STABILITY OF BASSET EQUATION

V. GOVINDARAJ, K. BALACHANDRAN

Abstract. In this paper, we discuss the stabilizability of Basset equation by using the duality results of controllability and observability of linear fractional dynamical systems and feedback control. The numerical results are provided to illustrate the effectiveness of the analytical approach.

1. Introduction

Consider the composite fractional relaxation equation with the proper initial condition

\[ a \dot{y}(t) + b C_{0+}^{\alpha} y(t) + c y(t) = f(t), \quad y(0) = y_0, \quad 0 < \alpha < 1, \quad t \geq 0, \]

where \( a \neq 0, b, c \) and \( y_0 \) are arbitrary real constants and \( f(t) \) denotes the forcing function [23]. Qualitative behavior of this system is very important and becomes more complicated making it being realized that fractional order systems are generalizations that include the integer-order systems as special cases. The fractional differential equation (1) with \( \alpha = 1/2 \) corresponds to the Basset problem, a classical problem in fluid dynamics concerning the unsteady motion of a particle accelerating a viscous fluid under the action of gravity. The qualitative behaviors of dynamical systems are controllability, observability and stability. The study of such qualitative behaviors in fractional dynamical systems are important issues for many applied problems because the use of fractional order derivatives and integrals in control theory leads to better results than those by integer order ones [16].

Stability is an important qualitative behavior of a dynamical system. The question of stability is of main interest in fractional order systems because the analysis of stability in fractional order systems is more complex than in integer order systems. The stability of solution is important in physical applications, because deviations in mathematical model inevitably result from errors in measurement. A stable solution will be usable despite such deviations. The analysis of stability of fractional system can be carried out by studying the solutions of the differential equations that characterize them. Recently the stability of linear fractional dynamical systems has attracted many researchers [14, 15, 20, 22, 26, 29–31].

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In applications of control theory, one of the basic objectives is to construct stable systems. If a system is not stable by itself, then the question arises whether it can be stabilized by choosing appropriate control inputs. This is called the stabilizability problem. In particular, we wish to construct a control law such that the system is brought to rest from any given initial position. In the field of fractional order control systems, one of the challenging problems related to stability theory is stabilizability. To the best of our knowledge, the results on the stabilizability of fractional-order linear and nonlinear systems are still relatively few. For example, Cheng and Hwang [10] studied the stabilization of unstable first-order time-delay systems using fractional-order proportional derivative controllers. Also they investigate how the fractional derivative of order $\mu$ in the range $(0, 2)$ affects the stabilizability of unstable fractional-order time-delay processes. Hamamci [12] discussed the stabilization of fractional-order time delay systems by using fractional-order $PI^\mu D^\mu$ controllers. Lu and Chen [17,18] derived the sufficient conditions for the robust stability and stabilization of a class of fractional-order linear systems as well as interval systems by using linear matrix inequalities. Chen et al. [9] established the asymptotic stability and asymptotic stabilizability of a class of nonlinear fractional-order systems by using feedback controls. However, there is no analytical proof to obtain the feedback gain matrix.

Motivated by the above, in this paper, we derive the stabilizability criteria of Basset equation by using the duality results of controllability and observability of linear fractional dynamical systems and the feedback control. This stabilization by using linear feedback control was well known and used before the development of modern control theory. The fundamental idea of feedback control is to use the information on inputs and outputs to choose the current value of the input. Finally numerical examples are provided to illustrate the theory.

2. Preliminaries

In this section, we introduce the definitions and preliminary results from fractional calculus which are used throughout this paper [11,13,25,27].

2.1. Basic Definitions. Let $\alpha \in \mathbb{C}$ with $n - 1 < \alpha \leq n$, $n \in \mathbb{N}$, $[a, b]$ be a finite interval of the real line $\mathbb{R}$ and $f(t) \in AC^n[a, b]$. The following definitions give well known fractional operators from fractional calculus.

