} \left[ \log^{m+1} |h| (r) - \log^{q+1} |g| (r) \right] - \lambda_{m,p} (f, \varphi). \tag{10}
\]

By taking $A = \left( \log^{m+1} |h| (r) - \log^{q+1} |g| (r) \right)$ and $B = \left( \log^{m+1} |h| (r) - \log^{q+1} |g| (r) \right)$, we obtain from above that
\[
\log \rho_g^{(p,q)} (f, \varphi) \geq \max \left( \liminf_{r \to \infty} A + \limsup_{r \to \infty} - B, \limsup_{r \to \infty} A + \liminf_{r \to \infty} - B \right).
\]

Therefore in view of (3), (4), (5) and (6), it follows from above that
\[
\log \rho_g^{(p,q)} (f, \varphi) \geq \max \left( \lambda_{m,p}^{(m,q)} (f, \varphi) - \lambda_{m,p}^{(m,p)} (g), \log \rho_h^{(m,q)} (f, \varphi) - \log \rho_h^{(m,p)} (g) \right). \tag{10}
\]

Thus the theorem follows from (7), (8), (9) and (10).

The conclusion of the following remark can be carried out after applying the same technique of Theorem 1 and therefore its proof is omitted.

**Remark 1.** Let $f, g \in \mathcal{A} (\mathbb{R})$. Also let $0 < \lambda^{(m,q)} (f, \varphi) \leq \rho^{(m,q)} (f, \varphi) < \infty$ and $0 < \lambda^{(m,p)} (g) \leq \rho^{(m,p)} (g) < \infty$. Then
\[
\frac{\lambda^{(m,q)} (f, \varphi)}{\rho^{(m,p)} (g)} \leq \frac{\lambda_{m,p}^{(m,q)} (f, \varphi)}{\rho_{h}^{(m,p)} (g)}, \min \left\{ \frac{\lambda^{(m,q)} (f, \varphi)}{\lambda_{m,p}^{(m,q)} (g)}, \frac{\rho^{(m,q)} (f, \varphi)}{\rho_{h}^{(m,p)} (g)} \right\} \leq \frac{\lambda^{(m,q)} (f, \varphi)}{\lambda_{m,p}^{(m,q)} (g)}.
\]

**Remark 2.** From the conclusion of Theorem 1, one may write $\rho_g^{(p,q)} (f, \varphi) = \frac{\lambda^{(m,q)} (f, \varphi)}{\lambda_{m,p}^{(m,q)} (g)}$ and $\lambda_g^{(p,q)} (f, \varphi) = \frac{\lambda^{(m,q)} (f, \varphi)}{\lambda_{m,p}^{(m,q)} (g)}$ when $\lambda_{m,p}^{(m,q)} (g) = \rho_{h}^{(m,p)} (g)$. Similarly $\rho_h^{(p,q)} (f, \varphi) = \frac{\lambda^{(m,q)} (f, \varphi)}{\lambda_{h}^{(m,p)} (g)}$ and $\lambda_h^{(p,q)} (f, \varphi) = \frac{\lambda^{(m,q)} (f, \varphi)}{\lambda_{h}^{(m,p)} (g)}$ when $\lambda_{m,p}^{(m,q)} (f, \varphi) = \rho_{h}^{(m,q)} (f, \varphi)$.

**Theorem 2.** Let $f, g, h \in \mathcal{A} (\mathbb{R})$. If $g \sim h$ then $\rho_g^{(p,q)} (f, \varphi) = \rho_h^{(p,q)} (f, \varphi)$ and $\lambda_g^{(p,q)} (f, \varphi) = \lambda_h^{(p,q)} (f, \varphi)$.
Proof. Let $\varepsilon > 0$. Since $g \sim h$, for any $l$ ($0 < l < \infty$) it follows for all sufficiently large positive numbers of $r$ that
\[
|g| (r) < (l + \varepsilon)|h| (r) .
\]

