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NOTES ON SOME FRACTIONAL CALCULUS OPERATORS AND THEIR PROPERTIES

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Abstract. Here we state the main properties of the Caputo, Riemann-Liouville and the Caputo via Riemann-Liouville fractional derivatives and give some general notes on these properties. Some properties given in some recent literatures and used to solve fractional nonlinear partial differential equations will be proved that they are incorrect by giving some counter examples.

1. Introduction

The classical calculus provides a power tool to model and explain many important dynamically processes in most parts of applied areas of the sciences. But There are many complex systems in nature with anomalous dynamics, including biology, chemistry, physics, geology, astrophysics and social sciences, and more in particular in transport of chemical contaminant through water around rocks, dynamics of viscoelastic materials as polymers, signals theory, control theory, electromagnetic theory, and many more their dynamics cannot be characterized by classical derivative models. (for detail [4, 9, 15, 18, 20, 25]).

Fractional calculus is one of the generalizations of the classical calculus and it has been used successfully in various fields of science and engineering.

Really there are New possibilities in mathematics and theoretical physics appear, when the order of the differential operator or the integral operator becomes an arbitrary parameter. The fractional calculus is a powerful tool to describe physical systems that have long-term memory and long-range spatial interactions (see [9, 15, 18, 20, 19, 22]).

In this paper we are concerned with general properties and some notes on Riemann-Liouville fractional integral, Riemann-Liouville fractional derivative, Caputo fractional derivative and Caputo via Riemann-Liouville fractional derivative which are the most famous definitions in fractional calcuclus (see [18, 20, 19, 22]).

This paper is organized as follows. In Section 2 some basic definitions and properties of fractional derivatives are given. Fractional mild solution will be given in

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section 3 with giving the difference between it and strong solution. In section 4 we will give some incorrect properties of the Caputo via Riemann-Liouville fractional derivatives. Finally two incorrect methods for solving fractional differential equations will be given in Section 5.

2. Preliminaries

Here we give the definitions of the famous four fractional operators with giving the main properties of them (for detail see [18, 20, 22]).

Definition 2.1 (Left and right Riemann-Liouville fractional integral)
If $f(t) \in L_1(a,b)$, the set of all integrable functions, and $\alpha > 0$ then the left and right Riemann-Liouville (RL) fractional integral of order α, denoted respectively by ${}_aI_t^\alpha$ and ${}_tI_b^\alpha$, are defined by

$$\begin{aligned} {}_aI_t^\alpha f(t) &= \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1} f(\tau)d\tau, \\ {}_tI_b^\alpha f(t) &= \frac{1}{\Gamma(\alpha)}\int_t^b (\tau-t)^{\alpha-1} f(\tau)d\tau. \end{aligned}$$

Definition 2.2 (Left and right RL fractional derivative)
For $\alpha > 0$ the left and right RL fractional derivative of order α, denoted respectively by ${}_a^RD_t^\alpha$ and ${}_t^RD_b^\alpha$, are defined by

$$\begin{aligned} {}_a^RD_t^\alpha f(t) &= \frac{1}{\Gamma(n-\alpha)} D^n \int_a^t (t-\tau)^{n-\alpha-1} f(\tau)d\tau, \\ {}_t^RD_b^\alpha f(t) &= \frac{1}{\Gamma(n-\alpha)} (-D)^n \int_t^b (\tau-t)^{n-\alpha-1} f(\tau)d\tau \end{aligned}$$

where n is such that $n-1 < \alpha < n$ and $D = \frac{d}{dt}$

Definition 2.3 (Left and right Caputo fractional derivative)
For $\alpha > 0$ the left and right Caputo fractional derivative of order α, denoted respectively by ${}_a^CD_t^\alpha$ and ${}_t^CD_b^\alpha$, are defined by

$$\begin{aligned} {}_a^CD_t^\alpha f(t) &= \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} D^n f(\tau)d\tau, \\ {}_t^CD_b^\alpha f(t) &= \frac{1}{\Gamma(n-\alpha)} \int_t^b (\tau-t)^{n-\alpha-1} (-D)^n f(\tau)d\tau \end{aligned}$$