**Definition 1** The left-sided and right-sided Caputo fractional derivative of order $\alpha$ is defined as

\[
(CD^\alpha_{a+}f)(t) = \frac{1}{\Gamma(n - \alpha)} \int_a^t (t - s)^{n-\alpha-1} f^n(s)ds,
\]

\[
(CD^\alpha_{b-}f)(t) = \frac{(-1)^n}{\Gamma(n - \alpha)} \int_t^b (s - t)^{n-\alpha-1} f^n(s)ds,
\]

provided they exist almost everywhere on $[a, b]$. In particular, when $0 < \Re(\alpha) < 1$,

\[
(CD^\alpha_{a+}f)(t) = \frac{1}{\Gamma(1 - \alpha)} \int_a^t (t - s)^{-\alpha} f'(s)ds,
\]

\[
(CD^\alpha_{b-}f)(t) = \frac{-1}{\Gamma(1 - \alpha)} \int_t^b (s - t)^{-\alpha} f'(s)ds.
\]
Definition 2 The Miller-Ross sequential fractional derivative is defined as
\[ f^{(k\alpha)}(x) := \left(D^{k\alpha} f\right)(x) = \left(D^\alpha D^{(k-1)\alpha} f\right)(x), \]
where \( k = 1, \cdots, n \), \((D^\alpha f)(x) = f(x)\) and \(D^\alpha\) is any fractional differential operator, for example, it could be \(D^\alpha_{a+}\).

Definition 3 The Mittag-Leffler function is a complex function which depends on two complex parameters \(\alpha, \beta \in \mathbb{C}\) defined by
\[ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C}, \quad \alpha, \beta > 0, \]
\[ E_{\alpha,1}(z) = E_{\alpha}(z) \text{ with } \beta = 1. \]

The Laplace transforms of some special types of Mittag-Leffler function are
\[ \mathcal{L}\left\{ t^{\beta-1} E_{\alpha,\beta}(\pm q t^\alpha) \right\}(s) = \frac{s^{\alpha-\beta}}{s^\alpha \mp q}, \quad q \in \mathbb{C}, \]
\[ \mathcal{L}\left\{ 2\sqrt{\frac{t}{\pi}} \mp 2q t E_{1/2}(\mp q \sqrt{t}) \right\}(s) = \frac{1}{s^{1/2} (s^{1/2} \pm q)^2}, \]
\[ \mathcal{L}\left\{ \mp 2\sqrt{\frac{t}{\pi}} + (1 + 2q^2 t) E_{1/2}(\mp q \sqrt{t}) \right\}(s) = \frac{1}{(s^{1/2} \pm q)^2}. \]

Definition 4 The Mittag-Leffler matrix function for an arbitrary square matrix \(A\) is
\[ E_{\alpha,\beta}(A) = \sum_{k=0}^{\infty} \frac{A^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0. \]

Consider the composite fractional relaxation equation represented by the following fractional differential equation:
\[ a\dot{y}(t) + b D_{0+}^{1/2} y(t) + cy(t) = d u(t), \quad t \in [0, T] = J, \]
\[ y(0) = y_0, \]
where \(d \in \mathbb{R}\) and the forcing function is taken in terms of control function \(u(t) \in L_2(J, \mathbb{R})\). Let us take the auxiliary variables as \(x_1 = y\) and \(x_2 = D_{0+}^{1/2} y\). Then
\[ C D_{0+}^{1/2} x_1 = x_2, \quad C D_{0+}^{1/2} x_2 = a^{-1} (du(t) - bx_2 - cx_1). \]
The system of equations can be written in the matrix form as
\[ C D_{0+}^{1/2} x(t) = Ax(t) + Bu(t), \quad t \in J, \]
where \(A = \begin{bmatrix} 0 & 1 \\ -c/a & -b/a \end{bmatrix}, B = \begin{bmatrix} 0 \\ d/a \end{bmatrix}\) and \(x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}\). Initial conditions are defined by
\[ x_1(0) = y(0) = y_0, \quad x_2(0) = C D_{0+}^{1/2} y_0 = 0. \]
Therefore \( x(0) = x_0 = \begin{bmatrix} y_0 \\ 0 \end{bmatrix} \). By using the Laplace transform of some special type of Mittag-Leffler functions, we obtain the Mittag-Leffler matrix function for a given matrix \( A \) as:

- When \( b^2 - 4ac \neq 0 \), we get \( E_{1/2}(At^{1/2}) = \begin{bmatrix} L_1(t) & L_2(t) \\ L_3(t) & L_4(t) \end{bmatrix} \) with

  \[
  L_1(t) = \frac{\mu_1 + b/a}{\mu_1 - \mu_2} E_{1/2}(\mu_1 t^{1/2}) + \frac{\mu_2 + b/a}{\mu_2 - \mu_1} E_{1/2}(\mu_2 t^{1/2})
  \]

  \[
  L_2(t) = \frac{1}{\mu_1 - \mu_2} \left[ E_{1/2}(\mu_1 t^{1/2}) - E_{1/2}(\mu_2 t^{1/2}) \right]
  \]