Now for $\alpha > \max \{1, (l + \varepsilon)\}$, we get by Lemma 1 and above for all sufficiently large positive numbers of $r$ that
\[
|g| (r) < |h| (\alpha r)
\]
\[
i.e., \quad \hat{h} (r) < \alpha \hat{g} (r) .
\]

Therefore we get from (11) that
\[
\rho_{h}^{(p,q)} (f, \varphi) = \limsup_{r \to \infty} \frac{\log^{[p]} \hat{h} (|f| (r))}{\log^{[q]} \varphi (r)} \leq \limsup_{r \to \infty} \frac{\log^{[p]} \alpha \hat{g} (|f| (r))}{\log^{[q]} \varphi (r)}
\]
\[
i.e., \quad \rho_{h}^{(p,q)} (f, \varphi) \leq \limsup_{r \to \infty} \frac{\log^{[p]} \alpha \hat{g} (|f| (r)) + O(1)}{\log^{[q]} \varphi (r)} .
\]

Therefore from above we get that $\rho_{h}^{(p,q)} (f, \varphi) \leq \rho_{g}^{(p,q)} (f, \varphi)$. The reverse inequality is clear because $h \sim g$ and so $\rho_{h}^{(p,q)} (f, \varphi) = \rho_{g}^{(p,q)} (f, \varphi)$. In a similar manner, $\lambda_{h}^{(p,q)} (f, \varphi) = \lambda_{g}^{(p,q)} (f, \varphi)$. This proves the theorem.

\[\square\]

Theorem 3. Let $f, g, h \in \mathcal{A} (K)$. If $f \sim h$ then $\rho_{g}^{(p,q)} (h, \varphi) = \rho_{g}^{(p,q)} (f, \varphi)$ and $\lambda_{g}^{(p,q)} (h, \varphi) = \lambda_{g}^{(p,q)} (f, \varphi)$ where $\varphi (r) : [0, +\infty) \to (0, +\infty)$ is a non-decreasing unbounded function with $\lim_{r \to \infty} \log^{[q]} \varphi (ar) = 1$ for any $\alpha > 0$.

Proof. Since $f \sim h$, for any $\varepsilon > 0$ we obtain that
\[
|f| (r) < (l + \varepsilon)|h| (r) ,
\]
where $0 < l < \infty$.

Therefore for $\alpha > \max \{1, (l + \varepsilon)\}$ and in view of Lemma 1, we get from above for all sufficiently large positive numbers of $r$ that
\[
|f| (r) < |h| (\alpha r) .
\]

Now we obtain from (12) that
\[
\rho_{g}^{(p,q)} (f, \varphi) = \limsup_{r \to \infty} \frac{\log^{[p]} \hat{g} (|f| (r))}{\log^{[q]} \varphi (r)}
\]
\[
\leq \limsup_{r \to \infty} \frac{\log^{[p]} \hat{g} (|h| (\alpha r))}{\log^{[q]} \varphi (\alpha r)} \cdot \frac{\log^{[q]} \varphi (\alpha r)}{\log^{[q]} \varphi (r)}
\]
\[
\leq \limsup_{r \to \infty} \frac{\log^{[p]} \hat{g} (|h| (r))}{\log^{[q]} \varphi (r)} \cdot \lim_{\sigma \to +\infty} \frac{\log^{[q]} \varphi (\sigma)}{\log^{[q]} \varphi (r)} .
\]

Now from above we get that $\rho_{g}^{(p,q)} (f, \varphi) \leq \rho_{g}^{(p,q)} (h, \varphi)$. Further $f \sim h \Rightarrow h \sim f$, so we also obtain that $\rho_{g}^{(p,q)} (h, \varphi) \leq \rho_{g}^{(p,q)} (f, \varphi)$ and therefore $\rho_{g}^{(p,q)} (h, \varphi) = \rho_{g}^{(p,q)} (f, \varphi)$. In a similar manner, $\lambda_{g}^{(p,q)} (h, \varphi) = \lambda_{g}^{(p,q)} (f, \varphi)$.

