THE MODIFIED FRACTIONAL POWER SERIES FOR SOLVING A CLASS OF FRACTIONAL STURM-LIOUVILLE EIGENVALUE PROBLEMS

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Abstract. This article is devoted to both theoretical and numerical studies of eigenvalues of regular fractional 2-order Sturm-Liouville problem where $\frac{1}{2} < \alpha \leq 1$. In this paper, we implement the modified fractional power series (MFPS) method to approximate the eigenvalues. To find the eigenvalues, we force the approximate solution produced by the MFPS method satisfies the boundary condition at $x = 1$. The fractional derivative is described in the Caputo sense. Numerical results demonstrate the accuracy of the present algorithm. In addition, we prove the existence of the eigenfunctions of the proposed problem. The convergence of the approximate eigenfunctions produced by the MFPS to the exact eigenfunctions is proven.

1. Introduction

Fractional differential equations (FDEs) appear as generalizations to existing models with integer derivative and they also present new models for some physical problems [1]. In recent years, great interests were devoted to the analytical and numerical treatments of fractional differential equations. In general, fractional differential equations don’t have exact solutions in closed form, and therefore, numerical methods such as, the variational iteration [2], the homotopy analysis method [3], and the Adomian decomposition method [4, 5, 6], have been implemented for several types of fractional differential equations. Also, the maximum principle and the method of lower and upper solutions have been extended to deal with FDEs and obtain analytical and numerical results [7]. The Tau method, the Pseudo-spectral method, and the wavelet method based on the legendre polynomials have been implemented for several types of FDEs [8].

The Sturm-Liouville eigenvalue problem has played an important role in modeling many physical problems [9]. The theory of the problem is well developed and many results have been obtained concerning the eigenvalues and corresponding eigenfunctions. Since finding analytical solutions for this problem is a difficult...
task, many numerical algorithms have been investigated to find approximate solutions.

The fractional Sturm-Liouville eigenvalue problem was studied earlier [10] and [11]. In [10], the existence of a solution to such boundary value problem was established. In [11], the aforementioned relation between eigenvalues and zeros of Mittag-Leffler function was shown. The Adomian decomposition method was established for estimating fractional second order eigenvalues [12, 13]. The Homotopy Analysis method has been used to numerically approximate the eigenvalues of the fractional Sturm-Liouville Problems [14]. In [15], fractional differential transform method was used to approximate the eigenvalues of Sturm–Liouville problems of fractional order. Fourier series was used in [16], the method of Haar wavelet operational matrix was used in [17] and [18]. In [19]-[22], extended some spectral properties of fractional Sturm-Liouville problem. Variational Methods and Inverse Laplace transform method applied in [23] and [24], respectively. Recently P. Antunes and R. Ferreira constructed numerical schemes using radial basis functions [25], B. Jin et al used Galerkin finite element method to solve fractional eigenvalue problems [26].

In this paper, we develop a numerical technique for approximating the eigenvalues of the following regular fractional Sturm-Liouville problem of the form

\[ D^\alpha[P(x)D^\alpha u(x)] + \lambda Q(x)u(x) = \mu(x)u(x), \quad 0 \leq x \leq 1, \quad \frac{1}{2} < \alpha \leq 1 \]  

subject to

\[ a_0 u(0) + a_1 D^\alpha u(0) = 0, \quad a_0^2 + a_1^2 > 0, \]  
\[ a_2 u(1) + a_3 D^\alpha u(1) = 0, \quad a_2^2 + a_3^2 > 0, \]

where \( a_0, a_1, a_2, a_3 \) are constants, \( p(x), q(x), r(x) \) are continuous functions with \( p(x), q(x) > 0 \) for all \( x \in [0, 1] \), and \( D^\alpha \) is the Caputo fractional derivative.

This paper is organized as follows. In section 2, we present some preliminaries which we will use in this paper. A description of the modified fractional power series (MFPS) method for discretization of the fractional 2\( \alpha \)-order Sturm-Liouville problem (1.1)-(1.2) is presented in section 3. Convergence analysis is presented in Section 4. Several numerical examples and conclusions are discussed in Section 5. Conclusions and closing remarks are given in Section 6.

2. Preliminaries

In this section, we review some preliminaries definitions and theorems which we use in this paper. First, we write the definition and some preliminary results of the Caputo fractional derivatives, as well as, the definition of the Riemann-Liouville fractional and their properties.