where n is such that $n-1 < \alpha < n$ and $D = \frac{d}{d\tau}$

Definition 2.4 (Caputo via Riemann-Liouville fractional derivative)
For $\alpha > 0$ the left and right Caputo via Riemann-Liouville fractional derivative of order α, denoted respectively by ${}_a^{CVR}D_t^\alpha$ and ${}_t^{CVR}D_b^\alpha$, are defined by

$$\begin{aligned} {}_a^{CVR}D_t^\alpha f(t) &= {}_a^RD_t^\alpha (f(t) - f(0)), \\ {}_t^{CVR}D_b^\alpha f(t) &= {}_t^RD_b^\alpha (f(t) - f(0)) \end{aligned}$$

where n is such that $n-1 < \alpha < n$ and $D = \frac{d}{d\tau}$

Now consider only the left fractional integral and derivative with $J=[0,T]$. For simplicity denote by $I^\alpha,\ {}^RD^\alpha,\ {}^CD^\alpha,\ {}^{CVR}D^\alpha$ instead of ${}_0I_t^\alpha,\ {}_0^RD_t^\alpha,\ {}_0^CD_t^\alpha,\ {}_0^{CVR}D_t^\alpha$.

First we give the spaces in which these operators can be defined (see [22]):

Lemma 2.5

Let $\alpha\in(0,1)$ and $J=[0,T]$. Denote by $AC(J)$ to the set of all absolutely continuous functions on J. we have that

(1) If $f\in AC(J)$, then $I^\alpha f\in AC(J)$;
(2) If $f\in C(J)$, set of all continuous functions on J, then $I^\alpha f\in C(J)$;
(3) If f is Riemann integrable, then $I^\alpha f$ exists for all $tinJ$;
(4) If $f\in L^1(J)$, then $I^\alpha f$ exists almost every where and $I^\alpha f\in L^1(J)$;
(5) The Riemann-Liouville fractional derivative of $f(t)$ exists if $I^{1-\alpha}f\in AC(J)$ and then we get ${}^RD^\alpha f\in L^1(J)$,
(6) The Caputo fractional derivative of $f(t)$ exists if $f\in AC(J)$ and then we get ${}^CD^\alpha f\in L^1(J)$,
(7) The Caputo via Riemann-Liouville fractional derivative of $f(t)$ exists if $I^{1-\alpha}f\in AC(J)$ and then we get ${}^{CVR}D^\alpha f\in L^1(J)$,

The RL fractional integral has a semigroup property and some limit properties given in the following lemma

Lemma 2.6

Let $\beta,\gamma\in\mathbb{R}^+$. Then we have

(1) If $f(t)\in L^1$, then $I^\gamma I^\beta f(t)=I^{\gamma+\beta}f(t)$,
(2) $\lim_{\beta\to n} I^\beta f(t)=I^n f(t)$ uniformly on $[a,b],\ n=1,2,3,...$ where $I^1f(t)=\int_0^t f(s)ds$,
(3) $\lim_{\beta\to 0} I_a^\beta f(t)=f(t)$, weakly (in the sense of distribution),
(4) If $f(t)$ is bounded and measurable the then $I^\alpha f(t)|_{t=0}=0$.

Now we give the relations between the three fractional derivatives for $\alpha\in(0,1)$.

Lemma 2.7

(1) If $f\in AC(J)$ and $f(0)=0$, then ${}^CD^\alpha f(t)={}^RD^\alpha f(t)$;
(2) If $f\in AC(J)$ then ${}^RD^\alpha f(t)={}^CD^\alpha f(t)+\frac{t^{-\alpha}}{\Gamma(1-\alpha)}f(0)$;
(3) If $f\in AC(J)$ then ${}^{CVR}D^\alpha f(t)={}^CD^\alpha f(t)$.