This proves the theorem. \[\square\]
Theorem 4. Let \( f, g, h \in \mathcal{A}(\mathbb{K}) \). If \( g \sim h \) and \( f \sim k \) then 
\[
\rho_f^{(p,q)}(h, \varphi) = \rho_f^{(p,q)}(h, \varphi) = \rho_k^{(p,q)}(h, \varphi) = \lambda_f^{(p,q)}(h, \varphi) = \lambda_k^{(p,q)}(h, \varphi) \]
where \( \varphi(r) : [0, +\infty) \rightarrow (0, +\infty) \) is a non-decreasing unbounded function with 
\[
\lim_{r \to \infty} \frac{\log^{[\alpha]}(\varphi(r))}{\log^{[\alpha]}(\varphi(r))} = 1 \text{ for any } \alpha > 0.
\]

Theorem 4 follows from Theorem 2 and Theorem 3.

Now we state the following four theorems which can easily be carried out from the definitions of relative \((p, q)\)-\( \varphi \) order and relative \((p, q)\)-\( \varphi \) lower order and with the help of Theorem 2, Theorem 3 and Theorem 4 and therefore their proofs are omitted.

Theorem 5. Let \( f, g, h \in \mathcal{A}(\mathbb{K}) \). Also let \( g \sim h \), \( 0 < \lambda_h^{(p,q)}(f, \varphi) \leq \rho_h^{(p,q)}(f, \varphi) < \infty \) and \( 0 < \lambda_h^{(p,q)}(f, \varphi) \leq \rho_h^{(p,q)}(f, \varphi) < \infty \). Then 
\[
\liminf_{\sigma \to +\infty} \frac{\log[p] \hat{g}(|f| (r))}{\log[p] |h| (|f| (r))} \leq 1 \leq \limsup_{\sigma \to +\infty} \frac{\log[p] \hat{g}(|f| (r))}{\log[p] |h| (|f| (r))}.
\]

Theorem 6. Let \( f, g, h \in \mathcal{A}(\mathbb{K}) \). Also let \( f \sim h \), \( 0 < \lambda_h^{(p,q)}(f, \varphi) \leq \rho_h^{(p,q)}(f, \varphi) < \infty \) and \( 0 < \lambda_h^{(p,q)}(h, \varphi) \leq \rho_h^{(p,q)}(h, \varphi) < \infty \) where \( \varphi(r) : [0, +\infty) \rightarrow (0, +\infty) \) is a non-decreasing unbounded function with 
\[
\lim_{r \to \infty} \frac{\log^{[\alpha]}(\varphi(r))}{\log^{[\alpha]}(\varphi(r))} = 1 \text{ for any } \alpha > 0.
\]

Theorem 7. Let \( f, g, h \in \mathcal{A}(\mathbb{K}) \). Also let \( f \sim h \) and \( g \sim k \), \( 0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty \) and \( 0 < \lambda_k^{(p,q)}(h) \leq \rho_k^{(p,q)}(h) < \infty \) where \( \varphi(r) : [0, +\infty) \rightarrow (0, +\infty) \) is a non-decreasing unbounded function with 
\[
\lim_{r \to \infty} \frac{\log^{[\alpha]}(\varphi(r))}{\log^{[\alpha]}(\varphi(r))} = 1 \text{ for any } \alpha > 0.
\]

Theorem 8. Let \( f, g, h \in \mathcal{A}(\mathbb{K}) \). Also let \( f \sim h \) and \( g \sim k \), \( 0 < \lambda_h^{(p,q)}(h) \leq \rho_h^{(p,q)}(h) < \infty \) and \( 0 < \lambda_k^{(p,q)}(f) \leq \rho_k^{(p,q)}(f) < \infty \) where \( \varphi(r) : [0, +\infty) \rightarrow (0, +\infty) \) is a non-decreasing unbounded function with 
\[
\lim_{r \to \infty} \frac{\log^{[\alpha]}(\varphi(r))}{\log^{[\alpha]}(\varphi(r))} = 1 \text{ for any } \alpha > 0.
\]

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References


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