NOTE ON GENERALIZATIONS OF A SYMMETRIC $q$-SERIES IDENTITY

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Dedicated to Professor G.E. Andrews on the occasion of his 80th birthday

ABSTRACT. The main object of this paper is to generalize a symmetric identity which is given in a recent work [Discrete Math. 339 (2016), 2994–2997.] by the method of $q$-difference equation. In addition, we generalize symmetric identity by fractional integral. Moreover, we generalize symmetric identity by moment integrals. Finally, we generalize symmetric identity by generating function for Al-Salam–Carlitz polynomial $\Phi^{(a,b)}_n(x,y|q)$.

1. Introduction

In this paper, we follow the notations and terminology in [16] and suppose that $0 < q < 1$. We first show a list of various definitions and notations in $q$-calculus which are useful to understand the subject of this paper. The basic hypergeometric series $\phi_s$

$$\phi_s\left[ a_1, a_2, \ldots, a_r \mid b_1, b_2, \ldots, b_s \mid q, z \right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(q, b_1, b_2, \ldots, b_s; q)_n} \left[ (-1)^n q \right]^{1+z-r} z^n,$$

converges absolutely for all $z$ if $r \leq s$ and for $|z| < 1$ if $r = s + 1$ and for terminating. The $q$-series and its compact factorials are defined respectively by

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - a q^k), \quad (a; q)_\infty = \prod_{k=0}^{\infty} (1 - a q^k),$$

where $a$ is a complex variable. For convenience, we always assume $0 < q < 1$ in the paper, $(a_1, a_2, \ldots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n$, where $m$ is a positive integer and $n$ is a non-negative integer or $\infty$.

In [9, 10], Chen and Liu introduced two $q$-exponential operators

$$\mathcal{T}(b D_a) = \sum_{n=0}^{\infty} \frac{1}{(q; q)_n} (b D_a)^n, \quad \mathcal{E}(b \theta_a) = \sum_{n=0}^{\infty} \frac{q^{(n)}}{(q; q)_n} (b \theta_a)^n.$$
The Rogers–Szegő polynomials [1] are given by

\[ h_n(b, c|q) = \sum_{k=0}^{n} \binom{n}{k} b^k c^{n-k}, \quad \text{and} \quad g_n(b, c|q) = \sum_{k=0}^{n} \binom{n}{k} q^{k(k-n)} b^k c^{n-k}. \] (3)

The Al-Salam–Carlitz polynomials [6, Eq. (4.4)]

\[ \Phi_n^{(a)}(b, c|q) = \sum_{k=0}^{n} \binom{n}{k} (a; q)_k b^k c^{n-k}, \quad \text{and} \quad \Psi_n^{(a)}(b, c|q) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k q^{k^2} \left( \frac{1}{a} \right)_k (ab)^k c^{n-k}. \] (4)

The Al-Salam–Carlitz polynomials reduce to the Rogers–Szegő polynomials with \( a = 0 \).

The Rogers–Szegő polynomials play important roles in the theory of orthogonal polynomials. Liu [18, 19] obtained several important results by the following \( q \)-difference equations. Liu and Zeng [23] studied relations between \( q \)-difference equations and \( q \)-orthogonal polynomials. For more information, please refer to [3, 12, 13, 14, 15, 17, 20, 21, 22, 27, 29, 30, 31, 32, 33].

**Proposition 1.** Let \( f(a, b) \) be a two-variable analytic function at \((0, 0) \in \mathbb{C}^2\). Then

(A) \( f \) can be expanded in terms of \( h_n(a, b|q) \) if and only if \( f \) satisfies the functional equation

\[ bf(aq, b) - af(a, bq) = (b - a) f(a, b). \] (5)

(B) \( f \) can be expanded in terms of \( g_n(a, b|q) \) if and only if \( f \) satisfies the functional equation

\[ af(aq, b) - bf(a, bq) = (a - b) f(aq, bq). \] (6)

In [4], Andrews gave a wonderful introduction of Ramanujan’s “lost” notebook, and listed some interesting identities contained therein. One of which is the following beautiful symmetric identity. Where if

\[ f(\alpha, \beta) := \frac{1}{1 - \alpha} + \sum_{n \geq 1} \frac{\beta^n}{(1 - \alpha x^n)(1 - \alpha x^{n-1} y)(1 - \alpha x^{n-2} y^2) \cdots (1 - \alpha y^n)}. \]

Then

\[ f(\alpha, \beta) = f(\beta, \alpha). \]

The identity we present here is a refinement of the case where \( x = q, y = q^2 \).