In the following lemma we give some important properties to the fractional Caputo derivative with the RL fractional integral

Lemma 2.8

Let $\alpha,\beta\in(0,1)$, then:

(1) $DI^\beta f(t)=I^\beta Df(t)$, if $f(0)=0$,
(2) If $Df(t)$ is absolutely continuous and $\alpha+\beta\le 1$, then ${}^CD^\alpha\,{}^CD^\beta f(t)={}^CD^{\alpha+\beta}f(t)$,
(3) If $1<\alpha+\beta\le 2$ and $Df(t)$ is absolutely continuous and $Df(t)|_{t=0}=0$, then ${}^CD^\alpha\,{}^CD^\beta f(t)={}^CD^{\alpha+\beta}f(t)$,
(4) If $f(t)$ is absolutely continuous and $\beta>\alpha$, then ${}^CD^\alpha I^\beta f(t)=I^{\beta-\alpha}f(t)$,
(5) If $f(t)$ is absolutely continuous, $f(0)=0$ and $\alpha>\beta$, then ${}^CD^\alpha I^\beta f(t)={}^CD^{\alpha-\beta}f(t)$,
(6) If $f(t)$ is absolutely continuous, then $I^\beta\,{}^CD^\alpha f(t)={}^CD^{\alpha-\beta}f(t),\alpha>\beta$,
(7) If $f(t)$ is absolutely continuous and $f(0)=0$, then $I^\beta\,{}^CD^\alpha f(t)=I^{\beta-\alpha}f(t),\ \beta>\alpha$.

A continuation property of Caputo fractional derivative is one of the most important properties of it which given in the following lemma

Lemma 2.9

Let $\alpha \in (0,1)$ then if if $f(t)$ is absolutely continuous on J we get

(1) $\lim_{\alpha\to 1^-}\ {}^{C}D^{\alpha} f(t) = \frac{d}{dt} f(t)$,
(2) $\lim_{\alpha\to 0}\ D^{\alpha}\ f(t) = f(t) - f(0)$.

Laplace transform is one of the methods used to solve fractional differential equations. In the following lemma we give the Laplace transform to each fractional operator

Lemma 2.10

The Laplace transform of the fractional operators with the parameter λ and $0 < \alpha < 1$ are given by

(1) $L(I^{\alpha} f(t)) = \lambda^{-\alpha} L(f(t))$
(2) $L(\ {}^{C}D^{\alpha} f(t)) = \lambda^{\alpha} L(f(t)) - \lambda^{\alpha-1} f(0)$
(3) for $f \in AC(J)$ we get $L(\ {}^{R}D^{\alpha} f(t)) = \lambda^{\alpha} L(f(t))$
(4) for $f \notin AC(J)$ and $I^{1-\alpha} f \in AC(J)$ then we have $L(\ {}^{R}D^{\alpha} f(t)) = \lambda^{\alpha} L(f(t)) - (I^{1-\alpha} f(t)|_{t=0}$
(5) for $f \in AC(J)$ we get $L(\ {}^{CVR}D^{\alpha} f(t)) = \lambda^{\alpha} L(f(t)) - \lambda^{\alpha-1} f(0)$
(6) for $f \notin AC(J)$ and $I^{1-\alpha} f \in AC(J)$ then we have $L(\ {}^{CVR}D^{\alpha} f(t)) = \lambda^{\alpha} L(f(t)) - \lambda^{\alpha-1} f(0) - (I^{1-\alpha}(f(t) - f(0)))|_{t=0}$

Finally we give the value of fractional integration and derivatives to the constant function and to the function t^{β} in the following example

Example 2.11

(1) if $f(t) = k \neq 0$, k is a constant, then ${}^{C}D^{\alpha} k = {}^{CVR}D^{\alpha} k = 0$ and ${}^{R}D^{\alpha} k = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} k$;
(2) ${}^{R}D^{\alpha} t^{\alpha-1} = 0$;
(3) for $\beta > -1$ we have $I^{\alpha} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta}$;
(4) for $\beta > -1 + \alpha$ we have ${}^{R}D^{\alpha} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(1+\beta-\alpha)} t^{\beta-\alpha}$;
(5) for $\beta > 0$ we have ${}^{C}D^{\alpha} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(1+\beta-\alpha)} t^{\beta-\alpha}$;
(6) for $\beta > 0$ we have ${}^{CVR}D^{\alpha} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(1+\beta-\alpha)} t^{\beta-\alpha}$.