  \[
  L_3(t) = \frac{-c/a}{\mu_1 - \mu_2} \left[ E_{1/2}(\mu_1 t^{1/2}) - E_{1/2}(\mu_2 t^{1/2}) \right]
  \]

  \[
  L_4(t) = \frac{1}{\mu_1 - \mu_2} \left[ \mu_1 E_{1/2}(\mu_1 t^{1/2}) - \mu_2 E_{1/2}(\mu_2 t^{1/2}) \right],
  \]

  where \( \mu_1 = -b + \sqrt{b^2 - 4ac} \) and \( \mu_2 = -b - \sqrt{b^2 - 4ac} \).

- When \( b^2 - 4ac = 0 \), we get \( E_{1/2}(At^{1/2}) = \begin{bmatrix} 2aM_1(t) & M_2(t) \\ M_3(t) & M_4(t) \end{bmatrix} \) with

  \[
  M_1(t) = -2 \sqrt{\frac{t}{\pi}} + \left( 1 + \frac{b^2}{2a^2} t \right) E_{1/2} \left( -\frac{bt^{1/2}}{2a} \right) + \frac{b}{a} \left[ 2 \sqrt{\frac{t}{\pi}} - \frac{bt}{a} E_{1/2} \left( -\frac{bt^{1/2}}{2a} \right) \right]
  \]

  \[
  M_2(t) = 2 \sqrt{\frac{t}{\pi}} - \frac{bt}{a} E_{1/2} \left( -\frac{bt^{1/2}}{2a} \right)
  \]

  \[
  M_3(t) = -\frac{c}{a} \left[ 2 \sqrt{\frac{t}{\pi}} - \frac{bt}{a} E_{1/2} \left( -\frac{bt^{1/2}}{2a} \right) \right]
  \]

  \[
  M_4(t) = -2 \sqrt{\frac{t}{\pi}} + \left( 1 + \frac{b^2}{2a^2} t \right) E_{1/2} \left( -\frac{bt^{1/2}}{2a} \right).
  \]

The solution of the system (3) is

\[
 x(t) = E_{1/2}(At^{1/2})x_0 + \int_0^t (t-s)^{-1/2} E_{1/2,1/2}(A(t-s)^{1/2}) Bu(s) ds.
\]

Using this solution representation, we have to study the qualitative behaviors of the dynamical system (3).

2.2. Controllability Result. First we study the controllability of the fractional dynamical system (3). The problem of controllability of fractional dynamical systems has been extensively studied by many authors [18, 21, 24, 32].

**Definition 5** The system (3) is controllable on \( J \) if, for every pair of vectors \( x_0, x_1 \in \mathbb{R}^2 \), there is a control \( u(t) \in L^2(J, \mathbb{R}) \) such that the solution \( x(t) \) of (3) satisfies the conditions \( x(0) = x_0 \) and \( x(T) = x_1 \).
Define the controllability Grammian matrix $M$ as
\[
M = \int_0^T E_{1/2,1/2} (A(T - \tau)^{1/2}) B B' E_{1/2,1/2} (A'(T - \tau)^{1/2}) d\tau
\]
and take the control as
\[
u(t) = (T - t)^{1/2} B' E_{1/2,1/2} (A'(T - t)^{1/2}) M^{-1} \left[ x_1 - E_{1/2} (A T^{1/2}) x_0 \right].
\] (4)
where the symbol $'$ denotes the matrix transpose. Observe that the control (4) steers the system (3) from $x_0$ to $x_1$. Further the system (3) is controllable if and only if the controllability Grammian $M$ is positive definite.

2.3. Observability Result. Observability is defined as the possibility to deduce the initial state of the system from observing its input-output behavior. This means that, from the system’s outputs, it is possible to determine the behavior of the entire system. Several authors (see, for instance [5, 8, 19, 33]), have established the results for observability of linear fractional dynamical systems using Grammian matrix and rank conditions.

Consider the linear time invariant fractional order system with right sided Caputo fractional derivative given by
\[C^{D_{\alpha}^1/2}_T x(t) = A' x(t), \quad t \in J.\] (5)
and the linear observation
\[z(t) = B' x(t).\] (6)

**Definition 6** The system (5), (6) is observable on an interval $J$ if
\[z(t) = B' x(t) = 0, \quad t \in J\]
implies
\[x(t) = 0, \quad t \in J.\]

**Lemma 1** The linear system (5), (6) is observable on $J$ if and only if the observability Grammian matrix
\[W = \int_0^T E_{1/2} (A(T - \tau)^{1/2}) B B' E_{1/2} (A'(T - \tau)^{1/2}) d\tau\] (7)
is positive definite.

