FRACTIONAL ORDER OF MATHEMATICAL SYSTEMS FOR SOME BIO-CHEMICAL APPLICATION

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Abstract. In this paper, we present an algorithm of the homotopy analysis method (HAM) to obtain symbolic approximate solutions for two systems of fractional ordinary differential equations that often appear in chemical applications. We show that the HAM is different from all analytical methods; it provides us with a simple way to adjust and control the convergence region of the series solution by introducing the auxiliary parameter ℏ, the auxiliary function \( H(t) \), the initial guess \( y_0(t) \) and the auxiliary linear operator \( \ell \). Two examples, the fractional systems of differential equations which appear in chemical applications. The algorithm of the homotopy analysis method (HAM) can be widely implemented to solve both linear and nonlinear differential equations of fractional order.

1. Introduction

Fractional differential equations have gained importance and popularity during the past three decades or so, mainly due to its demonstrated applications in numerous seemingly diverse fields of science and engineering. For example, the nonlinear oscillation of earthquake can be modeled with fractional derivatives, and the fluid-dynamic traffic model with fractional derivatives can eliminate the deficiency arising from the assumption of continuum traffic flow [1]. The differential equations with fractional order have recently proved to be valuable tools to the modeling of many physical phenomena [1] -[8]. This is because of the fact that the realistic modeling of a physical phenomenon does not depend only on the instant time, but also on the history of the previous time which can also be successfully achieved by using fractional calculus. Most nonlinear fractional equations do not have exact analytic solutions, so approximation and numerical techniques must be used. The Adomain decomposition method (ADM) [9] -[14], the homotopy perturbation method (HPM) [15] -[25], the variational iteration method (VIM) [26] -[28] and other methods have been used to provide analytical approximation to linear and nonlinear problems. However, the convergence region of the corresponding results is rather small as shown in [9] -[28]. The homotopy analysis method (HAM) is proposed first by Liao [29] -[33] for solving linear and nonlinear differential and integral equations. Different from perturbation techniques; the HAM doesn’t depend upon any small or

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large parameter. This method has been successfully applied to solve many types of nonlinear differential equations, such as projectile motion with the quadratic resistance law [34], Klein-Gordon equation [35], solitary waves with discontinuity [36], the generalized Hirota-Satsuma coupled KdV equation [37], heat radiation equations [38], MHD flows of an Oldroyd 8-constant fluid [39], Vakhnenko equation [40], unsteady boundary-layer flows [41]. Recently, Song and Zhang [42] used the HAM to solve fractional KdV-Burgers-Kuramoto equation, Cang and his co-authors [43] constructed a series solution of non-linear Riccati differential equations with fractional order using HAM. They proved that the Adomian decomposition method is a special case of HAM, and we can adjust and control the convergence region of solution series by choosing the auxiliary parameter close to zero. The objective of the present paper is to modify the HAM to provide symbolic approximate solutions for a systems of fractional differential equations which often appear in chemical applications.

Our modification is implemented on the fractional oscillation equation, the fractional Riccati equation and the fractional Lane-Emden equation. By choosing suitable values of the auxiliary parameter $h$, the auxiliary function $H(t)$, the initial guess $u_o(t)$ and the auxiliary linear operator $L$ function we can adjust and control the convergence region of solution series. Moreover, we illustrated for several examples that the Adomain decomposition, Variational iteration and homotopy perturbation solutions are special cases of homotopy analysis solution.

For the concept of fractional derivative we will adopt Caputo’s definition [7] which is a modification of the Riemann-Liouville definition and has the advantage of dealing properly with initial value problems in which the initial conditions are given in terms of the field variables and their integer order which is the case in most physical processes [1].

**Definition 1.** A real function $f(x), x > 0$, is said to be in the space $C^\mu, \mu \in \mathbb{R}$ if there exists a real number $p > \mu$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0, \infty)$ and it is said to be in the space $C^n\mu$ iff $f^{(n)}(x) \in C_\mu, n \in \mathbb{N}$.  

**Definition 2.** The Riemann-Liouville fractional integral operator of order $\alpha > 0$, of a function $f(x) \in C^\mu, \mu \geq -1$, is defined as:

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt, \quad \alpha > 0, \ x > 0$$ (1)

$$J^0 f(x) = f(x).$$ (2)

Properties of the operator $J^\alpha$ can be found in [5]-[8], we mention only the following: For $f \in C^\mu, \mu \geq -1$, $\alpha, \beta \geq 0$ and $\gamma \geq -1$

$$J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x) = J^\beta J^\alpha f(x)$$ (3)

$$J^\alpha x^\gamma = \frac{\Gamma[\gamma + 1]}{\Gamma[\alpha + \gamma + 1]} x^{\alpha+\gamma}$$ (4)