3. Mild and strong solutions of fractional differential equations

The fractional differential equation has attracted a lot of authors to discuss the existence and uniqueness of solution with finding it. Really there is a confusion in many literatures between fractional mild and strong solutions (see [1, 3, 8, 9, 10, 11, 12, 21, 27]). In this section we give the definition of each of them and give some examples to illustrate this definition.

Definition 3.1

The mild solution of any fractional differential equation is the solution of its corresponding integral equation. If this solution satisfies the fractional differential equation with its conditions this solution is said to be strong solution.

Example 3.2

Consider the fractional differential equation

$$^{C}D^{\alpha}x(t) = f(t),\ t \in (0,T],\ x(0) = x_0,\ \alpha \in (0,1) \tag{1}$$

we obtain the corresponding integral equation

$$x(t) \ = \ x(0) \ + \ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds \ \in C[0,T]. \tag{2}$$

Now we have the following cases

(1) If $f \in C[0,T]$, then the derivative of the solution of (2) does not exist. which proves that (2) is not a strong solution to (1) and there is no equivalent between the fractional differential equation (1) and the integral equation (2). In this case (2) is called a mild solution to (1).
(2) If $f \in L^1[0,T]$, then the derivative of the solution of (2) does not exist. which proves that (2) is not a strong solution to (1) and there is no equivalent between the fractional differential equation (1) and the integral equation (2). In this case (2) is called a mild solution to (1).
(3) If $f \in AC[0,T]$ we get that $x \in AC[0,T]$ and satisfies (1) which proves the equivalent between the (1) and (2). And in this case (2) is a unique strong solution to (1).

Example 3.3

Let $\alpha \in (0,1)$ and $f(t,x(t))$ satisfies the assumptions of the Banach fixed point Theorem or the Caratheodory Theorem or Peano Theorems. Integrate the initial-value problem

$$D^{\alpha}x(t) \ = \ f(t,x(t)),\ t \in (0,T],\ x(0) = x_0 \tag{3}$$

we obtain the corresponding integral equation

$$x(t) \ = \ x(0) \ + \ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds \ \in L^1[0,T] \ or \ C[0,T]. \tag{4}$$

But the derivative $x'(t)$ consequently the (**Caputo**) derivative $D^{\alpha}x(t)$ does not exist and there is no equivalent between the initial-value problem (3) and the integral equation (4). which prove that the problem can not be solved in this general case for strong solution but only for mild solutions.

4. Some incorrect properties of the Caputo via Riemann-Liouville fractional derivatives

In many recent papers the Caputo via Riemann-Liouville fractional derivative is called modified Riemann-Liouville fractional derivative. Jumarie in [16, 17] gave a definition to the fractional derivative, by limits as in classical derivative, as follows:

Definition 4.1

Let $f : \mathbb{R} \to \mathbb{R}$ denote a continuous (but not necessarily differentiable) function. and let $h > 0$ denote a constanst discretization span. Define the forward operator $FW(h)$ by the equality $FW(h)f(x) := f(x+h)$ then the fractional difference of order $\alpha, 0 < \alpha < 1$, of $f(x)$ is defined by the expression

$$\Delta^{\alpha} f(x) := (FW-1)^{\alpha} f(x) = \sum_{k=0}^{\infty} (-1)^k \begin{pmatrix} \alpha \\ k \end{pmatrix} f(x + (\alpha - k)h) \tag{5}$$

and its fractional derivative is the limit

$$f^{(\alpha)} := \lim_{h\to 0} \frac{\Delta^\alpha [f(x) - f(0)]}{h^\alpha}. \tag{6}$$

Really one can note that:

Jumarie fractional derivative near to the Grunwald-Letnikov fractional derivative.

