ON THE DYNAMICS OF SOME RECURSIVE SEQUENCES

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Abstract. In this paper we deal with the investigation of some qualitative behavior such as the global convergence, boundedness and periodicity of the solutions of the difference equation

\[ x_{n+1} = \frac{\alpha x_n^2 + \beta x_{n-1}x_n + \gamma x_{n-2}^2}{Ax_n^2 + Bx_{n-1}x_n + Cx_{n-2}^2}, \quad n = 0, 1, \ldots, \]

where the parameters \( \alpha, \beta, \gamma, A, B \) and \( C \) are nonnegative numbers with \( A + B + C > 0, \alpha + \beta + \gamma > 0 \) and the initial conditions \( x_{-2}, x_{-1} \) and \( x_0 \) are arbitrary positive real numbers.

1. Introduction

In this paper we deal with the behavior of the solutions of the following difference equation

\[ x_{n+1} = \frac{\alpha x_n^2 + \beta x_{n-1}x_n + \gamma x_{n-2}^2}{Ax_n^2 + Bx_{n-1}x_n + Cx_{n-2}^2}, \quad n = 0, 1, \ldots, \] (1)

where the parameters \( \alpha, \beta, \gamma, A, B, \) and \( C \) are nonnegative numbers with \( A + B + C > 0, \alpha + \beta + \gamma > 0 \) and the initial conditions \( x_{-2}, x_{-1} \) and \( x_0 \) are arbitrary positive real numbers.

In [11], Kulenović et al. investigated the solutions of the difference equation

\[ x_{n+1} = \frac{\alpha x_n + \beta x_{n-1} + \gamma x_{n-2}}{Ax_n + Bx_{n-1} + Cx_{n-2}}. \]

In [2], Elabbasy et al. studied the periodicity and the global stability of the difference equation

\[ x_{n+1} = \frac{\alpha x_n + \beta x_{n-1} + \gamma x_{n-2}}{Ax_n + Bx_{n-1} + Cx_{n-2}}. \]

In [3], El-Metwally, studied qualitative the properties of some higher order the difference equations of the form

Key words and phrases. Asymptotic stability; Global convergence; Periodic solutions; Rational difference equation.

2010 Mathematics Subject Classification. 39A10.
\[ x_{n+1} = \frac{a x_{n-k_0} x_{n-k_1} \ldots x_{n-k_i}}{c x_{n-k_0} x_{n-k_1} \ldots x_{n-k_i}} + b x_{n-r_0} x_{n-r_1} \ldots x_{n-r_j}, \]

and

\[ y_{n+1} = \frac{\alpha_0 y_n + \alpha_1 y_{n-1} + \ldots + \alpha_t y_{n-t}}{\beta_0 y_n + \beta_1 y_{n-1} + \ldots + \beta_t y_{n-t}}. \]

Other related results on rational difference equations can be found in [3], [8], [9].

Let \( I \) be some interval of real numbers and let \( f : I^{k+1} \to I \), be a continuously differentiable function. Then for every set of initial conditions \( x_{-k}, x_{-k+1}, \ldots, x_0 \in I \), the difference equation

\[ x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, \ldots, \] (2)

has a unique solution \( \{ x_n \}_{n=-k}^\infty \). A solution of Eq. (2) that is constant for all \( n \geq -k \) is called an equilibrium solution of Eq. (2). If \( x_n = \bar{x}, \) for all \( n \geq -k, \) is an equilibrium solution of Eq. (2) then \( \bar{x} \) is called an equilibrium point or simply an equilibrium of Eq. (2).

The following definitions and previous results will be useful for the proof of our results in this paper.

**Definition 1** [1] (stability)

(i) An equilibrium point \( \bar{x} \) of Eq. (2) is called locally stable if, for every \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that if \( \{ x_n \}_{n=-k}^\infty \) is a solution of Eq. (2) with

\[ |x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + \ldots + |x_0 - \bar{x}| < \delta, \]

then

\[ |x_n - \bar{x}| < \varepsilon, \quad \text{for all} \quad n \geq -k. \]

(ii) An equilibrium point \( \bar{x} \) of Eq. (2) is called locally asymptotically stable if \( \bar{x} \) is locally stable and if in addition there exists \( \gamma > 0 \) such that if \( \{ x_n \}_{n=-k}^\infty \) is a solution of Eq. (2) with

\[ |x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + \ldots + |x_0 - \bar{x}| < \gamma, \]

then

\[ \lim_{n \to \infty} x_n = \bar{x}. \]

(iii) An equilibrium point \( \bar{x} \) of Eq. (2) is called a global attractor if, for every solution \( \{ x_n \}_{n=-1}^\infty \) of Eq. (2) we have

\[ \lim_{n \to \infty} x_n = \bar{x}. \]

(iv) An equilibrium point \( \bar{x} \) of Eq. (2) is called globally asymptotically stable if \( \bar{x} \) is locally stable and \( \bar{x} \) is also a global attractor of Eq. (2).

(v) An equilibrium point \( \bar{x} \) of Eq. (2) is called unstable if \( \bar{x} \) is not locally stable.