**Definition 1** A real function \( f(t), t > 0 \), is said to be in the space \( C_\mu, \mu \in \mathbb{R} \) if there exists a real number \( \rho > \mu \), such that \( f(t) = t^\rho f_1(t) \), where \( f_1(t) \in C[0, \infty) \), and it is said to be in the space \( C_\mu^m \) if \( f^{(m)} \in C_\mu, \quad m \in \mathbb{N} \).
Definition 2 The left Riemann-Liouville fractional integral of order $\delta \geq 0$, of a function $f \in C_{\mu}, \mu \geq -1$, is defined by
\[
I^\delta f(t) = \begin{cases} 
\frac{1}{\Gamma(\delta)} \int_0^t (t - s)^{\delta - 1} f(s)ds, & \delta > 0, \\
\frac{f(t)}{\Gamma(\delta + 1)}, & \delta = 0.
\end{cases}
\] (4)

Definition 3 For $\delta > 0$, $m - 1 < \delta < m$, $m \in \mathbb{N}, t > 0$, and $f \in C^m$, the left Caputo fractional derivative is defined by
\[
D^\delta f(t) = \begin{cases} 
\frac{1}{\Gamma(m - \delta)} \int_0^t (t - s)^{m-1-\delta} f^{(m)}(s)ds, & \delta > 0, \\
\frac{f^{(m)}(t)}{\Gamma(m + 1)}, & \delta = 0,
\end{cases}
\] (5)
where $\Gamma$ is the well-known Gamma function.

The Caputo derivative defined in (5) is related to the Riemann-Liouville fractional integral, $I^\delta$, of order $\delta \in \mathbb{R}^+$, by $D^\delta f(t) = I^{m-\delta} f^{(m)}(t)$. The Caputo fractional derivative satisfy the following properties for $f \in C_{\mu}, \mu \geq -1$ and $\alpha \geq 0$, see [27].

1. $D^\alpha I^\alpha f(t) = f(t)$,
2. $I^\alpha D^\alpha f(t) = f(t) - \sum_{k=0}^{\alpha-1} f^{(k)}(0) \frac{t^k}{k!}$,
3. $D^\alpha c = 0$, where $c$ is constant,
4. $D^\alpha t^\gamma = \begin{cases} 
0, & \gamma \in \mathbb{R}^- \\
\frac{\Gamma(\gamma + 1)}{\Gamma(\gamma - \alpha + 1)} t^{\gamma - \alpha}, & \gamma < \alpha, \gamma \in \{0, 1, 2, \ldots\}, \\
\text{otherwise}
\end{cases}$,
5. $D^\alpha (\sum_{k=0}^m c_k f_k(t)) = \sum_{k=0}^m c_k D^\alpha f_k(t)$, where $c_1, c_2, \ldots, c_m$ are constants.

Second, we write the definition and one of the properties of the fractional power series which are used in this paper. More details can be found in [28]-[37].

Definition 4 A power series expansion of the form
\[
\sum_{m=0}^{\infty} c_m (x - x_0)^{m\alpha} = c_0 + c_1 (x - x_0)^\alpha + c_2 (x - x_0)^{2\alpha} + ...
\]
where $0 \leq m - 1 < \alpha \leq m$, is called fractional power series FPS about $x = x_0$.

Suppose that $f$ has a fractional FPS representation at $x = x_0$ of the form
\[
g(x) = \sum_{m=0}^{\infty} c_m (x - x_0)^{m\alpha}, \quad x_0 \leq x < x_0 + \beta.
\]
If $D^{m\alpha} g(x), \ m = 0, 1, 2, \ldots$ are continuous on $\mathbb{R}$, then $c_m = \frac{D^{m\alpha} g(x_0)}{\Gamma(1 + m\alpha)}$.