Then A.E. Patkowski [25] obtained the following symmetric \( q \)-series identity.

**Proposition 2 ([25, Eq. (1.3)]).** We have, for arbitrary \( a, b \), and \( |b| < 1, \ |t| < 1, \)

\[ \sum_{n=0}^{\infty} \frac{(-abq^{n+1}; q)_n t^n}{(bq^n; q)_{n+1}} = \sum_{n=0}^{\infty} \frac{(-atq^{n+1}; q)_n b^n}{(tq^n; q)_{n+1}}. \] (7)

In this paper, we first generalize this symmetric \( q \)-series identity by the method of \( q \)-difference equation.
Theorem 3. For arbitrary $|a| < 1$, $|b| < 1$ and $|t| < 1$, we have
\[
\sum_{n=0}^{\infty} \frac{(-atq^{n+1}; q)_{n} q^{n}}{(tq^n; q)_{n+1}} h_n(c, b|q) = \sum_{n=0}^{\infty} \frac{(-abq^{n+1}; q)_{n} a^n}{(bq^n; q)_{n+1}} \sum_{k=0}^{n} \frac{(q^n, bq^n; q)_k (-acq^{2n+1})^k}{(q, -abq^{n+1}, bq^{2n+1}; q)_k} 2\phi_1 \left[ \frac{q^{n+1}, 0}{bq^{2n+1+k}; q, cq^n} \right],
\]
(8)
\[
\sum_{n=0}^{\infty} \frac{(-atq^{n+1}; q)_{n} g_n(c, b|q)}{(tq^n; q)_{n+1}} = \sum_{n=0}^{\infty} \frac{(-abq^{n+1}; q)_{n} a^n}{(bq^n; q)_{n+1}} \sum_{k=0}^{n} \frac{(-aq, 1/(bq^{2n}); q)_k}{(q, 1/(abq^{2n}), 1/(bq^{2n-1}); q)_k} \left( \frac{cq^{n+1}}{b} \right)^k \times \sum_{n=0}^{\infty} \frac{(q^{n+1}, q)_{n}}{(q^{k+1-n}/b; q)_n} q^{(n+k)(n+k)} (c/b)^n.
\]
(9)

Proof of Theorem 3. Denoting the LHS of equation (8) can be written by
\[
f(b, c) = \sum_{n=0}^{\infty} \frac{(-abq^{n+1}; q)_{n} a^n}{(bq^n; q)_{n+1}} \sum_{k=0}^{n} \frac{(q^n, bq^n; q)_k (-acq^{2n+1})^k}{(q, -abq^{n+1}, bq^{2n+1}; q)_k} 2\phi_1 \left[ \frac{q^{n+1}, 0}{bq^{2n+1+k}; q, cq^n} \right]
\]
\[
= \sum_{n=0}^{\infty} \frac{(-abq^{n+1}, bq^{2n+1}; q)_{n} a^n}{(-abq^{2n+1}, bq^n; q)_\infty} \sum_{k=0}^{n} \frac{(q^n, bq^n; q)_k (-acq^{2n+1})^k}{(q, -abq^{n+1}, bq^{2n+1}; q)_k} 2\phi_1 \left[ \frac{q^{n+1}, 0}{bq^{2n+1+k}; q, cq^n} \right]
\]
\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{0}^{\infty} \frac{c^k}{(q; q)_k} D_b^k \left( \frac{(-abq^{n+1}, bq^{2n+1}; q)_{\infty}}{(-abq^{2n+1}, bq^n; q)_\infty} \right)
\]
\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{0}^{\infty} \frac{c^k}{(q; q)_k} D_b^k \left( \frac{(-atq^{n+1}; q)_{n} b^n}{(tq^n; q)_{n+1}} \right).
\]
By using equation (7), we have
\[
f(b, c) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{c^k}{(q; q)_k} D_b^k \left( \frac{(-atq^{n+1}; q)_{n} b^n}{(tq^n; q)_{n+1}} \right)
\]
\[
= \sum_{n=0}^{\infty} \frac{(-atq^{n+1}; q)_{n} b^n}{(tq^n; q)_{n+1}} \sum_{k=0}^{\infty} \frac{c^k}{(q; q)_k} D_b^k (b^n).
\]
We can verify that $f(a, b, c)$ satisfies equation (5). Then, we have
\[
f(b, c) = \sum_{n=0}^{\infty} u_n h_n(c, b|q),
\]
then, we have
\[
f(b, 0) = \sum_{n=0}^{\infty} u_n b^n = \sum_{n=0}^{\infty} \frac{(-atq^{n+1}; q)_{n} b^n}{(tq^n; q)_{n+1}}.
\]
Hence
\[
f(b, c) = \sum_{n=0}^{\infty} \frac{(-atq^{n+1}; q)_{n} h_n(c, b|q)}{(tq^n; q)_{n+1}}.
\]
Using the same way, we gain the equation (9). The proof is complete. □