**Definition 2** [1]

(1) A solution \( \{ x_n \} \) of Eq. (2) is said to be periodic with period \( p \) if

\[ x_{n+p} = x_n \quad \text{for all} \quad n \geq -1. \] (3)
(2) A solution \( \{x_n\} \) of Eq. (2) is said to be **periodic with prime period** \( p \), or a **\( p \)-cycle** if it is periodic with period \( p \) and \( p \) is the least positive integer for which (3) holds.

**Definition 3** \( [7] \) (permanence): Eq. (2) is said to be permanent and bounded if there exists number \( m \) and \( M \) with \( 0 < m < M < \infty \) such that for any initial condition \( x_{-\kappa}, x_{-\kappa+1}, ..., x_0 \in (0, \infty) \) there exists a positive integer \( N \) which depends on these initial conditions such that \( m < x_n < M \) for all \( n \geq N \).

The linearized equation of Eq. (2) about the equilibrium point \( \bar{x} \) is

\[
z_{n+1} = a_1z_{n} + a_2z_{n-1} + \ldots + a_{k+1}z_{n-k},
\]

where \( a_i = \frac{\partial f}{\partial x_{n-i}}(\bar{x}, \bar{x}, ..., \bar{x}) \), \( i = 0, 1, ..., k \). The characteristic equation of Eq. (4) is

\[
\lambda^{k+1} - \sum_{i=1}^{k+1} a_i \lambda^{k-i+1} = 0.
\]

**Theorem A.** \([10]\) Assume that \( p_i \in \mathbb{R}, i = 2, ..., k \), and \( k \in \{0, 1, 2, \ldots\} \). Then

\[
\sum_{i=0}^{k} |p_i| < 1,
\]

is a sufficient condition for the asymptotic stability of the difference equation

\[
y_{n+1} + p_1y_{n+k-1} + \ldots + p_ky_n = 0, \quad n = 0, 1, 2, \ldots
\]

consider the following equation

\[
x_{n+1} = q(x_n, x_{n-1}, x_{n-2}).
\]

**Theorem B.** \([6]\) Let \( J \) be some interval of real numbers, \( f \in C[J^{v+1}, J] \) and let \( \{x_n\}_{n=-v}^{\infty} \) be a bounded solution of the difference equation

\[
x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-v}), \quad n = 0, 1, ..., \tag{5}
\]

with

\[
I = \lim_{n \to \infty} \inf x_n, \quad S = \lim_{n \to \infty} \sup x_n \quad \text{and with } I, S \in J.
\]

Then there exist two solutions \( \{I_n\}_{n=-\infty}^{\infty} \) and \( \{S_n\}_{n=-\infty}^{\infty} \) of Eq. (5) with

\[
I_0 = I, \quad S_0 = S, \quad I_n, S_n \in [I, S] \quad \text{for all } n \in \mathbb{Z},
\]

and such that for every \( N \in \mathbb{Z}, I_N \) and \( S_N \) are limit points of \( \{x_n\}_{n=-v}^{\infty} \).

Furthermore for every \( m \leq -v \), there exist two subsequences \( \{x_{r_n}\} \) and \( \{x_{l_n}\} \)

of the solution \( \{x_n\}_{n=-v}^{\infty} \) such that the following are true:

\[
\lim_{n \to \infty} x_{r_{n+N}} = I_N \quad \text{and} \quad \lim_{n \to \infty} x_{l_{n+N}} = S_N \quad \text{for every } N \geq m.
\]

The solutions \( \{I_n\}_{n=-\infty}^{\infty} \) and \( \{S_n\}_{n=-\infty}^{\infty} \) are called **full limiting sequences** of Eq. (6).

Consider the scalar \( k \)th order linear difference equation

\[
x(n + k) + p_1(n)x(n + k - 1) + \ldots + p_k(n)x(n) = 0, \tag{6}
\]

where \( k \) is a positive integer and \( p_i : \mathbb{Z}^+ \to \mathbb{C} \) for \( i = 1, ..., k \). Eq. (6) is said to be of Poincaré type if the limits

\[
q_i = \lim_{k \to \infty} p_i(n), \quad i = 1, ..., k, \tag{7}
\]
exist in \( C \). Under this hypothesis, Eq. (6) can be regarded as a perturbation of the equation with constant coefficients
\[
x(n + k) + q_1x(n + k - 1) + \ldots + q_kx(n) = 0.
\]

**Theorem C.** (Poincaré’s Theorem). Suppose condition (7) holds.

Let \( \lambda_1, \ldots, \lambda_k \) be the roots of the characteristic equation
\[
\lambda^k + q_1\lambda^{k-1} + \ldots + q_k = 0
\]
(9) of Eq. (8) and suppose that
\[
|\lambda_i| \neq |\lambda_j| \quad \text{for} \quad i \neq j.
\]
(10)
If \( x(n) \) is a solution of (6) then either \( x(n) = 0 \) for all large \( n \) or there exists an index \( j \in \{1, \ldots, k\} \) such that
\[
\lim_{n \to \infty} \frac{x(n + 1)}{x(n)} = \lambda_j.
\]
(11)

2. Local Stability of the equilibrium points of Eq. (1)

In this section, we investigate the local stability character of the solutions of Eq. (1).