3. THE MFPS METHOD FOR A CLASS OF FRACTIONAL SECOND-ORDER STURM-LIOUVILLE PROBLEMS

In this section, we discuss the numerical solution of the following class of fractional 2\alpha-order Sturm-Liouville Problems using MFPS:
\[
D^\alpha [P(x)D^\alpha u(x)] + \lambda Q(x)u(x) = \mu(x)u(x), \quad 0 < x < 1, \quad \frac{1}{2} < \alpha \leq 1
\] (6)
subject to
\[ a_0 u(0) + a_1 D^\alpha u(0) = 0, \quad a_0^2 + a_1^2 > 0, \]  
\[ a_2 u(1) + a_3 D^\alpha u(1) = 0, \quad a_2^2 + a_3^2 > 0, \]  
where \( a_0, a_1, a_2, a_3 \) are constants, \( P(x), Q(x) \), and \( \mu(x) \) are continuous with \( P(x), \mu(x) > 0 \) for all \( x \in [0,1] \). Assume that \( u(0) = \theta \) and \( D^\alpha u(0) = \eta \). We find the values of \( \theta \) and \( \eta \) from the boundary conditions (3.2)-(3.3) later. Using the MRPS method, the solution problem (3.1)-(3.3) can be written in the fractional power series form as
\[ u(x) = \sum_{n=0}^{\infty} f_n \frac{x^{n\alpha}}{\Gamma(1+n\alpha)}. \]  
To obtain the approximate values of \( f_n \) in Eq. (9), we write the \( k \)-th truncated series \( u_k(x) \) in the form
\[ u_k(x) = \sum_{n=0}^{k} f_n \frac{x^{n\alpha}}{\Gamma(1+n\alpha)}. \]  
Since \( u(0) = f_0 = \theta \) and \( D^\alpha u(0) = f_1 = \eta \), we rewrite (10) as
\[ u_k(x) = \theta + \eta \frac{x^\alpha}{\Gamma(1+\alpha)} + \sum_{n=2}^{k} f_n \frac{x^{n\alpha}}{\Gamma(1+n\alpha)}, \quad k = 2, 3, \ldots \]
where \( u_1(x) = \theta + \eta \frac{x^\alpha}{\Gamma(1+\alpha)} \) is considered to be the 1-st MFPS approximate solution of \( u(x) \). Let \( P_k(x) = \sum_{n=0}^{k} p_n x^{n\alpha}, Q_k(x) = \sum_{n=0}^{k} q_n x^{n\alpha}, \) and \( \mu_k(x) = \sum_{n=0}^{k} m_n x^{n\alpha}. \) Then,
\[ P_k(x)D^\alpha u_k(x) = \left( \sum_{n=0}^{k} p_n x^{n\alpha} \right) \left( \sum_{n=0}^{k-1} f_{n+1} \frac{x^{n\alpha}}{\Gamma(1+n\alpha)} \right) \]
\[ = \sum_{n=0}^{k-1} \left( \sum_{i=0}^{n} \frac{f_{i+1}}{\Gamma(1+i\alpha)} p_{n-i} \right) x^{n\alpha} + h.o.t_1, \]
\[ Q_k(x)u_k(x) = \left( \sum_{n=0}^{k} q_n x^{n\alpha} \right) \left( \sum_{n=0}^{k} f_n \frac{x^{n\alpha}}{\Gamma(1+n\alpha)} \right) \]
\[ = \sum_{n=0}^{k} \left( \sum_{i=0}^{n} \frac{f_i}{\Gamma(1+i\alpha)} q_{n-i} \right) x^{n\alpha} + h.o.t_2, \]
and
\[ \mu_k(x)u_k(x) = \left( \sum_{n=0}^{k} m_n x^{n\alpha} \right) \left( \sum_{n=0}^{k} f_n \frac{x^{n\alpha}}{\Gamma(1+n\alpha)} \right) \]
\[ = \sum_{n=0}^{k} \left( \sum_{i=0}^{n} \frac{f_i}{\Gamma(1+i\alpha)} m_{n-i} \right) x^{n\alpha} + h.o.t_3, \]
where \( h.o.t_1 \) means a linear combination of \( \{x^k, x^{k+1}, \ldots, x^{2k-1}\} \) and \( h.o.t_2 \) and \( h.o.t_3 \) means linear combination of \( \{x^{k+1}, x^{k+2}, \ldots, x^{2k}\} \). Let \( \text{Res}_k(u(x)) \) be the \( k \)-th residual function which is defined by
\[ \text{Res}_k(u(x)) = D^\alpha[P_k(x)D^\alpha u_k(x)] + \lambda Q_k(x)u_k(x) - \mu_k(x)u_k(x). \]
For simplicity, we write the \( k \)-th residual function as