2. Fractional $q$-integrals for a symmetric $q$-series identity

In this section, we use the fractional $q$-integrals to deduce a new identity for a symmetric $q$-series. For more information, please refer to [2, 8, 26].
The $q$-gamma function is defined by [16]

$$
\Gamma_q(x) = \frac{(q^x q)_\infty}{(q^\infty q)_\infty} (1 - q)^{1-x}, \quad x \in \mathbb{R}\setminus\{0, -1, -2, \ldots\}.
$$

(10)

The Thomae–Jackson $q$-integral is defined by [16, 11, 28]

$$
\int_a^b f(x) \, dq(x) = (1 - q) \sum_{n=0}^{\infty} [bf(bq^n) - af(aq^n)] q^n.
$$

(11)

The Riemann–Liouville fractional $q$-integral operator is introduced in [2]

$$
(I^\alpha_q f)(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x (qt/x, q)_a^{-\alpha} f(t) \, dt.
$$

(12)

The generalized Riemann–Liouville fractional $q$-integral operator for $\alpha \in \mathbb{R}^+$ is given by [26]

$$
(I^\alpha_{q,a} f)(x) = \sum_{n=0}^{\infty} \frac{[k]_q! a^{-k}}{k \Gamma_q(\alpha+k+1)} x^{\alpha+k}(a/x, q)_a^{k}.\tag{13}
$$

Proposition 4. For $\alpha \in \mathbb{R}^+$, $0 < a < x < 1$, we have

$$
I^\alpha_{q,a} [x^n] = \sum_{k=0}^{n} \frac{[k]_q! a^{-k}}{k \Gamma_q(\alpha+k+1)} x^{\alpha+k}(a/x, q)_a^{k}.\tag{14}
$$

Theorem 5. For $\alpha \in \mathbb{R}^+$, $0 < c < b < 1$, we have we have

$$
\sum_{n=0}^{\infty} \frac{(-acq^{n+1}; q)_n}{(cq^n; q)_{n+1}} \sum_{k=0}^{\infty} \frac{b^{q+1}(c/b; q)_{n+k}}{c^k(q^n; q)_{n+k}} \frac{\phi_2}{\phi_2} \left[ q^{-k}, -acq^{2n+1}, cq^n \right]_n \frac{cq^{2n+1}, -acq^{n+1}}{q}.\tag{15}
$$

**Proof of Theorem 5.** Multiply $(1 - q)^\alpha$ on both sides of equation (15), the LHS of equation (15) become to

$$
\sum_{n=0}^{\infty} (1 - q)\frac{(-acq^{n+1}; q)_n q^n}{(-acq^{n+1}; q)_{n+1}} \sum_{k=0}^{\infty} \frac{b^{q+1}(c/b; q)_{n+k}}{c^k(q^n; q)_{n+k}} \frac{\phi_2}{\phi_2} \left[ q^{-k}, -acq^{2n+1}, cq^n \right]_n \frac{cq^{2n+1}, -acq^{n+1}}{q}.\tag{16}
$$

$$
= \sum_{n=0}^{\infty} \frac{(-abq^{n+1}; q)_n q^n}{(-abq^{n+1}; q)_{n+1}} \sum_{k=0}^{\infty} \frac{b^{q+1}(c/b; q)_{n+k}}{c^k(q^n; q)_{n+k}} \frac{\phi_2}{\phi_2} \left[ q^{-k}, -acq^{2n+1}, cq^n \right]_n \frac{cq^{2n+1}, -acq^{n+1}}{q}.\tag{17}
$$