**Definition 3.** The fractional derivative of $f(x)$ in the Caputo sense is defined as:

$$D^\alpha f(x) = J^{n-\alpha} D^n f(x) = \frac{1}{\Gamma[n-\alpha]} \int_0^x (x-t)^{n-\alpha-1} f^{(n)}(t) dt,$$ (5)

\[\text{For the concept of fractional derivative we will adopt Caputo’s definition [7] which is a modification of the Riemann-Liouville definition and has the advantage of dealing properly with initial value problems in which the initial conditions are given in terms of the field variables and their integer order which is the case in most physical processes [1].} \]
for \( n - 1 < \alpha \leq n, \ n \in \mathbb{N}, \ x > 0, \ f \in C^n_{-1}. \)

**Lemma 1.** If \( n - 1 < \alpha \leq n, \ n \in \mathbb{N} \) and \( f \in C^n_{\mu}, \ \mu > -1, \) then

\[
D_\alpha^n J^\alpha f(x) = f(x),
\]

\[
J^\alpha D_\alpha^n f(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0^+)}{k!} x^k, \quad x > 0.
\]

2. Homotopy analysis method

Let us consider the following system of differential equation

\[
N_i[u_1(r, t), u_2(r, t) \ldots u_n(r, t)] = 0, \quad i = 1, 2, 3, \ldots, n
\]

\[
u^{(k)}(r, t) = c_k, \quad k = 0, 1, 2, \ldots, n - 1
\]

where \( N_i \) are nonlinear operators that represent the whole equations, \( r \) and \( t \) denote the independent variables and \( u_{i}(r, t) \) are unknowns function respectively. By means of generalizing the traditional homotopy method Liao [44] constructed the so-called zero-order deformation equations for \( i = 1, 2, 3, \ldots \)

\[
(1 - q)\phi_i[\phi_i(r, t; q) - u_{i0}(r, t)] = q h_i(r, t) N_i[\phi_i(r, t; q)],
\]

where \( q \in [0, 1] \) is the embedding parameter, \( h_i \neq 0 \) are non-zero auxiliary parameters for \( H_i(r, t) \neq 0 \) are non-zero auxiliary functions, \( \phi_i = D^\alpha_i (n - 1 < \alpha \leq n) \) are auxiliary linear operator with the following property for \( i = 1, 2, 3, \ldots n \)

\[
\phi_i[\phi_i(r, t)] = 0, \text{ when } \phi_i(r, t) = 0
\]

\( u_{i0}(r, t) \) are initial guess of \( u_{i}(r, t) \) and \( u_{i}(r, t; q) \) are unknown function, respectively. It is important, that one has great freedom to choose auxiliary things in HAM. Obviously, when \( q = 0 \) and \( q = 1, \) it holds

\[
\phi_i(r, t; 0) = u_{i0}(r, t), \quad \phi_i(r, t; 1) = u_{i}(r, t), \quad i = 1, 2, 3, \ldots, n
\]

respectively. Thus, as \( q \) increases from 0 to 1, the solution \( \phi_i(r, t; q) \) varies from the initial guesses \( u_{i0}(r, t) \) to the solution \( u_{i}(r, t) . \)

Expanding \( \phi_i(r, t; q) \) in Taylor series with respect to \( q, \) we have

\[
\phi_i(r, t; q) = u_{i0}(r, t) + \sum_{m=1}^{\infty} u_{im}(r, t) q^m, \quad i = 1, 2, 3, \ldots, n
\]

where

\[
u_{im}(r, t) = \left. \frac{1}{m!} \frac{\partial^m \phi_i(r, t; q)}{\partial q^m} \right|_{q=0}, \quad i = 1, 2, 3, \ldots, n
\]

If the auxiliary linear operator, the initial guess, the auxiliary parameter \( h, \) and the auxiliary function are so properly chosen, the series Eq.(14) converges at \( q = 1, \) then we have

\[
\phi_i(r, t; q) = u_{i0}(r, t) + \sum_{m=1}^{\infty} u_{im}(r, t), \quad i = 1, 2, 3, \ldots, n
\]
Define the vector 
\[ \vec{u}_n = \{u_0(r, t), u_1(r, t), u_2(r, t), \ldots, u_n(r, t)\}, \quad i = 1, 2, 3, \ldots, n \]  
(16)

Differentiating Eq. (10) \(m\) times with respect to embedding parameter \(q\), then setting \(q = 0\) and dividing them by \(m!\), we have, using Eq. (12), the so-called \(m\)th order deformation equation for \(i = 1, 2, \ldots, n\)

\[ \phi_i[u_{im}(r, t) - u_{im-1}(r, t)] = h_i H_i(r, t) R_{im}(\vec{u}_{im-1}(r, t), i = 1, 2, \ldots, n \]  
(17)

where

\[ R_{im}(\vec{u}_{im-1}) = \frac{1}{(m - 1)!} \left. \frac{\partial^{m-1} N_i[\phi_i(r, t; q)]}{\partial q^{m-1}} \right|_{q=0} \]  
(18)

And

\[ \chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases} \]  
(19)

Applying the Riemann-Liouville integral operator \(J^{\alpha_i}\) on both side of Eq.(17), we have

\[ u_{im} = \chi_m u_{im-1} + h_i H_i(r, t) J^{\alpha_i}[R_{im}(\vec{u}_{im-1}(r, t)] \]  
(20)

In this way, it is easily to obtain \(u_{im}(r, t)\) for \(m \geq 1\), at \(M\)th order, we have

\[ u_i(r, t) = \sum_{m=0}^{\infty} u_{im}(r, t), \quad i = 1, 2, \ldots, n \]  
(21)

we get an accurate approximation of the original Eq.(8)

3. Examples

In this section we employ our algorithm of the homotopy analysis method to find out series solutions for some fractional initial value problems.