\[
\text{Res}_k(u(x)) = D^\alpha \left[ \sum_{n=0}^{k-1} \left( \sum_{i=0}^{n} \frac{f_{i+1}}{\Gamma(1 + i\alpha)} p_{n-i} \right) x^{n\alpha} \right] \\
+ \lambda \sum_{n=0}^{k} \left( \sum_{i=0}^{n} \frac{f_i}{\Gamma(1 + i\alpha)} q_{n-i} \right) x^{n\alpha} - \sum_{n=0}^{k} \left( \sum_{i=0}^{n} \frac{f_i}{\Gamma(1 + i\alpha)} m_{n-i} \right) x^{n\alpha}
\]

or

\[
\text{Res}_k(u(x)) = \sum_{n=0}^{k-1} \left( \sum_{i=0}^{n+1} \frac{f_{i+1}}{\Gamma(1 + i\alpha)} p_{n+i-1} \right) \frac{\Gamma(1 + (n + 1)\alpha)}{\Gamma(1 + n\alpha)} x^{n\alpha} \\
+ \lambda \sum_{n=0}^{k} \left( \sum_{i=0}^{n} \frac{f_i}{\Gamma(1 + i\alpha)} q_{n-i} \right) x^{n\alpha} - \sum_{n=0}^{k} \left( \sum_{i=0}^{n} \frac{f_i}{\Gamma(1 + i\alpha)} m_{n-i} \right) x^{n\alpha}.
\]

To find the values of the MFPS-coefficients \( f_j, \ j \in \{2, 3, \ldots, k\} \), we solve the fractional differential equation

\[
D^{(j-1)\alpha} \text{Res}_k(u(0)) = 0.
\]

For \( j \in \{0, 1, 2, \ldots, k-2\} \),

\[
0 = D^{(j-1)\alpha} \text{Res}_k(u(0)) = \left( \sum_{i=0}^{j+1} \frac{f_{i+1}}{\Gamma(1 + i\alpha)} p_{j+1-i} \right) \frac{\Gamma(1 + (j + 1)\alpha)}{\Gamma(1 + j\alpha)} \frac{\Gamma(1 + j\alpha)}{\Gamma(1 + (j - 1)\alpha)} \\
+ \lambda \left( \sum_{i=0}^{j} \frac{f_i}{\Gamma(1 + i\alpha)} q_{j-i} \right) \frac{\Gamma(1 + j\alpha)}{\Gamma(1 + (j - 1)\alpha)} = \left( \sum_{i=0}^{j} \frac{f_i}{\Gamma(1 + i\alpha)} m_{j-i} \right) \frac{\Gamma(1 + j\alpha)}{\Gamma(1 + (j - 1)\alpha)}
\]

or

\[
\frac{\Gamma(1 + (j + 1)\alpha)}{\Gamma(1 + j\alpha)} \sum_{i=0}^{j+1} \frac{f_{i+1}}{\Gamma(1 + i\alpha)} p_{j+1-i} + \lambda \sum_{i=0}^{j} \frac{f_i}{\Gamma(1 + i\alpha)} q_{j-i} - \sum_{i=0}^{j} \frac{f_i}{\Gamma(1 + i\alpha)} m_{j-i} = 0
\]

Thus,

\[
f_{j+2} = \frac{\Gamma(1 + j\alpha) \sum_{i=0}^{j} \frac{f_i}{\Gamma(1 + i\alpha)} m_{j-i} - \lambda \Gamma(1 + j\alpha) \sum_{i=0}^{j} \frac{f_i}{\Gamma(1 + i\alpha)} q_{j-i} - \Gamma(1 + (j + 1)\alpha) \sum_{i=0}^{j} \frac{f_{i+1}}{\Gamma(1 + i\alpha)} p_{j+1-i}}{p_0}
\]

Simple calculations imply that

\[
f_0 = \theta, \\
f_1 = \eta, \\
f_2 = \frac{m_0 - \lambda \eta_0}{p_0} \theta - \frac{\Gamma(1 + \alpha)}{p_0} \eta, \\
f_3 = \frac{\left( \frac{m_0 - \lambda \eta_1}{p_0} \Gamma(1 + \alpha) \frac{m_0 - \lambda \eta_0}{p_0} \theta - \frac{\Gamma(1 + 2\alpha)}{p_0} m_0 - \lambda \eta_0 + \Gamma(1 + \alpha) p_1 \eta \right)}{p_0} \frac{m_0 - \lambda \eta_2}{p_0} \Gamma(1 + \alpha) \frac{m_0 - \lambda \eta_1}{p_0} \theta - \frac{\Gamma(1 + 2\alpha)}{p_0} m_0 - \lambda \eta_0 + \Gamma(1 + \alpha) p_1 \eta}.
\]