Jumarie fractional derivative is strictly equivalent to the Caputo via Riemann fractional derivative (modified Riemann-Liouville derivative)

Jumarie gave two important properties to his fractional derivative:

(1) the derivative of product of two functions (fractional Liebenz rule) in the form
$$(u(x)v(x))^{(\alpha)} = u(\alpha)(x)v(x) + u(x)v^{(\alpha)}(x). \tag{7}$$
(2) fractional chain rule in the form
$$(f[u(x)])^{(\alpha)} = f'_u(u)u^{(\alpha)}(x) = f_u^{(\alpha)}(u)(u'_x)^\alpha, \tag{8}$$
(3) In [7] He and et. al. gave an improvement of the fractional chain rule given by
$$\frac{\partial^\alpha u}{\partial t^\alpha} = \sigma \frac{\partial u}{\partial s}\frac{\partial^\alpha s}{\partial t^\alpha} \tag{9}$$
where σ is a fractal index which is determined separately in each problem

Unfortunately, these results are incorrect. One can see if these properties were satisfied for Caput via Riemann-Liouville fractional derivative then they are satisfied for Caputo fractional derivative as seen from Lemma 2.7. Tarasov (see [26]) proved that there is no fractional derivative satisfies Liebenz rule.

Here we give three counter examples for these properties:

Example 4.2 In this example we show that the fractional Liebenz rule given in (7) is not correct

Let $f(t) = t^{1+\alpha}$ and $g(t) = t^{1-\alpha}$. Now using

$$D_t^\alpha(t^\mu) = \frac{\Gamma(1+\mu)}{\Gamma(1+\mu-\alpha)} t^{\mu-\alpha} \tag{10}$$

we get that

$$D_t^\alpha(f(t)) = \Gamma(2+\alpha)t, \text{ and } D_t^\alpha(g(t)) = \frac{\Gamma(2-\alpha)}{\Gamma(2-2\alpha)} t^{1-2\alpha} \tag{11}$$

which gives

$$f(t)D_t^\alpha g(t) + g(t)D_t^\alpha f(t) = \left(\frac{\Gamma(2-2\alpha)\Gamma(2+\alpha) + \Gamma(2-\alpha)}{\Gamma(2-2\alpha)}\right) t^{2-\alpha} \tag{12}$$

which is not equal to

$$D_t^\alpha(f(t)g(t)) = \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha} \tag{13}$$

as seen for example when $\alpha = \frac{1}{2}$.

Example 4.3 In this example we show that the first fractional chain rule given (8) is incorrect

Let $f(g(t)) = (g(t))^2,\ g(t) = \frac{t^\alpha}{\Gamma(1+\alpha)}$ we get that

$$D_t^\alpha f(g(t)) = D_t^\alpha \left(\frac{t^{2\alpha}}{(\Gamma(1+\alpha))^2} \right) = \frac{\Gamma(1+2\alpha)}{(\Gamma(1+\alpha))^3} t^\alpha, \tag{14}$$

$$f'_g(g(t) D_t^\alpha g(t) = \frac{2}{\Gamma(1+\alpha)} t^\alpha, \tag{15}$$

and

$$D_g^\alpha (f(g(t))(g'(t))^\alpha = \frac{2\alpha^\alpha}{\Gamma(3-\alpha)(\Gamma(1+\alpha))^2} t^\alpha. \tag{16}$$

From the above three equations we get that (8) is incorrect.

Finally in the following example we show that the second fractional chain rule given by (9) is incorrect

Example 4.4

Consider the function $u(s) = E_\alpha(s),\ s = t^\alpha$ where $E_\alpha(s) = \sum_{k=0}^\infty \frac{s^k}{\Gamma(\alpha k+1)}$ is the Mittag-Liffler function (for detail see [18, 20]).