**Example 1.** Consider the following nonlinear system of fractional differential equation in chemistry problem:

\[ \begin{align*}
\frac{d^{a_1} u}{dt^{a_1}} &= -k_1 u + k_2 vw, \\
\frac{d^{a_2} v}{dt^{a_2}} &= k_3 u - k_4 vw - k_5 v^2, \\
\frac{d^{a_3} w}{dt^{a_3}} &= k_6 v^2.
\end{align*} \]  
(22)

Subject to the initial conditions \(u(0) = 1, \ v(0) = 0\) and \(w(0) = 0\), where \(k_1, k_2, k_3, k_4, k_5\) and \(k_6\) are constant parameters \((k_1 = 0.04, k_2 = 0.01, k_3 = 400, k_4 = 100, k_5 = 30000, k_6 = 30)\). We choose the linear operator

\[ \begin{align*}
L_1[\phi(t; q)] &= D^{a_1}\phi(t; q), \quad L_1^{-1} = J^{a_1}\phi(t; q), \\
L_2[\phi(t; q)] &= D^{a_2}\phi(t; q), \quad L_2^{-1} = J^{a_2}\phi(t; q), \\
L_3[\phi(t; q)] &= D^{a_3}\phi(t; q), \quad L_3^{-1} = J^{a_3}\phi(t; q).
\end{align*} \]  
(23)
Since
\[ N_1[u(t)] = \frac{d^{\alpha_1} u}{dt^{\alpha_1}} + k_1 u - k_2 v w, \]
\[ N_2[v(t)] = \frac{d^{\alpha_2} v}{dt^{\alpha_2}} - k_3 u + k_4 v w + k_5 v^2, \]
\[ N_3[w(t)] = \frac{d^{\alpha_3} w}{dt^{\alpha_3}} - k_6 v^2. \tag{24} \]

According to Eqs. (8) and (18), we have
\[
R_{1m}(\bar{u}_{m-1}) = D^{\alpha_1} \bar{u}_{m-1} + k_1 \bar{u}_{m-1} - k_2 \sum_{i=0}^{m-1} v_{m-i-1} w_i, \\
R_{2m}(\bar{v}_{m-1}) = D^{\alpha_2} \bar{v}_{m-1} - k_3 \bar{u}_{m-1} + k_4 \sum_{i=0}^{m-1} v_{m-i-1} w_i + k_5 \sum_{i=0}^{m-1} v_{m-i-1} v_i, \\
R_{3m}(\bar{w}_{m-1}) = D^{\alpha_3} \bar{w}_{m-1} - k_6 \sum_{i=0}^{m-1} v_{m-i-1} v_i. \tag{25} \]

Now, the solution of the mth-order deformation equation
\[
u_m = \chi_m \nu_{m-1} + h J^{\alpha_1}[R_{1m}(\bar{u}_{m-1})], \\
v_m = \chi_m v_{m-1} + h J^{\alpha_2}[R_{2m}(\bar{v}_{m-1})], \\
w_m = \chi_m w_{m-1} + h J^{\alpha_3}[R_{3m}(\bar{w}_{m-1})]. \tag{26} \]

Consequently, the first few terms of the HAM
\[ u_0 = 1, \quad v_0 = 0, \quad w_0 = 0 \]
\[ u_1 = h k_1 \frac{t^{\alpha_1}}{\Gamma[\alpha_1 + 1]}, \quad v_1 = -h k_3 \frac{t^{\alpha_2}}{\Gamma[\alpha_2 + 1]}, \quad w_1 = 0 \]
\[ u_2 = (h + h^2) \frac{k_1 t^{\alpha_1}}{\Gamma[\alpha_1 + 1]} + h^2 \frac{k_2^2 t^{2\alpha_1}}{\Gamma[2\alpha_1 + 1]}, \quad w_2 = 0 \]
\[ u_3 = (h + 2h^2 + h^3) \frac{k_1 t^{\alpha_1}}{\Gamma[\alpha_1 + 1]} + 2(h^2 + h^3) \frac{k_2^2 t^{2\alpha_1}}{\Gamma[2\alpha_1 + 1]} + h^3 \frac{k_3^3 t^{3\alpha_1}}{\Gamma[3\alpha_1 + 1]}, \quad w_3 = 0 \]
\[ v_3 = -(h + 2h^2 + h^3) \frac{k_3 t^{\alpha_2}}{\Gamma[\alpha_2 + 1]} - 2(h^2 + h^3) \frac{k_1^2 k_3 t^{\alpha_1 + \alpha_2}}{\Gamma[\alpha_1 + \alpha_2 + 1]} \frac{\Gamma[\alpha_1 + \alpha_2 + 1]}{\Gamma[3\alpha_1 + 1]}, \quad w_3 = 0 \]
\[ u_3 = -h^3 \frac{k_2^2 k_3 \Gamma[2\alpha_2 + 1]}{(\Gamma[\alpha_2 + 1])^2} \frac{\Gamma[2\alpha_2 + 1]}{\Gamma[3\alpha_2 + 1]}. \]
Finally, we have

\[u(t) = \sum_{m=0}^{\infty} u_m(t), \quad v(t) = \sum_{m=0}^{\infty} v_m(t), \quad w(t) = \sum_{m=0}^{\infty} w_m(t).\]  \hfill (27)