...
Hence,

\[ u_k(x) = \theta + \eta \frac{x^{\alpha}}{\Gamma(1 + \alpha)} + \sum_{n=2}^{k} \left( \frac{\Gamma(1 + (n - 2)\alpha) \sum_{i=0}^{n-2} f_i^2 \Gamma(1 + i\alpha) m_{n-i}}{p_0} \right) + \sum_{n=2}^{k} \left( \frac{-\lambda \Gamma(1 + (n - 2)\alpha) \sum_{i=0}^{n-2} f_i^2 \Gamma(1 + i\alpha) q_{n-i}}{p_0} \right) \]

\[ x^{\alpha} \frac{\Gamma(1 + (n - 1)\alpha) \sum_{i=0}^{n-2} f_i^2 \Gamma(1 + i\alpha) p_{n-i}}{p_0} \]

\[ \Gamma(1 + n\alpha) \]  \hspace{1cm} (13)

for \( k = 1, 2, 3, \ldots \). Simple calculations imply that

\[ u_1(x) = \theta + \eta \frac{x^{\alpha}}{\Gamma(1 + \alpha)}, \]

\[ u_2(x) = \theta \left( 1 + \frac{m_0 - \lambda q_0}{p_0 \Gamma(1 + 2\alpha)} x^{2\alpha} \right) + \eta \left( \frac{x^{\alpha}}{\Gamma(1 + \alpha)} - \frac{\Gamma(1 + \alpha) p_1}{p_0 \Gamma(1 + 2\alpha)} x^{2\alpha} \right), \]

\[ u_3(x) = \theta \left( 1 + \frac{m_0 - \lambda q_0}{p_0 \Gamma(1 + 2\alpha)} x^{2\alpha} \right) + \eta \left( \frac{(m_0 - \lambda q_1) \Gamma(1 + \alpha)}{p_0 \Gamma(1 + 3\alpha)} x^{2\alpha} - \frac{\Gamma(1 + 2\alpha) \sum_{i=0}^{n-2} f_i^2 \Gamma(1 + i\alpha) m_{n-i}}{p_0 \Gamma(1 + 3\alpha)} \right) \]

\[ x^{\alpha} \frac{\Gamma(1 + \alpha) p_1}{p_0 \Gamma(1 + 2\alpha)} x^{2\alpha} - \frac{\Gamma(1 + 2\alpha) \sum_{i=0}^{n-2} f_i^2 \Gamma(1 + i\alpha) p_{n-i}}{p_0 \Gamma(1 + 3\alpha)} \]

\[ \Gamma(1 + n\alpha) \]

\[ \]  \hspace{1cm} (14)

It is easy to see that

\[ u_k(x) = \theta \ h_1(x, \lambda) + \eta \ h_2(x, \lambda). \]

Using the the first boundary conditions, we get

\[ 0 = a_0 u(0) + a_1 D^\alpha u(0) \]

\[ = a_0 \theta + a_1 \eta. \]

Since \( a_0^2 + a_1^2 > 0 \) and \( \theta a_0 + \eta a_1 = 0 \), we have the following two cases.

- If \( a_0 = 0, \ \eta = 0 \) and \( a_1 \neq 0 \). Thus,

\[ u_k(x) = \theta \ h_1(x, \lambda). \]

Using the second boundary condition, we get

\[ 0 = a_2 u(1) + a_3 D^\alpha u(1) \]

\[ = \theta (a_2 h_1(1, \lambda) + a_3 D^\alpha h_1(1, \lambda)) \]

\[ = a_2 h_1(1, \lambda) + a_3 D^\alpha h_1(1, \lambda) = 0. \]

- If \( a_0 \neq 0, \ \theta = -\frac{a_1 \eta}{a_0}. \ Thus,

\[ u_k(x) = \eta \left( -\frac{a_1}{a_0} \ h_1(x, \lambda) + \ h_2(x, \lambda) \right). \]

Using the second boundary condition, we get

\[ 0 = a_2 u(1) + a_3 D^\alpha u(1) \]

\[ = \eta \left( a_2 \left( -\frac{a_1}{a_0} \ h_1(1, \lambda) + \ h_2(1, \lambda) \right) + a_3 D^\alpha \left( -\frac{a_1}{a_0} \ h_1(1, \lambda) + \ h_2(1, \lambda) \right) \right) \]

\[ \]
or

\[ a_2 \left( -\frac{a_1}{a_0} h_1(1, \lambda) + h_2(1, \lambda) \right) + a_3 \ D^{\alpha} \left( -\frac{a_1}{a_0} h_1(1, \lambda) + h_2(1, \lambda) \right) = 0. \]  \tag{16}

To find the eigenvalues, we solve either Eqs. (3.10) or (3.11).