Similarly, the RHS of equation (15) become to

$$
\sum_{n=0}^{\infty} \frac{(-atq^{n+1}; q)_n}{(tq^n; q)_{n+1}} (1 - q)^\alpha \sum_{k=0}^{n} \frac{(q^n q)_{n-k}}{(q^n q)_{n-k}} \cdot \frac{b^{q+1}(c/b; q)_{n+k}}{(q^n q)_{n+k}} \tag{18}
$$

$$
= \sum_{n=0}^{\infty} \frac{(-atq^{n+1}; q)_n p^n}{(tq^n; q)_{n+1}} \cdot \frac{[b^n q^n]}{(tq^n; q)_{n+1}} = I^\alpha_{q,a} \left\{ \frac{(-atq^{n+1}; q)_n b^n}{(tq^n; q)_{n+1}} \right\}.\tag{19}
$$

then, we use Proposition 2 can obtain the equation (15). The proof is complete. \( \square \)
3. Moment integrals for a symmetric $q$-series identity

In this section, we use the moment integrals to deduce a new identity for a symmetric $q$-series.

Al-Salam and Carlitz [1] defined moments of two discrete distributions $d\alpha^{(a)}(x)$ and $d\beta^{(a)}(x)$ by Rogers-Szego polynomials as follow

$$\int_{-\infty}^{\infty} x^n d\alpha^{(a)}(x) = h_n(a|q) \quad \text{and} \quad \int_{-\infty}^{\infty} x^n d\beta^{(a)}(x) = g_n(a|q),$$

where $\alpha^{(a)}(x)$ is a step function whose jumps occur at the points $q^k$ and $aq^k$ for $k \in \mathbb{N}$, while the jumps of $\beta^{(a)}(x)$ occur at the points $q^{-k}$ for $k \in \mathbb{N}$. These jumps are given by

$$d\alpha^{(a)}(q^k) = \frac{q^k}{(a; q)_n(q, qa; q)_k} \quad \text{and} \quad d\alpha^{(a)}(aq^k) = \frac{q^k}{(1/a; q)_n(q, aq; q)_k}, \quad (19)$$

$$d\beta^{(a)}(q^{-k}) = \frac{a^k q^k (aq^{k+1})_{\infty}}{(q; q)_k}. \quad (20)$$

Liu gained the following expression of bivariate Rogers-Szego polynomials by the technique of partial fraction [18, Eq. (4.20)].

$$h_n(a, b|q) = \frac{\alpha^a}{(b/a; q)_n} \sum_{k=0}^{\infty} \frac{q^{n+k}}{(q, aq/b; q)_k} + \frac{b^n}{(a/b; q)_n} \sum_{k=0}^{\infty} \frac{q^{n+k}}{(q, qb/a; q)_k}. \quad (21)$$

So it’s natural to define the generalized discrete probability measure $\alpha^{(a,b)}$ by

$$\alpha^{(a,b)} = \sum_{k=0}^{\infty} \left[ \frac{q^k}{(a/b; q)_n(q, qb/a; q)_k} \epsilon_{bqk} + \frac{q^k}{(b/a; q)_n(q, aq/b; q)_k} \epsilon_{aqk} \right], \quad (22)$$

where the bivariate Rogers–Szegő polynomials expressed by

$$h_n(a, b|q) = \int_{-\infty}^{\infty} x^n d\alpha^{(a,b)}(x), \quad (23)$$

and their generating function are given [18, Eq. (2.3)]

$$\sum_{n=0}^{\infty} h_n(a, b|q) \frac{t^n}{(q; q)_n} = \frac{1}{(at, bt; q)_n} = \int_{-\infty}^{\infty} \frac{1}{(x; q)_n} d\alpha^{(a,b)}(x). \quad (24)$$


Proposition 6 ([5, Eq. (1.11)]). For $x \in \mathbb{N}$ and $d/c = q^{-1}$, if $\max\{|c|, |as|, |at|, |bs|, |bt|\} < 1$, we have

$$\int_{-\infty}^{\infty} \frac{dx}{(ax, bx, cx; q)_n} d\alpha^{(c,x)}(x) = (ds, abst; q)_n \frac{3\phi_2}{(cs, as, at, bs, bt; q)_n} \frac{d/c}{ds, abst : q, ct}. \quad (25)$$