The equilibrium points of Eq. (1) are given by the relation
\[
\Pi = \frac{\alpha \pi^2 + \beta \pi^2 + \gamma \pi^2}{A \pi^2 + B \pi^2 + C \pi^2}.
\]
Clearly, the unique positive equilibrium point of Eq. (1) is
\[
\Pi = (\alpha + \beta + \gamma) / (A + B + C).
\]

Let \( f : (0, \infty)^3 \to (0, \infty) \) be a function defined by
\[
f(u, v, w) = \frac{\alpha u^2 + \beta vu + \gamma w^2}{A u^2 + B u v + C w^2}.
\]
(12)

Then we see that
\[
\frac{\partial f(u, v, w)}{\partial u} = \frac{(2\alpha u + \beta v) (A u^2 + B u v + C w^2) - (2A u + B v) (\alpha u^2 + \beta vu + \gamma w^2)}{(A u^2 + B u v + C w^2)^2},
\]
\[
\frac{\partial f(u, v, w)}{\partial v} = \frac{\beta u (A u^2 + B u v + C w^2) - B u (\alpha u^2 + \beta vu + \gamma w^2)}{(A u^2 + B u v + C w^2)^2},
\]
and
\[
\frac{\partial f(u, v, w)}{\partial w} = \frac{2\gamma w (A u^2 + B u v + C w^2) - 2C w (\alpha u^2 + \beta vu + \gamma w^2)}{(A u^2 + B u v + C w^2)^2}.
\]

Then we see that
\[
\frac{\partial f(\Pi, \Pi, \Pi)}{\partial u} = \frac{(\alpha B + 2\alpha C + \beta C) - (A \beta + 2A \gamma + \gamma B)}{(A + B + C)(\alpha + \beta + \gamma)} = -p_2,
\]
\[
\frac{\partial f(\Pi, \Pi, \Pi)}{\partial v} = \frac{(A \beta + \beta C) - (\alpha B + \gamma B)}{(A + B + C)(\alpha + \beta + \gamma)} = -p_1,
\]
and
\[
\frac{\partial f(\Pi, \Pi, \Pi)}{\partial w} = \frac{2\gamma (A + B) - 2C (\alpha + \beta)}{(A + B + C)(\alpha + \beta + \gamma)} = -p_0.
Then the linearized equation of Eq.(1) about the positive equilibrium point $\pi$ is
\[ y_{n+1} + p_2 y_n + p_1 y_{n-1} + p_0 y_{n-2} = 0. \] (13)
The characteristic equation of the linearized equation is given by
\[ \lambda^3 + p_2 \lambda^2 + p_1 \lambda + p_0 = 0. \] (14)

**Theorem 1** Assume that
\[
\begin{cases}
(A + B + C) (\alpha + \beta + \gamma) > \max \left\{ \frac{4(\alpha C - A \gamma) + (\beta C - B \gamma)}{2(\alpha B - A \beta) + (B \gamma - \beta C)}, \right. \\
\left. 2(\alpha B - A \beta) + 2(\alpha C - A \gamma) + (\beta C - B \gamma) \right\}.
\end{cases}
\] (15)
Then the positive equilibrium point of Eq.(1) is locally asymptotically stable.

**proof.** It is follows by Theorem A that Eq.(13) is locally asymptotically stable if all roots of the Eq.(14) lie in the open disc $|\lambda| < 1$ that is if
\[ |p_2| + |p_1| + |p_0| < 1. \] (16)

We consider the following different possibilities:
(1) If $p_2 > 0$, $p_1 > 0$ and $p_0 > 0$. In this case we see from (16) that
\[
(A + B + C) (\alpha + \beta + \gamma) > 0,
\]
which is always true.
(2) If $p_2 > 0$, $p_1 > 0$ and $p_0 < 0$. It follows from (16) that
\[
(A + B + C) (\alpha + \beta + \gamma) < 0,
\]
which is satisfied by (15).

The proofs of the following cases are similar to the proof of cases 1 and 2 and will be left to the reader.

3. **Boundedness of the Solutions of Eq.(1)**

In the section we study the boundedness of the solutions of Eq.(1).