2.4. Duality Result. In order to discuss the duality results of controllability and observability, the adjoint of fractional derivative $C^{D_{\alpha}^\alpha}_0$ is $C^{D_{\alpha}^\alpha}_{T-}$ for $0 < \alpha \leq 1$ defined on $[0, T]$ (see [21] [28]). Note that the adjoint of the system $C^{D_{\alpha}^1/2}_0 x(t) = A x(t)$ is $C^{D_{\alpha}^1/2}_T x(t) = A' x(t)$.

**Lemma 2** The system (3) is controllable on $J$ if and only if the adjoint linear system
\[C^{D_{\alpha}^1/2}_T z(t) = A' z(t),\] (8)
\[w(t) = B' z(t),\] (9)
is observable on $J$. 

2.5. Stability Result. Consider the uncontrolled homogeneous system of (3) namely
\[ C D_1^{1/2} x(t) = Ax(t), \] (10)
where the matrix \( A \) is defined as above.

**Definition 7** The system (10) is said to be
- asymptotically stable iff \( |\arg(spec(A))| > \pi/4 \). In this case, the components of the state decay towards 0 like \( t^{-\alpha} \).
- stable iff either it is asymptotically stable or those critical eigenvalues which satisfy \( |\arg(spec(A))| = \pi/4 \) have geometric multiplicity one.

In order to study the stability of system (10), we find the eigenvalues of \( A \) to be
\[ \lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \]

Our aim is to find for what values of \( a, b \) and \( c \), the eigenvalues must satisfy \( |\arg(\lambda_1)| \geq \pi/4 \) and \( |\arg(\lambda_2)| \geq \pi/4 \). Stability results of system (10) with initial condition \( x_0 = [1 \ 0]' \) can be classified in the following three cases:

**CASE(1):** Let \( b^2 - 4ac = 0 \), then eigenvalues of \( A \) are real and identical.
- If \( a > 0 \) and \( b > 0 \), then the eigenvalues \( \lambda_1 = \lambda_2 = \frac{-b}{a} \) satisfy respectively \( |\arg(\lambda_1)| > \pi/4 \) and \( |\arg(\lambda_2)| > \pi/4 \). Hence the system (10) is asymptotically stable (see Fig. 1).
- If \( a < 0 \) and \( b < 0 \), then the eigenvalues \( \lambda_1 = \lambda_2 = \frac{-b}{a} \) satisfy respectively \( |\arg(\lambda_1)| > \pi/4 \) and \( |\arg(\lambda_2)| > \pi/4 \). Hence the system (10) is asymptotically stable (see Fig. 2).

In this case, for other combination of \( a \) and \( b \) the system (10) is unstable. In the following figures, \( x_1 \) indicates the state variable and \( x_2 \) indicates the pseudostate variable without any real interpretation till the moment.

**Fig. 1.** If \( a = 1, b = 2 \) and \( c = 1 \), then the system (10) is asymptotically stable.

**Fig. 2.** If \( a = -2, b = -1 \) and \( c = -1/8 \), then the system (10) is asymptotically stable.
Case(iii): Let \( b^2 - 4ac > 0 \), then eigenvalues of \( A \) are real and distinct.

- If \( a > 0, b > 0 \) and \( b > \sqrt{b^2 - 4ac} \), then the eigenvalues \( \lambda_1 \) and \( \lambda_2 \) satisfy
  \[ |\arg(\lambda_1)| > \pi/4 \] and \[ |\arg(\lambda_2)| > \pi/4 \]. Hence the system \( (10) \) is asymptotically stable (see Fig. 3).

- If \( a < 0, b < 0 \) and \( -b > \sqrt{b^2 - 4ac} \), then the eigenvalues \( \lambda_1 \) and \( \lambda_2 \) satisfy
  \[ |\arg(\lambda_1)| > \pi/4 \] and \[ |\arg(\lambda_2)| > \pi/4 \]. Hence the system \( (10) \) is asymptotically stable (see Fig. 4).

In this case, for other combination of \( a \) and \( b \) the system \( (10) \) is unstable.