4. Convergence Analysis

In this section, we study the convergence of the series (3.5) of the eigenfunction of problem (3.1)-(3.3).

**Theorem 3.1:** Suppose that \( \sum_{n=0}^{\infty} f_n \ x^{\alpha n} \) converges to \( u(x) \) on \( (0, 1) \) where \( 0 < \alpha < 1 \). Then, \( \sum_{n=0}^{\infty} f_n \frac{\Gamma(n \alpha + 1)}{\Gamma((n-1) \alpha + 1)} x^{\alpha(n-1)} \) converges to \( D^{\alpha} \) on \( (0, 1) \).

**Proof:** For \( x \in (0, 1) \),

\[
D^{\alpha} u(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-s)^{-\alpha} \frac{du(s)}{ds} \ ds
\]

\[
= \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-s)^{-\alpha} \frac{d}{ds} \left( \sum_{n=0}^{\infty} f_n s^{\alpha n} \right) \ ds
\]

\[
= \frac{1}{\Gamma(1-\alpha)} \sum_{n=0}^{\infty} f_n \int_0^x (x-s)^{-\alpha} \frac{d}{ds} (s^{\alpha n}) \ ds
\]

\[
= \sum_{n=0}^{\infty} f_n D^{\alpha} x^{\alpha n}
\]

\[
= \sum_{n=0}^{\infty} f_n \frac{\Gamma(n \alpha + 1)}{\Gamma((n-1) \alpha + 1)} x^{\alpha(n-1)}.
\]

Thus, \( \sum_{n=0}^{\infty} f_n \frac{\Gamma(n \alpha + 1)}{\Gamma((n-1) \alpha + 1)} x^{\alpha(n-1)} \) converges to \( D^{\alpha} u(x) \).

**Theorem 3.2.** Let \( P(x) = \sum_{n=0}^{\infty} p_n x^{\alpha n} \), \( Q(x) = \sum_{n=0}^{\infty} q_n x^{\alpha n} \), and \( \mu(x) = \sum_{n=0}^{\infty} m_n x^{\alpha n} \). Then, the sequence \( \{u_k\} \) defined in Eq. (3.5) is convergent to \( u(x) \).

**Proof:** Let

\[
D^{\alpha} \left[ \left( \sum_{n=0}^{\infty} p_n x^{\alpha n} \right) D^{\alpha} \left( \sum_{n=0}^{\infty} f_n \frac{x^{\alpha n}}{\Gamma(1+n \alpha)} \right) \right]
\]

\[
+ \lambda \left( \sum_{n=0}^{\infty} q_n x^{\alpha n} \right) \left( \sum_{n=0}^{\infty} f_n \frac{x^{\alpha n}}{\Gamma(1+n \alpha)} \right) - \sum_{n=0}^{\infty} m_n x^{\alpha n}
\]

\[
= \sum_{n=0}^{\infty} \delta_n x^{\alpha n}.
\]

Since \( P(x) = \sum_{n=0}^{\infty} p_n x^{\alpha n} \), \( Q(x) = \sum_{n=0}^{\infty} q_n x^{\alpha n} \), and \( \mu(x) = \sum_{n=0}^{\infty} m_n x^{\alpha n} \),

\[
\sum_{n=0}^{\infty} \delta_n x^{\alpha n} = 0.
\]

Let

\[
S_m = \sum_{n=m}^{\infty} \delta_n x^{\alpha n}.
\]
Then, the sequence \( \{S_m\} \) converges to zero. From Eq. (3.7), we see that if we substitute Eq. (3.5) into Eq (3.1),

\[
Res_m(u(x)) = S_m.
\]
Thus,

\[
\lim_{m \to \infty} Res_m(u(x)) = \lim_{m \to \infty} S_m = 0.
\]
Hence, the sequence \( \{u_k\} \) defined in Eq. (3.5) is convergent to \( u(x) \).

5. Numerical results

In this section, we present three examples to show the efficiency of the proposed method.