**Lemma 1** Every positive solution of Eq.(1) is bounded and persists.
proof. Let \( \{x_n\}_{n=-2}^\infty \) be a positive solution of Eq.\( (1) \). Then it follows that
\[
x_{n+1} = \frac{\alpha x_n^2 + \beta x_{n-1} x_n + \gamma x_{n-2}^2}{Ax_n^2 + Bx_{n-1} x_n + Cx_{n-2}^2} \leq \frac{\alpha x_n^2}{Ax_n^2 + Bx_{n-1} x_n + Cx_{n-2}^2} + \frac{\beta x_{n-1} x_n}{Ax_n^2 + Bx_{n-1} x_n + Cx_{n-2}^2}
\]
\[
+ \frac{\gamma x_{n-2}^2}{Ax_n^2 + Bx_{n-1} x_n + Cx_{n-2}^2}
\]
\[
\leq \frac{\alpha}{A} + \frac{\beta}{B} + \frac{\gamma}{C} := M.
\]
Then
\[
x_n \leq M \text{ for all } n \geq 1.
\]
By the change of variables \( x_n = \frac{1}{z_n} \) for all \( n \geq 1 \), Eq.\( (1) \) can be rewritten in the form
\[
z_{n+1} = \frac{A z_{n-1}^2 + B z_{n-2}^2 + C z_{n-1}^2 z_{n-1}}{\alpha z_{n-1}^2 + \beta z_{n-2}^2 + \gamma z_{n-1}^2 z_{n-1}} \leq \frac{B z_{n-2}^2}{\alpha z_{n-1}^2 + \beta z_{n-2}^2 + \gamma z_{n-1}^2 z_{n-1}}
\]
\[
+ \frac{C z_{n-1}^2}{\alpha z_{n-1}^2 + \beta z_{n-2}^2 + \gamma z_{n-1}^2 z_{n-1}}
\]
\[
\leq \frac{A}{\alpha} + \frac{B}{\beta} + \frac{C}{\gamma} := \bar{m}.
\]
That is
\[
x_n \geq \frac{1}{\bar{m}} = m \text{ for all } n \geq 1,
\]
and this completes the proof.

Theorem 2 Every solution of Eq.\( (1) \) is bounded and persists.

proof. Let \( \{x_n\}_{n=-2}^\infty \) be solution of Eq.\( (1) \). Then
\[
x_{n+1} = \frac{\alpha x_n^2 + \beta x_{n-1} x_n + \gamma x_{n-2}^2}{Ax_n^2 + Bx_{n-1} x_n + Cx_{n-2}^2} \leq \frac{\max \{\alpha, \beta, \gamma\} \max \{x_n^2 + x_{n-1} x_n + x_{n-2}^2\}}{\min \{A, B, C\} \max \{x_n^2 + x_{n-1} x_n + x_{n-2}^2\}} \leq \frac{\max \{\alpha, \beta, \gamma\}}{\min \{A, B, C\}}.
\]
Similarly it is easy to see that
\[
x_n \geq \frac{\min \{\alpha, \beta, \gamma\}}{\max \{A, B, C\}}.
\]
Thus we get
\[
0 < m := \frac{\min \{\alpha, \beta, \gamma\}}{\max \{A, B, C\}} \leq x_n \leq \frac{\max \{\alpha, \beta, \gamma\}}{\min \{A, B, C\}} := M < \infty.
\]
Therefore every solution of Eq.\( (1) \) is bounded and persists. Hence the result holds.
4. Global attractor of the equilibrium points of Eq. (1)

In this section we study the global attractor of the equilibrium points of Eq. (1).

**Lemma 2** For any partial order of the quotients \( \frac{A}{B} \), \( \frac{B}{C} \) and \( \frac{C}{A} \), the function \( f(u, v, w) \) defined by the equation (12) has the monotonicity behavior in at least one or two of its arguments.

**proof.** The proof follows by some direct substitutions and will be omitted.

**Remark 1** It follows from Eq. (1) when \( A = B = C \), that
\[
 x_n = \sigma, \quad n \geq -2 \quad \text{for some constant } \sigma.
\]

Whenever the quotients \( \frac{A}{B} \), \( \frac{B}{C} \) and \( \frac{C}{A} \) are not equal we get the following result.

**Theorem 3** The equilibrium point \( \bar{x} \) is global attractor of Eq. (1) if one of the following statements holds
1. \( \alpha B \geq A \beta, \beta C \geq \gamma B \) and
\[
(AB)^2 \left( \gamma^2 + \gamma CC^2 \bar{x} + AC \gamma C^2 \bar{x} + B \gamma C \right) \\
\geq \left( C^2 \bar{x} (\alpha B + A \beta) \right) \frac{AB (\alpha + \beta + C \bar{x}) + C (\alpha B + A \beta)}{AB} \cdot (17)
\]
2. \( \alpha B \geq A \beta, \alpha C \leq A \gamma \) and
\[
\frac{C \gamma^2 + C \bar{x} (A + C) (\gamma + \bar{x} (A + C))}{(A + C)\gamma^2} \\
\geq \frac{(B \alpha + A \beta) [(A + B) \bar{x} + (\alpha + \gamma)] + AB (\alpha + \gamma) \bar{x}}{AB} \cdot (18)
\]
3. \( \alpha B \geq A \beta, \beta C \leq \gamma B \) and
\[
\frac{\gamma^2 + C^2 \bar{x}^2 + C \gamma \bar{x}}{C} \\
\geq \frac{(B \alpha + A \beta) [(A + B) \bar{x} + (\alpha + \gamma)] + AB (\alpha + \gamma) \bar{x}}{AB} \cdot (19)
\]
4. \( \alpha B \leq A \beta, \alpha C \geq A \gamma \) and
\[
\frac{\gamma \beta^2 + \gamma \bar{x} (A + B) [\bar{x} (A + B) + \beta]}{(A + B)\gamma^2} \\
\geq \frac{\bar{x} (\alpha C + A \gamma)}{(AC)\gamma^2} [(AC) (\alpha + \beta + (A + C) \bar{x}) + (A + C) (C \alpha + A \gamma)]. \cdot (20)
\]
5. \( \alpha B \leq A \beta, \beta C \geq \gamma B \) and
\[
\frac{\gamma \beta^2 + \gamma \bar{x} (A + B) (\beta + \bar{x} (A + B))}{(A + B)\beta^2} \geq \frac{\gamma \bar{x}}{C^2} [C (\alpha + \beta) + (A + C) (C \bar{x} + \gamma)]. \cdot (21)
\]
6. \( \alpha B \leq A \beta, \beta C \leq \gamma B \) and
\[
\frac{\gamma \beta^2 + \gamma \bar{x} (A + B) + (\alpha - \gamma) (\gamma + C \bar{x})}{(A + B)\beta^2} \geq C \bar{x} (A + B) (\alpha B + A \beta) . \cdot (22)
\]