![Fig. 3](image1.png)

**Fig. 3.** If \( a = 1, b = 3 \) and \( c = 1 \), then the system \( (10) \) is asymptotically stable.

![Fig. 4](image2.png)

**Fig. 4.** If \( a = -2, b = -4 \) and \( c = -1 \), then the system \( (10) \) is asymptotically stable.

Case(iii): Let \( b^2 - 4ac < 0 \), then eigenvalues of \( A \) are complex.

- If \( a > 0 \) and \( b > 0 \), then the eigenvalues \( \lambda_1 \) and \( \lambda_2 \) satisfy
  \[ |\arg(\lambda_1)| > \pi/4 \] and \[ |\arg(\lambda_2)| > \pi/4 \]. Hence the system \( (10) \) is asymptotically stable (see Fig. 5).

- If \( a < 0 \) and \( b < 0 \), then the eigenvalues \( \lambda_1 \) and \( \lambda_2 \) satisfy
  \[ |\arg(\lambda_1)| > \pi/4 \] and \[ |\arg(\lambda_2)| > \pi/4 \]. Hence the system \( (10) \) is asymptotically stable (see Fig. 6).

- If \( a > 0, b < 0 \) and \( \frac{\pi}{2} < |\tan^{-1}\left(\frac{\pm\sqrt{b^2 - 4ac}}{b}\right)| < \frac{\pi}{2} \), then the eigenvalues \( \lambda_1 \)
  and \( \lambda_2 \) satisfy \[ |\arg(\lambda_1)| > \pi/4 \] and \[ |\arg(\lambda_2)| > \pi/4 \]. Hence the system \( (10) \)
  is asymptotically stable (see Fig. 7).

- If \( a < 0, b > 0 \) and \( \frac{\pi}{2} < |\tan^{-1}\left(\frac{\pm\sqrt{b^2 - 4ac}}{b}\right)| < \frac{\pi}{2} \), then the eigenvalues \( \lambda_1 \)
  and \( \lambda_2 \) satisfy \[ |\arg(\lambda_1)| > \pi/4 \] and \[ |\arg(\lambda_2)| > \pi/4 \]. Hence the system \( (10) \)
  is asymptotically stable (see Fig. 8).

- Let \( b = 0 \), then the eigenvalues becomes \( \lambda_1 = i\sqrt{\frac{4ac}{b}} \) and \( \lambda_2 = -i\sqrt{\frac{4ac}{b}} \). Then
  they satisfy \[ |\arg(\lambda_1)| > \pi/4 \] and \[ |\arg(\lambda_2)| > \pi/4 \]. Hence the system \( (10) \)
  is asymptotically stable (see Fig. 9).

- If \( a > 0, b < 0 \) and the eigenvalues \( \lambda_1 \) and \( \lambda_2 \) satisfy
  \[ |\arg(\lambda_1)| = \pi/4 \] and \[ |\arg(\lambda_2)| = \pi/4 \], then the system \( (10) \) is periodically stable (see Fig. 10).
• If $a < 0$, $b > 0$ and the eigenvalues $\lambda_1$ and $\lambda_2$ satisfy $|\arg(\lambda_1)| = \pi/4$ and $|\arg(\lambda_2)| = \pi/4$, then the system (10) is periodically stable (see Fig. 11).

In this case, for other combination of $a$ and $b$ the system (10) is unstable.

Fig. 5. If $a = 5$, $b = 2$ and $c = 3$, then the system (10) is asymptotically stable.

Fig. 6. If $a = -3$, $b = -2$ and $c = -4$, then the system (10) is asymptotically stable.

Fig. 7. If $a = 2$, $b = -3$ and $c = 3$, then the system (10) is asymptotically stable.

Fig. 8. If $a = -3$, $b = 2$ and $c = -1$, then the system (10) is asymptotically stable.
### 3. Main Result

Suppose the uncontrolled homogeneous system (10) fails to be asymptotically stable. Any system described, for example, by a differential equation of the evolution type, can be converted into a control system by adding an input variable representing the action of some controller upon the system. Then expression (10) turns into the form of (3).

One of the tasks of the control analyst is to use the control $u$ in such a way that the system is made as stable. Because of simplicity for both implementation and analysis, the traditionally favored means for accomplishing this objective is the use of a linear feedback control

$$u(t) = Kx(t), \quad (11)$$

where the control $u(t)$ is determined as a linear function of the current state $x(t)$. The state variable summarizes all past information that is relevant for the future. The problem now becomes that of choosing the $1 \times 2$ feedback matrix $K$ in such a way that the modified homogeneous system realized by substituting (11) into (3),
namely,

\[ C_D^{1/2} x(t) = (A + BK)x(t) \]

is asymptotically stable, that is, \(|\arg(spec(A + BK))| > \pi/4\).