Example 1. Consider the following fractional Sturm-Liouville problem

\[
D^\alpha[P(x)D^\alpha u(x)] + \lambda Q(x)u(x) = \mu(x)u(x), \quad 0 \leq x \leq 1, \quad \frac{1}{2} < \alpha \leq 1
\]
subject to

\[
u(0) = 0, u(1) = 0
\]
where \( P(x) = Q(x) = 1 \), and \( \mu(x) = 0 \). Figure 1 shows the graph of the eigenfunctions for \( \alpha = 0.75 \) and \( \lambda_1 \) and \( \lambda_2 \).

![Graph of the eigenfunctions for \( \alpha = 0.75 \) and \( \lambda_1 \) and \( \lambda_2 \).](image)
Table 1: Eigenvalues for different values of $\alpha$

For $\alpha = 1$, the exact eigenvalues are well-known and they are given by

$$\lambda_n = n^2 \pi^2, \quad n = 1, 2, 3, \ldots$$

It is worth mentioning that the eigenvalues of the problem in this example approaches to $n^2 \pi^2$ when $\alpha$ approaches to 1. We noticed that the eigenvalue problem in Example 1 does not have any eigenvalue for $\alpha = 0.5$. For this reason, we look for the numerical value of $\alpha^*$ such that the eigenvalue problem of this example does not have any eigenvalue for $\frac{1}{2} < \alpha < \alpha^*$. We noticed that $\alpha^* = 0.7355$. Let

$$\delta_{i,j} = \left| \int_0^1 u_i(x) u_j(x) Q(x) dx \right|.$$

For $\alpha = 0.75$, $\delta_{1,2} = 5.7 \times 10^{-16}$. Sample of these values for $\alpha = 0.95$ are given as $\delta_{1,2} = 5.7 \times 10^{-16}$, $\delta_{4,6} = 2.6 \times 10^{-16}$, $\delta_{1,6} = 8.3 \times 10^{-16}$.

Similarly for $\alpha = 0.99$,

$$\delta_{1,2} = 3.1 \times 10^{-16}, \quad \delta_{4,6} = 4.2 \times 10^{-16}, \quad \delta_{1,7} = 2.0 \times 10^{-16}.$$

This means, the orthogonality relation holds. We notice that the eigenvalues satisfy the property

$$\lambda_1 \leq \lambda_2 \leq \ldots$$

**Example 2.** Consider the following fractional Sturm-Liouville problem

$$D^\alpha [P(x)D^\alpha u(x)] + \lambda Q(x)u(x) = \mu(x)u(x), \quad 0 \leq x \leq 1, \quad \frac{1}{2} < \alpha \leq 1$$

subject to

$$u(0) = 0, \quad u(1) = 0$$

where $P(x) = 1$, $Q(x) = 1 + x^\alpha$, and $\mu(x) = 0$. The available results for $\lambda$ obtained by the present are summarized in Table 2.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
<th>$\lambda_3$</th>
<th>$\lambda_4$</th>
<th>$\lambda_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.501</td>
<td>3.7449685</td>
<td>4.9059651</td>
<td>5.8271193</td>
<td>8.0523428</td>
<td>11.388603</td>
</tr>
<tr>
<td>0.75</td>
<td>5.935961</td>
<td>9.9542383</td>
<td>21.863978</td>
<td>35.807427</td>
<td>51.211594</td>
</tr>
<tr>
<td>0.95</td>
<td>25.4751193</td>
<td>14.2468657</td>
<td>100.867952</td>
<td>234.2256821</td>
<td>439.2009128</td>
</tr>
</tbody>
</table>

Table 2: Eigenvalues for different values of $\alpha$
Let
\[ \delta_{i,j} = \left| \int_0^1 u_i(x) \ u_j(x) \ Q(x)dx \right|. \]
For \( \alpha = 0.502 \), \( \delta_{1,2} = 3.3 \times 10^{-16} \) and \( \delta_{2,4} = 4.9 \times 10^{-16} \). Sample of these values for \( \alpha = 0.75 \) are given as
\[ \delta_{1,2} = 2.2 \times 10^{-16}, \ \delta_{4,5} = 4.1 \times 10^{-16}, \ \delta_{1,5} = 6.9 \times 10^{-16}. \]
Similarly for \( \alpha = 0.95 \),
\[ \delta_{1,2} = 1.2 \times 10^{-16}, \ \delta_{4,6} = 2.1 \times 10^{-16}, \ \delta_{1,7} = 4.6 \times 10^{-16}. \]
This means, the orthogonality relation holds. We notice that the eigenvalues satisfy the property
\[ \lambda_1 \leq \lambda_2 \leq .... \]