**proof.** Let \( \{x_n\}_{n=-2}^{\infty} \) be a solution of Eq. (1) and let \( f \) be a function defined by Eq. (12). We will prove the theorem in the first case and the proofs of the other cases are similar and will be omitted.
Assume that (17) is true then it is easy to see that the function \( f(u,v,w) \) is non-decreasing in its first argument and non-increasing in its third argument. Thus from Eq. (1) we see that

\[
x_{n+1} = \frac{\alpha x_n^2 + \beta x_{n-1}x_n + \gamma x_{n-2}}{Ax_n + Bx_{n-1} + Cx_{n-2}} \leq \frac{\alpha x_n^2 + \beta x_{n-1}x_n + \gamma (0)}{Ax_n + Bx_{n-1} + C(0)}
\]

Then

\[
x_n \leq \frac{\alpha}{A} + \frac{\beta}{B} = H \quad \text{for all} \quad n \geq 1.
\]

(23)

Then

\[
x_{n+1} = \frac{\alpha x_n^2 + \beta x_{n-1}x_n + \gamma x_{n-2}}{Ax_n + Bx_{n-1} + Cx_{n-2}} \geq \frac{\alpha (0)^2 + \beta (0) (H) + \gamma (H)^2}{A(0)^2 + B(0) (H) + C(0)^2}
\]

Then

\[
x_n \geq \frac{\gamma}{C} = h \quad \text{for all} \quad n \geq 1.
\]

(24)

From Eq. (23) and (24) we see that

\[
h = \frac{\gamma}{C} \leq x_n \leq \frac{\alpha}{A} + \frac{\beta}{B} = H \quad \text{for all} \quad n \geq 1.
\]

It follows by the Method of Full Limiting Sequences that there exist solutions \( \{I_n\}_{n=-\infty}^{\infty} \) and \( \{S_n\}_{n=-\infty}^{\infty} \) of Eq. (1) with

\[
\frac{\gamma}{C} \leq I = I_0 = \lim_{n \to -\infty} \inf x_n \leq \lim_{n \to -\infty} \sup x_n = S_0 = S \leq \frac{\alpha}{A} + \frac{\beta}{B},
\]

where

\[
I_n, S_n \in [I,S], \quad n = 0, -1, ...
\]

It suffices to show that \( I = S \). Now it follows from Eq. (1) that

\[
I = \frac{\alpha I_{-1}^2 + \beta I_{-2} + \gamma I_{-3}}{AI_{-1}^2 + BI_{-2} + CI_{-3}} \geq f(I, I_{-2}, S)
\]

\[
= \frac{\alpha I^2 + \beta I_{-2} + \gamma S}{AI^2 + BI_{-2} + CS^2} \geq \frac{(\alpha + \beta) I^2 + \gamma S^2}{AI^2 + BSI + CS^2},
\]

and so

\[
AI^3 + BSI^2 + CS^2 I \geq (\alpha + \beta) I^2 + \gamma S^2.
\]

Then we obtain

\[
AI^3 S + BSI^2 + CS^2 I \geq (\alpha + \beta) I^2 S + \gamma S^3,
\]

or

\[
(\alpha + \beta) I^2 S + \gamma S^3 - AI^3 S - CS^2 I \leq BS^2 I^2.
\]

(25)

Similarly it is easy to see from Eq. (1) that

\[
(\alpha + \beta) IS^2 + \gamma I^3 - AIS^3 - CSI^3 \geq BS^2 I^2.
\]

(26)

Therefore it follows from Eqs. (25) and (26) that

\[
((\alpha + \beta) IS^2 - (\alpha + \beta) SI^2) + (\gamma I^3 - \gamma S^3) + (AI^3 S - AIS^3) + (CS^3 I - CSI^3) \geq 0
\]
\[ (\alpha + \beta) IS (S - I) + (A - C) IS (I^2 - S^2) + (\gamma I^3 - \gamma S^3) \geq 0 \]
\[ (\alpha + \beta) IS (S - I) + (A - C) IS (I^2 - S^2) + (\gamma I^3 - \gamma I^2 S) + (\gamma I^2 S - \gamma S^3) \geq 0, \]
or
\[ (I - S) [\gamma I^2 - (\alpha + \beta) IS] + (\gamma S + (A - C) SI) (I^2 - S^2) \geq 0, \]
or equivalently
\[ (I - S) [\gamma I^2 - (\alpha + \beta) IS + (\gamma S + (A - C) SI) (I + S)] \geq 0, \]
and so \( I \geq S \) if
\[ \gamma I^2 + (\gamma S + ASI) (I + S) \geq (\alpha + \beta) IS + CIS (I + S). \] (27)