**Definition 8**  The system \((3)\) is stabilizable if there exists an \(1 \times 2\) matrix \(K\) such that \(|\arg(spec(A + BK))| > \pi/4\).

**Theorem 1**  If the system \((3)\) is controllable, then it is stabilizable.

**Proof.**  Assume that the system \((3)\) is controllable. Then the adjoint systems \((8)\) and \((9)\) are observable and the corresponding observability Grammian matrix \((7)\) is positive definite for \(T > 0\). Then, for each \(T > 0\), the linear feedback control law

\[ u = B'W^{-1}E_{1/2}(A'T^{1/2})x = Kx \]

stabilizes \((3)\). Now we compute

\[
AW + WA' = \int_0^T AE_{1/2}(A(T - \tau)^{1/2})BB'E_{1/2}(A'(T - \tau)^{1/2})d\tau \\
+ \int_0^T E_{1/2}(A(T - \tau)^{1/2})BB'E_{1/2}(A'(T - \tau)^{1/2})A'd\tau. \quad (12)
\]

Since \(W\) is symmetric and positive definite for \(T > 0\), we can modify \((12)\) as follows:

\[
\left[ A + BB'W^{-1}E_{1/2}(A'T^{1/2}) \right] W + W \left[ A + BB'W^{-1}E_{1/2}(A'T^{1/2}) \right]' \\
= BB'W^{-1}E_{1/2}(A'T^{1/2})W + WE_{1/2}(A'T^{1/2})W^{-1}BB' + \int_0^T AE_{1/2}(A(T - \tau)^{1/2})BB' \\
\times E_{1/2}(A'(T - \tau)^{1/2})d\tau + \int_0^T E_{1/2}(A(T - \tau)^{1/2})BB'E_{1/2}(A'(T - \tau)^{1/2})A'd\tau \\
\geq BB'W^{-1}E_{1/2}(A'T^{1/2})W + WE_{1/2}(A'T^{1/2})W^{-1}BB' + \int_0^T AE_{1/2}(A(T - \tau)^{1/2})BB' \\
\times E_{1/2}(A'(T - \tau)^{1/2})d\tau + \int_0^T E_{1/2}(A(T - \tau)^{1/2})BB'E_{1/2}(A'(T - \tau)^{1/2})A'd\tau. \quad (13)
\]

We add and subtract the integrals \(\int_0^T AE_{1/2}(A(T - \tau)^{1/2})BB'E_{1/2}(A'(T - \tau)^{1/2})d\tau\) and \(\int_0^T E_{1/2}(A(T - \tau)^{1/2})BB'E_{1/2}(A'(T - \tau)^{1/2})A'd\tau\) on R.H.S of \((13)\). Then we get
where $A$ have negative real part. That is
\[
A + BB'W^{-1}E_{1/2}(A'T^{1/2})W + W \left[ A + BB'W^{-1}E_{1/2}(A'T^{1/2}) \right]' \geq BB'W^{-1}E_{1/2}(A'T^{1/2})W + WE_{1/2}(AT^{1/2})W^{-1}BB' + \int_{0}^{T} AE_{1/2}(A(T - \tau)^{1/2}) \times BB'E_{1/2}(A'(T - \tau)^{1/2})d\tau + \int_{0}^{T} E_{1/2}(A(T - \tau)^{1/2})BB'E_{1/2}(A'(T - \tau)^{1/2})A'd\tau + \int_{0}^{T} \left[ E_{1/2}(A(T - \tau)^{1/2}) - E_{1/2}(A(T - \tau)^{1/2}) \right] BB'E_{1/2}(A'(T - \tau)^{1/2})d\tau + \int_{0}^{T} \left[ E_{1/2}(A(T - \tau)^{1/2}) - E_{1/2}(A(T - \tau)^{1/2}) \right] BB'E_{1/2}(A'(T - \tau)^{1/2})A'd\tau \geq \int_{0}^{T} A \left[ E_{1/2}(A(T - \tau)^{1/2}) - E_{1/2}(A(T - \tau)^{1/2}) \right] BB'E_{1/2}(A'(T - \tau)^{1/2})d\tau + \int_{0}^{T} \left[ E_{1/2}(A(T - \tau)^{1/2}) - E_{1/2}(A(T - \tau)^{1/2}) \right] BB'E_{1/2}(A'(T - \tau)^{1/2})A'd\tau = \int_{0}^{T} A \left[ E_{1/2}(A(T - \tau)^{1/2}) - E_{1/2}(A(T - \tau)^{1/2}) \right] BB'E_{1/2}(A'(T - \tau)^{1/2})d\tau + \int_{0}^{T} \left[ E_{1/2}(A(T - \tau)^{1/2}) - E_{1/2}(A(T - \tau)^{1/2}) \right] BB'E_{1/2}(A'(T - \tau)^{1/2})A'd\tau + \int_{0}^{T} AE_{1/2}(A(T - \tau)^{1/2})BB' \left[ E_{1/2}(A'(T - \tau)^{1/2}) - E_{1/2}(A'(T - \tau)^{1/2}) \right] d\tau - \int_{0}^{T} AE_{1/2}(A(T - \tau)^{1/2})BB' \left[ E_{1/2}(A'(T - \tau)^{1/2}) - E_{1/2}(A'(T - \tau)^{1/2}) \right] d\tau \geq \int_{0}^{T} \left[ E_{1/2}(A(T - \tau)^{1/2}) - E_{1/2}(A(T - \tau)^{1/2}) \right] BB'E_{1/2}(A'(T - \tau)^{1/2})A'd\tau + \int_{0}^{T} AE_{1/2}(A(T - \tau)^{1/2})BB' \left[ E_{1/2}(A'(T - \tau)^{1/2}) - E_{1/2}(A'(T - \tau)^{1/2}) \right] d\tau.
\]