Then

\[u(t) = 1 + (3h + 3h^2 + h^3) \frac{k_1 t^{\alpha_1}}{\Gamma[\alpha_1 + 1]} + (3h^2 + 2h^3) \frac{k_2^2 t^{2\alpha_1}}{\Gamma[2\alpha_1 + 1]} + h^3 \frac{k_3^3 t^{3\alpha_1}}{\Gamma[3\alpha_1 + 1]} + \ldots.\]  \hfill (28)

\[v(t) = -(3h + 3h^2 + h^3) \frac{k_3 t^{\alpha_2}}{\Gamma[\alpha_2 + 1]} - (3h^2 + 2h^3) \frac{k_1 k_3 t^{\alpha_1+\alpha_2}}{\Gamma[\alpha_1 + \alpha_2 + 1]} - h^3 \frac{k_1^2 k_3 t^{2\alpha_1+\alpha_2}}{\Gamma[2\alpha_1 + \alpha_2 + 1]} + h^3 \frac{k_3^3 \Gamma(2\alpha_2 + 1)}{\Gamma[2\alpha_2 + 3] \Gamma[3\alpha_2 + 1]} + \ldots.\]  \hfill (29)

\[w(t) = -h^3 \frac{k_3^3}{\Gamma[\alpha_2 + 1]^2} \frac{t^{2\alpha_2+\alpha_3}}{\Gamma[2\alpha_2 + \alpha_3 + 1]} + \ldots.\]  \hfill (30)

If we put \(\alpha_1 = \alpha_2 = \alpha_3 = 1,\) and \(h = -1,\) we have \[u(t) = 1 - k_1 t + \frac{1}{2} k_1^2 t^2 - \frac{1}{6} k_1^3 t^3 + \ldots.\]  \hfill (31)

\[v(t) = k_3 t - \frac{1}{2} k_1 k_3 t^2 + \frac{1}{6} k_3 (2k_1^2 + k_2^2) t^3 + \ldots.\]  \hfill (32)

\[w(t) = \frac{1}{3} k_3^2 k_4 t^3 + \ldots.\]  \hfill (33)

see Figs. 1-3

**Example 2.** Consider the following nonlinear system of fractional differential equation in chemistry problem:

\[\frac{d^{\alpha_1} u}{dt^{\alpha_1}} = -u, \quad \frac{d^{\alpha_2} v}{dt^{\alpha_2}} = u - v^2, \quad \frac{d^{\alpha_3} w}{dt^{\alpha_3}} = v^2.\]  \hfill (34)

Subject to the initial conditions \(u(0) = 1, \quad v(0) = 0\) and \(w(0) = 0.\) We choose the linear operator

\[L_1 [\phi(t; q)] = D^{\alpha_1} \phi(t; q), \quad L_1^{-1} = J^{\alpha_1} \phi(t; q),\]

\[L_2 [\phi(t; q)] = D^{\alpha_2} \phi(t; q), \quad L_2^{-1} = J^{\alpha_2} \phi(t; q),\]

\[L_3 [\phi(t; q)] = D^{\alpha_3} \phi(t; q), \quad L_3^{-1} = J^{\alpha_3} \phi(t; q).\]  \hfill (35)
Consequently, the first few terms of the HAM

\[ N_1[u(t)] = \frac{d^{\alpha_1} u}{dt^{\alpha_1}} + u, \]
\[ N_2[v(t)] = \frac{d^{\alpha_2} v}{dt^{\alpha_2}} - u + v^2, \]
\[ N_3[w(t)] = \frac{d^{\alpha_3} w}{dt^{\alpha_3}} - v^2. \]  

(36)

According to Eqs. (8) and (18), we have

\[ R_{1m}(\vec{u}_{m-1}) = D^{\alpha_1} u_{m-1} + u_{m-1}, \]
\[ R_{2m}(\vec{v}_{m-1}) = D^{\alpha_2} v_{m-1} - u_{m-1} + \sum_{i=0}^{m-1} v_{m-i-1} v_i, \]
\[ R_{3m}(\vec{w}_{m-1}) = D^{\alpha_3} w_{m-1} - \sum_{i=0}^{m-1} v_{m-i-1} v_i. \]  

(37)

Now, the solution of the \( m \)-th order deformation equation

\[ u_m = \chi_m u_{m-1} + h J^{\alpha_1} R_{1m}(\vec{u}_{m-1}), \]
\[ v_m = \chi_m v_{m-1} + h J^{\alpha_2} R_{2m}(\vec{v}_{m-1}), \]
\[ w_m = \chi_m w_{m-1} + h J^{\alpha_3} R_{3m}(\vec{w}_{m-1}). \]  