Now it follows from (17) that
\[ \frac{\gamma^3 + \gamma C^2 x^2 + A \gamma^2 x + AC \gamma x^2 + C \gamma^2 x}{C^2} \]
\[ \geq \left( \frac{\pi (\alpha B + A \beta)}{(AB)^2} \right) \left[ AB (\alpha + \beta + C x) + C (\alpha B + A \beta) \right] \]
\[ \Leftrightarrow \frac{\gamma^3}{C^2} + \gamma x^2 + \frac{\gamma^2 x}{C} + \frac{A \gamma^2 x}{C} + \frac{A \gamma^2 x}{C^2} \]
\[ \geq \left( \frac{\pi (\alpha B + A \beta)}{AB} \right) \left( \frac{AB (\alpha + \beta + C A B x + C (\alpha B + A \beta))}{AB} \right) \]
\[ \Leftrightarrow \gamma \left( \frac{\gamma}{C} \right)^2 + \left( \frac{A \gamma x}{C} + \gamma x \right) \left( \frac{\gamma}{C} + x \right) \]
\[ \geq \left( \frac{(\alpha B + A \beta) x}{AB} \right) \left( \alpha + \beta + C \left( x + \frac{(\alpha B + A \beta)}{AB} \right) \right). \]
Now
\[ \gamma \left( \frac{\gamma}{C} \right)^2 + \left( \frac{A \gamma x}{C} + \gamma x \right) \left( \frac{\gamma}{C} + x \right) \leq \gamma I^2 + (AIS + \gamma S) (I + S), \] (28)
and
\[ \left( \frac{(\alpha B + A \beta) x}{AB} \right) \left( \alpha + \beta + C \left( x + \frac{(\alpha B + A \beta)}{AB} \right) \right) \geq (\alpha + \beta) IS + CIS (I + S). \] (29)

Then we see from (28) and (29) that
\[ \gamma I^2 + (\gamma S + ASI) (I + S) \geq (\alpha + \beta) IS + CIS (I + S), \]
which yields that (27) is satisfied and then the proof is complete.

5. Rate of Convergence of Eq. (1)

In this section we will recognize the rate of convergence of the solutions that converge to the unique positive equilibrium point of Eq. (1).

**Theorem 4** Assume that (15) holds. Then all solutions of Eq. (1) which are eventually approach to the equilibrium satisfy the following.
\[ \lim_{n \to \infty} \frac{x_{n+1} - \bar{x}}{x_n - \bar{x}} = \lambda_i, \]
where \( \lambda_i \) are the roots of Eq. (14).
proof. We have

\[
x_{n+1} - \overline{x} = \frac{A\beta x_n x_{n-1} - A\beta x_n^2 + A\gamma x_{n-2}^2 - A\gamma x_n x_{n-2}}{(Ax_n^2 + Bx_{n-1}x_n + Cx_{n-2}^2)(A + B + C)} + \frac{B\gamma x_{n-1} - B\gamma x_n x_{n-1} + C\alpha x_n^2 - C\alpha x_{n-2}^2 + C\beta x_n x_{n-1} - C\beta x_{n-2}}{(Ax_n^2 + Bx_{n-1}x_n + Cx_{n-2}^2)(A + B + C)}
\]

\[
= \frac{(B\alpha - A\beta) x_n^2 - (B\alpha - A\beta) x_n \overline{x} + (B\alpha - A\beta) x_n \overline{\overline{x}}}{(Ax_n^2 + Bx_{n-1}x_n + Cx_{n-2}^2)(A + B + C)} + \frac{(C\alpha - A\gamma) x_n^2 - (C\alpha - A\gamma) x_n \overline{x} + (C\alpha - A\gamma) x_n \overline{\overline{x}}}{(Ax_n^2 + Bx_{n-1}x_n + Cx_{n-2}^2)(A + B + C)} + \frac{(A\gamma - C\alpha) x_{n-2}^2 - (A\gamma - C\alpha) x_n x_{n-2} - (A\gamma - C\alpha) x_{n-2} \overline{x}}{(Ax_n^2 + Bx_{n-1}x_n + Cx_{n-2}^2)(A + B + C)} + \frac{(A\gamma - C\alpha) x_{n-2} \overline{x} + (C\alpha - A\gamma) x_n x_{n-2} + (C\alpha - A\gamma) x_n \overline{\overline{x}}}{(Ax_n^2 + Bx_{n-1}x_n + Cx_{n-2}^2)(A + B + C)} + \frac{(B\gamma - C\beta) x_n x_{n-1} - (B\gamma - C\beta) x_n \overline{x} - (B\gamma - C\beta) x_n \overline{\overline{x}} + (B\gamma - C\beta) x_n \overline{\overline{\overline{x}}}}{(Ax_n^2 + Bx_{n-1}x_n + Cx_{n-2}^2)(A + B + C)} + \frac{(A\beta - B\alpha) x_n x_{n-1} - (A\beta - B\alpha) x_n \overline{x} - (A\beta - B\alpha) x_n \overline{\overline{x}} + (A\beta - B\alpha) x_n \overline{\overline{\overline{x}}}}{(Ax_n^2 + Bx_{n-1}x_n + Cx_{n-2}^2)(A + B + C)} + \frac{(A\beta - B\alpha) x_n \overline{x} - (A\beta - B\alpha) x_n \overline{\overline{x}}}{(Ax_n^2 + Bx_{n-1}x_n + Cx_{n-2}^2)(A + B + C)} + \frac{(A\beta - B\alpha) x_n \overline{\overline{x}} - (A\beta - B\alpha) x_n \overline{\overline{\overline{x}}}}{(Ax_n^2 + Bx_{n-1}x_n + Cx_{n-2}^2)(A + B + C)}.
\]