Then
\[
A + BB'W^{-1}E_{1/2}(A'T^{1/2})W + W \left[ A + BB'W^{-1}E_{1/2}(A'T^{1/2}) \right]' + Q \geq 0,
\]
where
\[
Q = \int_{0}^{T} \left[ E_{1/2}(A(T - \tau)^{1/2}) - E_{1/2}(A(T - \tau)^{1/2}) \right] BB'E_{1/2}(A'(T - \tau)^{1/2})A'd\tau + \int_{0}^{T} AE_{1/2}(A(T - \tau)^{1/2})BB' \left[ E_{1/2}(A'(T - \tau)^{1/2}) - E_{1/2}(A'(T - \tau)^{1/2}) \right] d\tau.
\]

Since the $n \times n$ matrix $Q$ is symmetric. If $Q$ is positive definite, for $T > 0$, then $A + BB'W^{-1}E_{1/2}(A'T^{1/2})$ is the stability matrix. So all the eigenvalues of $A + BK$ have negative real part. That is
\[
\text{arg}(\text{spec}(A + BK)) > \pi/2 > \pi/4
\]

Therefore, system is stabilizable.

**Remark 1** Theorem (1) shows that every controllable system is stabilizable. The converse is not true. A system
\[
^cD_{0+}^{1/2}x(t) = Ax(t) + 0u(t)
\]
with $|\arg(\text{spec}(A))| = \pi/4$ is stable but clearly not controllable. For integer order system, see [2].

### 4. Example

In this section we apply the results established in the previous section to the following fractional dynamical systems.

**Example 1** Consider the linear control fractional dynamical system (2) with constants $a = 1, b = -2, c = -1$ and $y_0 = 1$. The problem can be expressed as

$$^CD_{0+}^{1/2}x(t) = Ax(t) + Bu(t), \quad t \in [0, 1],$$

(15)

where $A = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, B = \begin{bmatrix} 0 \\ d \end{bmatrix}$ with initial condition $x(0) = x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then the Mittag-Leffler matrix function for given matrix $A$ is

$$E_{1/2}(A^{1/2}) = \begin{bmatrix} N_1(t) \\ N_2(t) \\ N_3(t) \end{bmatrix},$$

where

$$N_1(t) = -\frac{1 + \sqrt{2}}{2\sqrt{2}}E_{1/2}((1 + \sqrt{2})t^{1/2}) + \frac{1 + \sqrt{2}}{2\sqrt{2}}E_{1/2}((1 - \sqrt{2})t^{1/2}),$$

$$N_2(t) = \frac{1}{2\sqrt{2}}[E_{1/2}((1 + \sqrt{2})t^{1/2}) - E_{1/2}((1 - \sqrt{2})t^{1/2})],$$

$$N_3(t) = \frac{1 + \sqrt{2}}{2\sqrt{2}}E_{1/2}((1 + \sqrt{2})t^{1/2}) - \frac{1 - \sqrt{2}}{2\sqrt{2}}E_{1/2}((1 - \sqrt{2})t^{1/2}).$$