(38)

Consequently, the first few terms of the HAM

\[ u_0 = 1, \quad v_0 = 0, \quad w_0 = 0 \]
\[ u_1 = h \frac{t^{\alpha_1}}{\Gamma[\alpha_1 + 1]}, \quad v_1 = -h \frac{t^{\alpha_2}}{\Gamma[\alpha_2 + 1]}, \quad w_1 = 0 \]
\[ u_2 = (h + h^2) \frac{t^{\alpha_1}}{\Gamma[\alpha_1 + 1]} + h^2 \frac{t^{2\alpha_1}}{\Gamma[2\alpha_1 + 1]}, \quad v_2 = -(h + h^2) \frac{t^{\alpha_2}}{\Gamma[\alpha_2 + 1]} - h^2 \frac{t^{\alpha_1 + \alpha_2}}{\Gamma[\alpha_1 + \alpha_2 + 1]}, \quad w_2 = 0 \]
\[ u_3 = (h + 2h^2 + h^3) \frac{t^{\alpha_1}}{\Gamma[\alpha_1 + 1]} + 2(h^2 + h^3) \frac{t^{2\alpha_1}}{\Gamma[2\alpha_1 + 1]} + h^3 \frac{t^{3\alpha_1}}{\Gamma[3\alpha_1 + 1]}, \]
\[ v_3 = -(h + 2h^2 + h^3) \frac{t^{\alpha_2}}{\Gamma[\alpha_2 + 1]} - 2(h^2 + h^3) \frac{t^{2\alpha_2}}{\Gamma[2\alpha_2 + 1]} - h^3 \frac{t^{\alpha_1 + \alpha_2}}{\Gamma[\alpha_1 + \alpha_2 + 1]} \]
\[ -h^3 \frac{t^{2\alpha_1 + \alpha_2}}{\Gamma[2\alpha_1 + \alpha_2 + 1]} + h^3 \frac{t^{2\alpha_2 + \alpha_3}}{\Gamma[2\alpha_2 + \alpha_3 + 1]}, \]
\[ w_3 = -h^3 \frac{t^{\alpha_2}}{\Gamma[\alpha_2 + 1]} - h^3 \frac{t^{\alpha_3}}{\Gamma[\alpha_3 + 1]} + h^3 \frac{t^{2\alpha_2 + \alpha_3}}{\Gamma[2\alpha_2 + \alpha_3 + 1]} \]

Finally, we have

\[ u(t) = \sum_{m=0}^{\infty} u_m(t), \quad v(t) = \sum_{m=0}^{\infty} v_m(t), \quad w(t) = \sum_{m=0}^{\infty} w_m(t). \]  

(39)
Then
\[ u(t) = 1 + (3h^3 + 3h^2 + h^3) \frac{t^\alpha_1}{\Gamma[\alpha_1 + 1]} + (3h^2 + 2h^3) \frac{t^{2\alpha_1}}{\Gamma[2\alpha_1 + 1]} + h^3 \frac{t^{3\alpha_1}}{\Gamma[3\alpha_1 + 1]} + \ldots. \] (40)

\[ v(t) = -(3h^3 + 3h^2 + h^3) \frac{t^\alpha_2}{\Gamma[\alpha_2 + 1]} - (3h^2 + 2h^3) \frac{t^{\alpha_1 + \alpha_2}}{\Gamma[\alpha_1 + \alpha_2 + 1]} - h^3 \frac{t^{2\alpha_1 + \alpha_2}}{\Gamma[2\alpha_1 + \alpha_2 + 1]} + \ldots. \] (41)

\[ w(t) = -h^3 \frac{t^{2\alpha_2 + \alpha_3}}{\Gamma[2\alpha_2 + 1]} \frac{t^{2\alpha_2 + \alpha_3}}{\Gamma[2\alpha_2 + 2\alpha_3 + 1]} + \ldots. \] (42)

If we put \( \alpha_1 = \alpha_2 = \alpha_3 = 1 \) and \( h = -1 \), we have
\[ u(t) = 1 - t + \frac{1}{2} t^2 - \frac{1}{6} t^3 + \ldots. \] (43)

\[ v(t) = t - \frac{1}{2} t^2 + \frac{1}{6} t^3 + \ldots. \] (44)

\[ w(t) = \frac{1}{3} t^3 + \ldots. \] (45)

see Figs. 4-6

**Example 3.** Consider the following nonlinear system of fractional differential equation in chemistry problem:

\[ \frac{d^{\alpha_1}}{dt^{\alpha_1}} u = -u + vw, \]
\[ \frac{d^{\alpha_2}}{dt^{\alpha_2}} v = u - vw - 2v^2, \]
\[ \frac{d^{\alpha_3}}{dt^{\alpha_3}} w = v^2. \] (46)