Put

\[
e_n = x_n - \overline{x}, \quad e_{n-1} = x_{n-1} - \overline{x}, \quad e_{n-2} = x_{n-2} - \overline{x}.
\]

Then we obtain

\[
e_{n+1} - \gamma_n e_n - \delta_n e_{n-1} - \zeta_n e_{n-2} = 0,
\]

where

\[
\gamma_n = \frac{((B\alpha - A\beta) + (C\alpha - A\gamma)) x_n + ((C\beta - B\gamma)) x_{n-2}}{(Ax_n^2 + Bx_{n-1}x_n + Cx_{n-2}^2)(A + B + C)},
\]

\[
\delta_n = \frac{((C\beta - B\gamma) + (A\beta - B\alpha)) x_n}{(Ax_n^2 + Bx_{n-1}x_n + Cx_{n-2}^2)(A + B + C)},
\]

\[
\zeta_n = \frac{((C\beta - B\gamma)) x_{n-2}}{(Ax_n^2 + Bx_{n-1}x_n + Cx_{n-2}^2)(A + B + C)}.
\]
and
\[ \zeta_n = \frac{((A \gamma - C \alpha) + (B \gamma - C \beta)) x_{n-2} + ((B \gamma - C \beta) + (A \gamma - C \alpha)) x_n}{(Ax_n^2 + Bx_{n-1} + Cx_{n-2})(A + B + C)}. \]

As the positive equilibrium is a global attractor, we get
\[ \lim_{n \to \infty} \gamma_n = \frac{(B\alpha - A\beta) + 2(C\alpha - A\gamma) + (C\beta - B\gamma)}{\bar{x}(A + B + C)^2} = \frac{(\alpha \beta + \gamma)(A + B + C)}{(A + B + C) (\alpha + \beta + \gamma)}, \]
\[ \lim_{n \to \infty} \delta_n = \frac{(C\beta - B\gamma) + (A\beta - B\alpha)}{\bar{x}(A + B + C)^2} = \frac{(A\beta + C\beta) - (\alpha \beta + \gamma B)}{(A + B + C)(\alpha + \beta + \gamma)}, \]
and
\[ \lim_{n \to \infty} \zeta_n = \frac{2(A \gamma - C \alpha) + 2(B \gamma - C \beta)}{\bar{x}(A + B + C)^2} = \frac{2\gamma (A + B) - 2C (\alpha + \beta)}{(A + B + C)(\alpha + \beta + \gamma)}. \]

Thus the limiting equation of Eq. (1) is the linearized equation (13). Then the result follows as an immediate consequence of Poincare's theorem C and the proof is complete.

6. Periodic solutions

In this section we present some results for the existence of minimal period-two solutions of Equation (1).

**Theorem 5** Eq. (1) has a positive periodic solution of prime period two if and only if
\[ (A + C - B)(\beta - (\alpha + \gamma)) > 4B (\alpha + \gamma). \] (30)

**Proof.** First assume that there exists a periodic solution of prime period two \{\ldots, p, q, p, q, \ldots\} of Eq. (1). From (30) we get
\[ p = \frac{(\alpha + \gamma)q^2 + \beta pq}{(A + C)q^2 + Bpq}, \]
and
\[ q = \frac{(\alpha + \gamma)p^2 + \beta pq}{(A + C)p^2 + Bpq}. \]

Then
\[ (A + C) pq + Bp^2 = (\alpha + \gamma) q + \beta p, \] (31)
and
\[ (A + C) pq + Bq^2 = (\alpha + \gamma) p + \beta q. \] (32)

Subtracting (32) from (31)
\[ B(p^2 - q^2) = (\beta - (\alpha + \gamma))(p-q). \]

Since \( p \neq q \), it follows that
\[ p + q = \frac{\beta - (\alpha + \gamma)}{B}. \] (33)
Substituting (33) in (31) we obtain
\[(B - (C + A))p^2 + \frac{(B(C + A) - B)(\beta - (\alpha + \gamma))}{B} p - \frac{(\beta - (\alpha + \gamma))(\alpha + \gamma)}{B} = 0,\]
from which
\[p_\pm = \frac{(\beta - (\alpha + \gamma))}{2B} \left(1 \pm \sqrt{1 - 4B(\alpha + \gamma)/(A + C - B)(\beta - (\alpha + \gamma))}\right)\]
and so
\[1 - 4B(\alpha + \gamma)/(A + C - B)(\beta - (\alpha + \gamma)) > 0,\]
or
\[4B(\alpha + \gamma) < (A + C - B)(\beta - (\alpha + \gamma)).\]
Since \(A + C - B\) and \(\beta - \alpha - \gamma\), have the same sign then (30) holds.