**Example 3.** Consider the following fractional Sturm-Liouville problem
\[ D^\alpha [P(x)D^\alpha u(x)] + \lambda Q(x)u(x) = \mu(x), \ 0 \leq x \leq 1, \frac{1}{2} < \alpha \leq 1 \]
subject to
\[ u(0) - D^\alpha y(0) = 0, \ u(1) + D^\alpha y(1) = 0 \]
where \( P(x) = Q(x) = 1 \), and \( \mu(x) = 0 \). The available results for \( \lambda \) obtained by the present method are summarized in Table 3.

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>0.75</th>
<th>0.8</th>
<th>0.99</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5022831</td>
<td>1.5178861</td>
<td>1.6932293</td>
<td></td>
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<tr>
<td>12.051004</td>
<td>10.3426409</td>
<td>13.1985857</td>
<td></td>
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<tr>
<td>14.5035813</td>
<td>21.0815838</td>
<td>41.8043677</td>
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<tr>
<td>44.2389076</td>
<td>88.7795093</td>
<td>153.9545688</td>
<td></td>
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<td></td>
<td>237.42247398</td>
<td>338.92086546</td>
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</tr>
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<td></td>
<td>458.58377853</td>
<td>596.14001070</td>
<td></td>
</tr>
<tr>
<td></td>
<td>752.04906711</td>
<td>918.71319420</td>
<td></td>
</tr>
</tbody>
</table>

Table 3: Eigenvalues for different values of \( \alpha \)

It worth mention that, there are eigenvalues for all \( \frac{1}{2} < \alpha \leq 1 \). For example, the first eigenvalue for \( \alpha = 0.5001 \) is 1.68861. Let
\[ \delta_{i,j} = \left| \int_0^1 u_i(x) \ u_j(x) \ Q(x)dx \right|. \]
For \( \alpha = 0.75 \), \( \delta_{1,2} = 3.3 \times 10^{-16} \) and \( \delta_{2,3} = 4.9 \times 10^{-16} \). Sample of these values for \( \alpha = 0.8 \) are given as
\[ \delta_{1,2} = 1.6 \times 10^{-16}, \ \delta_{2,4} = 1.9 \times 10^{-16}, \ \delta_{3,4} = 2.8 \times 10^{-16}. \]
Similarly for \( \alpha = 0.99 \),
\[ \delta_{1,2} = 3.2 \times 10^{-16}, \ \delta_{4,6} = 4.5 \times 10^{-16}, \ \delta_{1,7} = 4.1 \times 10^{-16}. \]
This means, the orthogonality relation holds. We notice that the eigenvalues satisfy the property
\[ \lambda_1 \leq \lambda_2 \leq .... \]
6. Conclusion

In this paper, we have developed a numerical technique to find the eigenvalues of regular 2\(\alpha\)-order fractional Sturm-Liouville problem for \(\frac{1}{2} < \alpha \leq 1\). The method of solution is based on MFPS. The numerical results for the examples demonstrate the efficiency and accuracy of the present method. From the three examples which we mentioned in the previous section, we notice that our technique is very efficient for computing the eigenvalues of the fractional second order problems. We end this section by the following remarks.

- From Examples 1-3, we find that the generated eigenvalues satisfy the following property

\[ \lambda_1 < \lambda_2 < \lambda_3 < \ldots < \lambda_n < \ldots \]

- From Examples 1-3, the orthogonality property

\[ \int_0^1 u_i(x) \ u_j(x) \ Q(x) = 0, \ i \neq j \]

holds.

- Example 1 has eigenvalues when \(\alpha > 0.7355\).
- The results in this paper confirm that MFPS is a powerful and efficient method for solving fractional Sturm-Liouville problems in different fields of sciences and engineering.
- MFPS is excellent tool due to rapid convergent.
- The existence and the convergent are proven in Theorems (3.2).
- We do not compare our results with others because we are the first who discuss this class of eigenvalues.

Future work:

- We state the following conjecture for the future work:

**Conjecture:** The eigenvalue problem in Example 1 does not have any eigenvalue for \(\alpha < 0.7355\).

We will work to prove it.

- Generalize the proposed method for higher order fractional Sturm-Liouville problems.

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References


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