When there is no control term in (15), it becomes

$$^CD_{0+}^{1/2}x(t) = Ax(t).$$

(16)

The eigenvalues of the matrix $A$ are $\lambda_1 = -0.4142$ and $\lambda_2 = 2.4142$.

$$|\arg(\lambda_1)| = \left|\tan^{-1}\left(\frac{0}{-0.4142}\right)\right| = \pi > \pi/4,$$

$$|\arg(\lambda_2)| = \left|\tan^{-1}\left(\frac{0}{2.4142}\right)\right| = 0 < \pi/4.$$

Therefore, the system $^CD_{0+}^{1/2}x(t) = Ax(t)$ is unstable (see Fig. 12).
Let \( d = 1 \); then the controllability Grammian matrix for the system (15) is

\[
M = \int_0^1 E_{1/2,1/2}(A(1 - \tau)^{1/2})BB' E_{1/2,1/2}(A'(1 - \tau)^{1/2}) d\tau \\
= \begin{bmatrix}
6639.9244 & 16018.5853 \\
16018.5853 & 386442.9987
\end{bmatrix},
\]

since it is positive definite, then the system (15) is controllable on \([0, 1]\). This implies that the adjoint system of (15) is observable on \([0, 1]\). Then it’s observability Grammian matrix is

\[
W = \int_0^1 E_{1/2}(A(1 - \tau)^{1/2})BB' E_{1/2}(A'(1 - \tau)^{1/2}) d\tau \\
= \begin{bmatrix}
1784.8502 & 4327.7131 \\
4327.7131 & 10493.7562
\end{bmatrix} > 0.
\]

From, by Theorem (1), the matrix

\[
Q = \int_0^1 \left[ E_{1/2}(A(1 - \tau)^{1/2}) - E_{1/2,2}(A(1 - \tau)^{1/2}) \right] BB' E_{1/2}(A'(1 - \tau)^{1/2}) A' d\tau \\
+ \int_0^1 AE_{1/2}(A(1 - \tau)^{1/2})BB' \left[ E_{1/2}(A'(1 - \tau)^{1/2}) - E_{1/2,2}(A'(1 - \tau)^{1/2}) \right] d\tau.
\]

is symmetric and positive definite; then the stabilizing feedback matrix \( K \) is

\[
B' W^{-1} E_{1/2}(A'T^{1/2}) = \begin{bmatrix}
-5.6031 \\
-1.6495
\end{bmatrix}.
\]

Using the control law

\[
u = B' W^{-1} E_{1/2}(A'T^{1/2}) x(t) = \begin{bmatrix}
-5.6031 \\
-1.6495
\end{bmatrix} x(t),
\]

we obtain the system with matrix \( A + BB' W^{-1} E_{1/2}(A'T^{1/2}) \) as

\[
CD_{0+}^{1/2} x(t) = \begin{bmatrix}
0 & 1 \\
-4.6031 & 0.3505
\end{bmatrix} x(t).
\]

The eigenvalues of the above matrix are \( \lambda_3 = 0.1753 + 2.1383 i \) and \( \lambda_4 = 0.1753 - 2.1383 i \).

\[
|\arg(\lambda_3)| = \left| \tan^{-1} \left( \frac{2.1383}{0.1753} \right) \right| = 1.4890 > \pi/4,
\]

\[
|\arg(\lambda_4)| = \left| \tan^{-1} \left( \frac{-2.1383}{0.1753} \right) \right| = 1.4890 > \pi/4.
\]

Therefore, the system (17) is stable (see Fig. 13) and consequently, the system (15) is stabilizable.
5. Conclusion

The stability results of Basset equation with different kinds of arbitrary real constants have been investigated in detail. For other values, this equation fails to be stable. So we wish to construct a control law such that the unstable system is made to be stable. This process is called the stabilizing process. The stabilizability results of Basset equation is established by using the duality results of controllability and observability of linear fractional dynamical systems and feedback control. Numerical examples are provided to illustrate the theory.

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V. Govindaraj
Department of Mathematics, Bharathiar University, Coimbatore, 641 046, India.
E-mail address: govindaraj.maths@gmail.com

K. Balachandran
Department of Mathematics, Bharathiar University, Coimbatore, 641 046, India.
E-mail address: kb.maths.bu@gmail.com