Corollary 7. For $x \in \mathbb{N}$, if $\max\{|as|, |at|, |bs|, |bt|, |abst|\} < 1$, we have

$$\int_{-\infty}^{\infty} \frac{dx}{(ax, bx, q)_n} d\alpha^{(c,x)}(x) = (cs, abst; q)_n \frac{2\phi_2}{(cs, as, abt; q)_n} \frac{as, bs}{cs, abst : q, ct}. \quad (26)$$

Proposition 8 ([7, Eq. (2.10)]). For $n \in \mathbb{N}$, we have

$$\mathbb{E}(\theta_d) [\{at, q\}_n] = \{at, bt, q\}_n, \quad (27)$$

$$\mathbb{E}(\theta_d) [\alpha^n(at, bt, q)_{\infty} \phi_1] \left[ q^n, q/(at) ; 0 : q, bt \right]. \quad (28)$$
Proposition 9. For $x \in \mathbb{N}$, if $\max \{ |as|, |at|, |bs|, |bt|, |abst!| \} < 1$, we have
\[
\int_{-\infty}^{\infty} \frac{(cx, dx; q)_\infty}{(ax, bx; q)_\infty} d\alpha^{(x, y)}(x) = \frac{(cs, ds, abst; q)_\infty}{(as, at, bs, bt; q)_\infty} \sum_{j=0}^{\infty} q^{(j)}(-ct)^j(bx, as; q)_j \frac{q^{-j}, 1/(csq^{j-1})}{0} ; q, dsq^j.
\]
(29)

Proof of Proposition 9. By using the equation (26), we have
\[
\mathbb{E}(s; t) \left\{ \int_{-\infty}^{\infty} \frac{(cx, q)_\infty}{(ax, bx; q)_\infty} d\alpha^{(x, y)}(x) \right\} = \frac{(ds, abst; q)_\infty}{(as, at, bs, bt; q)_\infty} \sum_{j=0}^{\infty} q^{(j)}(-1)^j(bx, as; q)_j \frac{q^{(j)}, 1/(csq^{j-1})}{0} ; q, dsq^j.
\]
(30)

Then the LHS of the equation (30) can be written by
\[
\mathbb{E}(s; t) \left\{ \int_{-\infty}^{\infty} \frac{(cx, q)_\infty}{(ax, bx; q)_\infty} d\alpha^{(x, y)}(x) \right\} = \int_{-\infty}^{\infty} \frac{1}{(ax, bx; q)_\infty} \mathbb{E}(s; t) \left\{ (cx, q)_\infty \right\} d\alpha^{(x, y)}(x) = \int_{-\infty}^{\infty} \frac{(cx, dx; q)_\infty}{(ax, bx; q)_\infty} d\alpha^{(x, y)}(x).
\]
(31)

Using the equation (28), the RHS of the equation (30) becomes
\[
\frac{(ds, abst; q)_\infty}{(as, at, bs, bt; q)_\infty} \sum_{j=0}^{\infty} q^{(j)}(-1)^j(bx, as; q)_j \frac{q^{(j)}, 1/(csq^{j-1})}{0} ; q, dsq^j
\]
\[
= \frac{(cs, ds, abst; q)_\infty}{(as, at, bs, bt; q)_\infty} \sum_{j=0}^{\infty} q^{(j)}(-ct)^j(bx, as; q)_j \frac{q^{-j}, 1/(csq^{j-1})}{0} ; q, dsq^j.
\]
The proof is complete.

\[
\Box
\]

Theorem 10. For $x \in \mathbb{N}$, if $\max \{ |-axq^{2n+1}|, |ayq^{2n+1}|, |xq^n|, |yq^n| \} < 1$, we have
\[
\sum_{n=0}^{\infty} \frac{(-abq^{n+1}; q)_n}{(bq^n; q)_{n+1}} h_n(x, y|q) = \sum_{n=0}^{\infty} \frac{(-axq^{3n+1}; q)_n}{(xq^n; q)_{n+1}} \sum_{j=0}^{\infty} \frac{q^{(j)}(-yq^{2n+1})^j(xq^n, -axq^{2n+1}; q)_j}{(q, -axq^{2n+1}, xq^{2n+1}, -axyq^{3n+1}; q)_j} \times \frac{q^{-j}, 1/(xq^{2n+j})}{0} ; q, -axq^{2n+1+j}.
\]
(32)

Remark 11. Let $y = 0$ in Theorem 10, equation (32) reduces to (7).