Subject to the initial conditions \( u(0) = 1 \), \( v(0) = 2 \) and \( w(0) = 0 \). We choose the linear operator

\[ L_1 [\phi(t; q)] = D^{\alpha_1} \phi(t; q), \quad L_1^{-1} = J^{\alpha_1} \phi(t; q), \]
\[ L_2 [\phi(t; q)] = D^{\alpha_2} \phi(t; q), \quad L_2^{-1} = J^{\alpha_2} \phi(t; q), \]
\[ L_3 [\phi(t; q)] = D^{\alpha_3} \phi(t; q), \quad L_3^{-1} = J^{\alpha_3} \phi(t; q). \] (47)

Since

\[ N_1[u(t)] = \frac{d^{\alpha_1}}{dt^{\alpha_1}} u + u - vw, \]
\[ N_2[v(t)] = \frac{d^{\alpha_2}}{dt^{\alpha_2}} v - u + vw + v^2, \]
\[ N_3[w(t)] = \frac{d^{\alpha_3}}{dt^{\alpha_3}} v - v^2. \] (48)

According to Eqs. (8) and (18), we have
\begin{align*}
R_1(m(\vec{u}_{m-1}) &= D^{\alpha_1}u_{m-1} + u_{m-1} - \sum_{i=0}^{m-1} v_{m-i-1}v_i, \\
R_2(m(\vec{v}_{m-1}) &= D^{\alpha_2}v_{m-1} - u_{m-1} + \sum_{i=0}^{m-1} v_{m-i-1}w_i + 2\sum_{i=0}^{m-1} v_{m-i-1}v_i, \\
R_3(m(\vec{w}_{m-1}) &= D^{\alpha_3}w_{m-1} - \sum_{i=0}^{m-1} v_{m-i-1}v_i. \quad (49) 
\end{align*}

Now, the solution of the \( m \)-th order deformation equation

\begin{align*}
u_m &= \chi_m u_{m-1} + hJ^{\alpha_1}[R_1(m(\vec{u}_{m-1})], \\
v_m &= \chi_m v_{m-1} + hJ^{\alpha_2}[R_2(m(\vec{v}_{m-1})], \\
w_m &= \chi_m w_{m-1} + hJ^{\alpha_3}[R_3(m(\vec{w}_{m-1})]. \quad (50)
\end{align*}

Consequently, the first few terms of the HAM

\begin{align*}
u_0 &= 1, \quad v_0 = 2, \quad w_0 = 0 \\
u_1 &= h \frac{\ell^{\alpha_1}}{\Gamma(\alpha_1 + 1)}, \quad v_1 = 7h \frac{\ell^{\alpha_2}}{\Gamma(\alpha_2 + 1)}, \quad w_1 = -4h \frac{\ell^{\alpha_3}}{\Gamma(\alpha_3 + 1)} \\
u_2 &= (h + h^2) \frac{\ell^{\alpha_1}}{\Gamma(\alpha_1 + 1)} + h^2 \frac{\ell^{2\alpha_1}}{\Gamma(2\alpha_1 + 1)} + 8h^2 \frac{\ell^{\alpha_1+\alpha_3}}{\Gamma(\alpha_1 + \alpha_3 + 1)}, \\
v_2 &= 7(h + h^2) \frac{\ell^{\alpha_2}}{\Gamma(\alpha_2 + 1)} - h^2 \frac{\ell^{\alpha_2+\alpha_2}}{\Gamma(\alpha_1 + \alpha_2 + 1)} - 8h^2 \frac{\ell^{\alpha_1+\alpha_3}}{\Gamma(\alpha_1 + \alpha_3 + 1)} + 56h^2 \frac{\ell^{2\alpha_2}}{\Gamma(2\alpha_2 + 1)}, \\
w_2 &= -4(h + h^2) \frac{\ell^{\alpha_3}}{\Gamma(\alpha_3 + 1)} - 28h^2 \frac{\ell^{\alpha_2+\alpha_3}}{\Gamma(\alpha_2 + \alpha_3 + 1)},
\end{align*}
\[ u_3 = (h + 2h^2 + h^3) \frac{\Gamma(\alpha_1 + 1)}{\Gamma[\alpha_1 + 1]} + 2(h^2 + h^3) \frac{\Gamma(2\alpha_1)}{\Gamma[2\alpha_1 + 1]} + 12(h + h^2) \frac{\Gamma(\alpha_1 + \alpha_3 + 1)}{\Gamma[\alpha_1 + \alpha_3 + 1]} + 32h^3 \frac{\Gamma(\alpha_2 + \alpha_3 + 1)}{\Gamma[\alpha_2 + \alpha_3 + 1]} \]