Second suppose that the condition (30) is true. We will show that Eq.(1) has positive prime period two solutions. Assume that \(p\) and \(q\) are distinct positive real numbers.

Now choose \(x_{-2} = p, x_{-1} = q\) and \(x_0 = p\). It is easy to prove that
\[x_1 = x_{-1} \quad \text{and} \quad x_2 = x_0.\]
Then it follows by induction that
\[x_{2n} = p \quad \text{and} \quad x_{2n+1} = q \quad \text{for all} \quad n \geq -1.\]
Thus Eq.(1) has the positive prime period two solution
\[\ldots, p, q, p, q, \ldots\]
where \(p\) and \(q\) are the distinct roots of the quadratic equation (34) and the proof is completed.

**Theorem 6** Assume that \(\{x_n\}_{n=-\infty}^\infty\) be solution of Eq.(1) with \(x_{N+2} > x_N > x_{N-1} > x_{N+1} > x_{N+3}\) for some \(N \geq 0\). Then \(\{x_n\}_{n=-\infty}^\infty\) converges to a two cycle solution of Eq.(1). 

**proof.** We have
\[x_{N+4} = f(x_{N+3}, x_{N+2}, x_{N+1}) > f(x_{N+1}, x_N, x_{N-1}) = x_{N+2},\]
and
\[x_{N+5} = f(x_{N+4}, x_{N+3}, x_{N+2}) < f(x_{N+2}, x_{N+1}, x_N) = x_{N+3}.\]
By induction on \(N \geq 0\) one can easy establishes the monotonic of the two sequences \(\{x_{N+2k}\}_{n=-\infty}^\infty\), \(\{x_{N+2k}\}_{n=-\infty}^\infty\) and since \(\{x_n\}_{n=-\infty}^\infty\) is bounded, the sequence \(\{x_{N+2k}\}\) is increasing and bounded above by \(M\) and the sequence \(\{x_{N+2k}\}\) is decreasing and bounded below by \(m\), i.e
\[x_{N+2k} \leq M \quad \text{and} \quad x_{N+2k+1} \geq m \quad \text{for} \quad k = 0, 1, \ldots.\]
Therefore, both subsequences \(\{x_{N+2k}\}_{n=-\infty}^\infty, \{x_{N+2k}\}_{n=-\infty}^\infty\) converges to, say \(\psi\) and \(\varphi\) respectively, that is
\[\lim_{k \to \infty} x_{N+2k} = \psi \quad \text{and} \quad \lim_{k \to \infty} x_{N+2k+1} = \varphi.\]
This completes the proof.
7. Numerical Examples

Example 1 We assume $x_{-2} = 4, x_{-1} = 5, x_0 = 0.5, \alpha = 2, \beta = 9, \gamma = 6,$ $A = 12, B = 7, C = 9.$ (See Figure 1).

Example 2 (See Figure 2) since $x_{-2} = 0.5, x_{-1} = 0.6, x_0 = 0.6, \alpha = 0.9,$ $\beta = 3.9, \gamma = 3, A = 8, B = 1.5, C = 2.$

Example 3 (See Figure 3) since $x_{-2} = 5, x_{-1} = 1, x_0 = 5, \alpha = 0.02, \beta = 5,$ $\gamma = 0.9, A = 1.2, B = 2, C = 0.8.$

Example 4 We consider $x_{-2} = 9, x_{-1} = 5, x_0 = 9, \alpha = 0.05, \beta = 15,$ $\gamma = 0.3, A = 2, B = 0.5, C = 2.$ (See Figure 4).

Example 5 (See Figure 5) since $x_{-2} = 3, x_{-1} = 2, x_0 = 1, \alpha = 5, \beta = 0.8,$ $\gamma = 3, A = 2, B = 0.4, C = 0.02.$

Example 6 (See Figure 6) since $x_{-2} = 15, x_{-1} = 5, x_0 = 1, \alpha = 13, \beta = 15,$ $\gamma = 0.2, A = 2, B = 0.5, C = 2.$
The given sequence is defined by the recursive relation:

\[ x_{n+1} = \frac{(a \cdot x_n^2 + b \cdot x_{n-1} \cdot x_n + c \cdot x_{n-2}^2)}{(A \cdot x_n^2 + B \cdot x_{n-1} \cdot x_n + C \cdot x_{n-2}^2)} \]

The figures (3), (4), and (5) illustrate the behavior of this sequence for different initial conditions and parameter values. Each graph shows the evolution of the sequence over a range of iterations, with the equilibrium point indicated for reference.
x(n+1) = (a * x(n)^2 + b * x(n-1) * x(n) + c * x(n-2)^2) / (A * x(n)^2 + B * x(n-1) * x(n) + C * x(n-2)^2)

Equilibrium point

Figure (6)

REFERENCES


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