Proof of Theorem 10. From a symmetric $q$-series identity
\[
\sum_{n=0}^{\infty} \frac{(-abq^{n+1}; q)_n}{(bq^n; q)_{n+1}} t^n = \sum_{n=0}^{\infty} \frac{(-atq^{n+1}; q)_n}{(aq^n; q)_{n+1}} t^n.
\]
(33)

Acting moment integral on both sides of the equation (33), we have
\[
\sum_{n=0}^{\infty} \frac{(-abq^{n+1}; q)_n}{(bq^n; q)_{n+1}} \int_{-\infty}^{\infty} t^n d\alpha^{(x, y)}(t) = \sum_{n=0}^{\infty} \frac{(-atq^{n+1}; q)_n}{(aq^n; q)_{n+1}} \int_{-\infty}^{\infty} t^n d\alpha^{(x, y)}(t).
\]
(34)

Then use the equation (23) and (29), we obtain equation (32). The proof is complete.
4. Generating functions for a symmetric $q$-series identity

In this section, motivated by the results of Liu’s [24], we use the generating function for Al-Salam–Carlitz polynomial $\Phi_n^{(a,b)}(x, y|q)$ to generalize symmetric $q$-series identity.

The homogeneous polynomials $\Phi_n^{(a,b)}(b, c|q)$ is defined by

$$\Phi_n^{(a,b)}(x, y|q) = \sum_{k=0}^{\infty} \binom{n}{k} (\alpha; q)_k (\beta; q)_{n-k} x^k y^{n-k}. \quad (35)$$

**Proposition 12 ([24, Proposition 3.2]).** If $\max \{|x|, |y|\} < 1$, we have

$$\sum_{n=0}^{\infty} \frac{\Phi_n^{(a,b)}(x, y|q)}{(q^n; q)_n} t^n = \frac{(ax, by; q)_\infty}{(ab, tx, ty; q)_\infty}. \quad (36)$$

**Theorem 13.** If $\max \{|x|, |y|, |aq2^{n+1}|, |bq^n|, |b|, |t|\} < 1$, we have

$$\sum_{n=0}^{\infty} \frac{(-atq^{n+1}; q)_n(c; q)_k b^n}{(aq^n; q)_{n+1}(q; q)_k} = \sum_{n=0}^{\infty} \frac{(c, -aq^{n+1}, bq^{2n+1}; q)_{n+1} t^n}{(q; -aq^{2n+1}, bq^n; q)_{n+1}} \emptyset(\frac{q/c, -aq^{2n+1}, bq^n}{-abq^{n+1}, bq^{2n+1}; q}) \emptyset(\frac{x}{t}). \quad (37)$$

**Remark 14.** Let $c = 0$ in Theorem 13, equation (37) reduces to (7).

**Proof of Theorem 13.** By Using equation (36), let $a = q^{-n}, b = q^{n+1}, x = -aq^{2n+1}, y = q^n, t = b$ and $\max \{|-aq^{2n+1}|, |bq^n|\} < 1$, then we have

$$\sum_{n=0}^{\infty} t^n \sum_{k=0}^{\infty} \frac{\Phi_k^{(q^{-n}q^{n+1})}(-aq^{2n+1}, q^n|q)}{(q; q)_k} \frac{b^k}{(q; q)_k} = \sum_{n=0}^{\infty} \frac{(-aq^{n+1}, bq^{2n+1}; q)_{n+1}}{(-aq^{2n+1}, bq^n; q)_{n+1}} \emptyset\sum_{k=0}^{\infty} \frac{(-abq^{n+1}; q)_k t^n}{(bq^n; q)_{n+1}}. \quad (38)$$

Then, the LHS of equation (37) can be written by

$$\sum_{n=0}^{\infty} \frac{(c, -aq^{n+1}, bq^{2n+1}; q)_{n+1} t^n}{(aq^n; q)_{n+1}} \emptyset(\frac{q/c, -aq^{2n+1}, bq^n}{-abq^{n+1}, bq^{2n+1}; q}) \emptyset(\frac{x}{t}).$$

Then, the proof is complete. \qed

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