\[ v_3 = 7(h + 2h^2 + h^3) \frac{\Gamma(\alpha_2 + 1)}{\Gamma[\alpha_2 + 1]} - 2(h^2 + h^3) \frac{\Gamma(\alpha_1 + \alpha_2 + 1)}{\Gamma[\alpha_1 + \alpha_2 + 1]} - 16(h^2 + h^3) \frac{\Gamma(\alpha_2 + \alpha_3 + 1)}{\Gamma[\alpha_2 + \alpha_3 + 1]} + 112(h^2 + h^3) \frac{\Gamma(\alpha_2 + 1)}{\Gamma[\alpha_2 + 1]} - 8h^3 \frac{\Gamma(\alpha_1 + \alpha_2 + \alpha_3 + 1)}{\Gamma[\alpha_1 + \alpha_2 + \alpha_3 + 1]} - 8h^3 \frac{\Gamma(\alpha_2 + \alpha_3 + 1)}{\Gamma[\alpha_2 + \alpha_3 + 1]} - 4h^3 \frac{\Gamma(\alpha_1 + \alpha_2 + \alpha_3 + 1)}{\Gamma[\alpha_1 + \alpha_2 + \alpha_3 + 1]} - 14h^3 \frac{\Gamma(\alpha_1 + \alpha_2 + \alpha_3 + 1)}{\Gamma[\alpha_1 + \alpha_2 + \alpha_3 + 1]} + 32h^3 \frac{\Gamma(\alpha_2 + \alpha_3 + 1)}{\Gamma[\alpha_2 + \alpha_3 + 1]} - 7h^3 \frac{\Gamma(\alpha_2 + 1)}{\Gamma[\alpha_2 + 1]} - 32h^3 \frac{\Gamma(\alpha_2 + \alpha_3 + 1)}{\Gamma[\alpha_2 + \alpha_3 + 1]} - 7h^3 \frac{\Gamma(\alpha_2 + 1)}{\Gamma[\alpha_2 + 1]} + 32h^3 \frac{\Gamma(\alpha_2 + \alpha_3 + 1)}{\Gamma[\alpha_2 + \alpha_3 + 1]} \]

Finally, we have

\[ u(t) = \sum_{m=0}^{\infty} u_m(t), \quad v(t) = \sum_{m=0}^{\infty} v_m(t), \quad w(t) = \sum_{m=0}^{\infty} w_m(t). \] (51)

Then

\[ u(t) = 1 + 3(h + 2h^2 + h^3) \frac{\Gamma(\alpha_1 + 1)}{\Gamma[\alpha_1 + 1]} + 3(h^2 + 2h^3) \frac{\Gamma(2\alpha_1)}{\Gamma[2\alpha_1 + 1]} + 32h^3 \frac{\Gamma(\alpha_2 + \alpha_3 + 1)}{\Gamma[\alpha_2 + \alpha_3 + 1]} \]

\[ v(t) = 2 + 7(3h + 3h^2 + h^3) \frac{\Gamma(\alpha_2 + 1)}{\Gamma[\alpha_2 + 1]} + 7(3h + 3h^2 + h^3) \frac{\Gamma(\alpha_2 + \alpha_3 + 1)}{\Gamma[\alpha_2 + \alpha_3 + 1]} + 4(5h^2 + 3h^3) \frac{\Gamma(\alpha_1 + \alpha_2 + \alpha_3 + 1)}{\Gamma[\alpha_1 + \alpha_2 + \alpha_3 + 1]} + 32h^3 \frac{\Gamma(\alpha_2 + \alpha_3 + 1)}{\Gamma[\alpha_2 + \alpha_3 + 1]} + 4h^3 \frac{\Gamma(\alpha_1 + \alpha_2 + \alpha_3 + 1)}{\Gamma[\alpha_1 + \alpha_2 + \alpha_3 + 1]} + 14h^3 \frac{\Gamma(\alpha_1 + \alpha_2 + \alpha_3 + 1)}{\Gamma[\alpha_1 + \alpha_2 + \alpha_3 + 1]} \] (52)
\[ w(t) = -4(3h + 3h^2 + h^3) \frac{t^{\alpha_1}}{\Gamma(\alpha_1 + 1)} - 28(3h^2 + h^3) \frac{t^{\alpha_2 + \alpha_3}}{\Gamma(\alpha_2 + \alpha_3 + 1)} + 4h^3 \frac{t^{\alpha_1 + \alpha_2 + \alpha_3}}{\Gamma(\alpha_1 + \alpha_2 + \alpha_3 + 1)} + 32h^3 \frac{t^{\alpha_2 + 2\alpha_3}}{\Gamma(\alpha_2 + 2\alpha_3 + 1)} - 7h^3 \left( \frac{7\Gamma(2\alpha_2)}{\Gamma(\alpha_2 + 1)^2} + 32 \right) \frac{t^{2\alpha_2 + \alpha_3}}{\Gamma(2\alpha_2 + \alpha_3 + 1)} + \ldots \] (54)

If we put \( \alpha_1 = \alpha_2 = \alpha_3 = 1 \), and \( h = -1 \), we have

\[ u(t) = 1 - t + \frac{9}{2}t^2 - \frac{65}{6}t^3 + \ldots \] (55)

\[ v(t) = 2 - 7t - 51t^2 + \frac{127}{2}t^3 + \ldots \] (56)

\[ w(t) = 4t - 28t^2 + \frac{29}{2}t^3 + \ldots \] (57)

see Figs. 7-9

4. Conclusion

In this work, we carefully proposed an efficient algorithm of the HAM which introduces an efficient tool for solving some systems of differential equations of fractional order in chemical applications. The work emphasized our belief that the method is a reliable technique to handle systems of nonlinear differential equations of fractional order. As an advantage of this method over the other analytical methods, such as ADM and HPM, in this method we can choose a proper value for the auxiliary parameter \( \hbar \), the auxiliary function \( H(t) \), the auxiliary linear operator \( \mathcal{L} \) and the initial guess \( u_{i0}(r, t) \) to adjust and control convergence region of the series solutions.

References

Figure 1. Ex. 1, the comparison of the results of $u$ Eq. (28).

Where $h = -1, k_1 = 0.04, k_2 = 0.01, k_3 = 400, k_4 = 100, k_5 = 30000$ and $k_6 = 30$. For different values of the HAM solution $\alpha_1, \alpha_2, \alpha_3$ ($\alpha_1 = \alpha_2 = \alpha_3 = 1$ solid black line, $\alpha_1 = \alpha_2 = \alpha_3 = 0.98$ red dashed line and $\alpha_1 = \alpha_2 = \alpha_3 = 0.95$ blue dotted line)


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Figure 2. Ex. 1, the comparison of the results of $v$ Eq. (29). Where $h = -1, k_1 = 0.04, k_2 = 0.01, k_3 = 400, k_4 = 100, k_5 = 30000$ and $k_6 = 30$. For different values of the HAM solution $\alpha_1, \alpha_2, \alpha_3$ ($\alpha_1 = \alpha_2 = \alpha_3 = 1$ solid black line, $\alpha_1 = \alpha_2 = \alpha_3 = 0.98$ red dashed line and $\alpha_1 = \alpha_2 = \alpha_3 = 0.95$ blue dotted line).

Figure 3. Ex. 1, the comparison of the results of $w$ Eq. (30). Where $h = -1, k_1 = 0.04, k_2 = 0.01, k_3 = 400, k_4 = 100, k_5 = 30000$ and $k_6 = 30$. For different values of the HAM solution $\alpha_1, \alpha_2, \alpha_3$ ($\alpha_1 = \alpha_2 = \alpha_3 = 1$ solid black line, $\alpha_1 = \alpha_2 = \alpha_3 = 0.98$ red dashed line and $\alpha_1 = \alpha_2 = \alpha_3 = 0.95$ blue dotted line).
Figure 4. Ex. 2, the comparison of the results of $u$ Eq. (40). Where $h = -1$. For different values of the HAM solution $\alpha_1, \alpha_2, \alpha_3$ ($\alpha_1 = \alpha_2 = \alpha_3 = 1$ solid black line, $\alpha_1 = \alpha_2 = \alpha_3 = 0.98$ red dashed line and $\alpha_1 = \alpha_2 = \alpha_3 = 0.95$ blue dotted line).

Figure 5. Ex. 2, the comparison of the results of $v$ Eq. (41). Where $h = -1$. For different values of the HAM solution $\alpha_1, \alpha_2, \alpha_3$ ($\alpha_1 = \alpha_2 = \alpha_3 = 1$ solid black line, $\alpha_1 = \alpha_2 = \alpha_3 = 0.98$ red dashed line and $\alpha_1 = \alpha_2 = \alpha_3 = 0.95$ blue dotted line).
Figure 6. Ex. 2, the comparison of the results of \( w \) Eq. (42). Where \( h = -1 \). For different values of the HAM solution \( \alpha_1, \alpha_2, \alpha_3 \) \((\alpha_1 = \alpha_2 = \alpha_3 = 1 \text{ solid black line, } \alpha_1 = \alpha_2 = \alpha_3 = 0.98 \text{ red dashed line and } \alpha_1 = \alpha_2 = \alpha_3 = 0.95 \text{ blue dotted line})\)

Figure 7. Ex. 3, the comparison of the results of \( u \) Eq. (52). Where \( h = -1 \). For different values of the HAM solution \( \alpha_1, \alpha_2, \alpha_3 \) \((\alpha_1 = \alpha_2 = \alpha_3 = 1 \text{ solid black line, } \alpha_1 = \alpha_2 = \alpha_3 = 0.98 \text{ red dashed line and } \alpha_1 = \alpha_2 = \alpha_3 = 0.95 \text{ blue dotted line})\)
Figure 8. Ex. 3, the comparison of the results of \( v \) Eq. (53). Where \( h = -1 \). For different values of the HAM solution \( \alpha_1, \alpha_2, \alpha_3 \) 
\( (\alpha_1 = \alpha_2 = \alpha_3 = 1 \) solid black line, \( \alpha_1 = \alpha_2 = \alpha_3 = 0.98 \) red dashed line and \( \alpha_1 = \alpha_2 = \alpha_3 = 0.95 \) blue dotted line).

Figure 9. Ex. 3, the comparison of the results of \( w \) Eq. (40). Where \( h = -1 \). For different values of the HAM solution \( \alpha_1, \alpha_2, \alpha_3 \) 
\( (\alpha_1 = \alpha_2 = \alpha_3 = 1 \) solid black line, \( \alpha_1 = \alpha_2 = \alpha_3 = 0.98 \) red dashed line and \( \alpha_1 = \alpha_2 = \alpha_3 = 0.95 \) blue dotted line).