We get the left hand side of (9) in the form

$$\begin{aligned}
\frac{\partial^\alpha u}{\partial t^\alpha} &= \frac{\partial^\alpha}{\partial t^\alpha} E_\alpha(t^\alpha) \\
&= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\xi)^{-\alpha} \left(\sum_{k=0}^\infty \left(\frac{\xi^{\alpha k}}{\Gamma(\alpha k+1)} - 1 \right) \right) d\xi \\
&= \frac{d}{dt} \sum_{k=0}^\infty \left(\frac{t^{\alpha k - \alpha + 1}}{\Gamma(1-\alpha)\Gamma(\alpha k+1)} \int_0^1 (1-\xi)^{-\alpha} \xi^{\alpha k} d\xi - \frac{t^{1-\alpha}}{\Gamma(\alpha)} \right) \\
&= t^{-\alpha} \sum_{k=0}^\infty \left(\frac{t^{\alpha k}}{\Gamma(\alpha k + 1 - \alpha)} - \frac{1}{\Gamma(1-\alpha)} \right) \\
&= t^{-\alpha} \sum_{k=1}^\infty \left(\frac{t^{\alpha k}}{\Gamma(\alpha k + 1 - \alpha)} \right) = E_\alpha(t^\alpha) \qquad (17)
\end{aligned}$$

Now

$$\frac{\partial u}{\partial s} = \frac{\partial}{\partial s} \sum_{k=0}^\infty \frac{s^k}{\Gamma(\alpha k+1)} = \sum_{k=1}^\infty \frac{k s^{k-1}}{\Gamma(\alpha k+1)} \tag{18}$$

and

$$\frac{\partial^\alpha s}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\xi)^{-\alpha} \xi^\alpha d\xi = \Gamma(\alpha+1) \tag{19}$$

then the right hand side of (9) takes the form

$$\begin{aligned}
\sigma' \frac{\partial u}{\partial s} \frac{\partial^\alpha s}{\partial t^\alpha} &= \sigma' \Gamma(\alpha+1) \sum_{k=1}^\infty \frac{k t^{\alpha(k-1)}}{\Gamma(\alpha k+1)} \\
&= \sigma' \Gamma(\alpha+1) \sum_{k=0}^\infty \frac{(k+1) t^{\alpha k}}{\Gamma(\alpha k + \alpha + 1)}. \qquad (20)
\end{aligned}$$

It is obvious that for each value of σ' the left and right hand side of (9) are not equal.

5. Some incorrect methods for solving fractional differential equations

Here we consider two methods which are used in recent literature to find the solution of the fractional differential equations. These two methods depend on transform the fractional differential equations to classical differential equations and then solve the obtained classical differential equations by different methods (for example G'/G-expansion method and the functional variable method).

(1) **complex and modified complex transform methods**
Depending on the first fractional chain rule and the fractional Leibniz rule the authors of [5, 6] gave the complex transform method which transform the fractional partial differential equation to an ordinary partial differential equation.

Using a substitution with a complex variable defined by

$$\xi = \frac{pt^{\alpha}}{\Gamma(\alpha+1)} + \frac{qx^{\beta}}{\Gamma(\beta+1)},\ p, q \text{are constants}$$

they transform the fractional partial differential equation

$${}^{CVR}D_t^{2\alpha} u(x,t) = k^2\ {}^{CVR}D_x^{2\beta} u(x,t)$$

to the ordinary differential equation

$$(q^2 - k^2 p^2) u_{\xi\xi} = 0$$

The authors of [7] gave a modification to this complex transform method by using the second fractional chain rule.

These two methods are used in many papers to solve the fractional differential equations (for example see [2] and the references there in).

After proving that the fractional chain rule and the fractional Libniez rule are incorrect we conclude that this method gives incorrect solutions because the solved transformed classical problems were not related to the original fractional problem.

(2) **Fractional normal mode analysis** In some literature (see for example [23, 24, 13, 14]) which studied fractional partial differential equations where the fractional exist only for time their authors used a substitution depends on the exponential function to transform the fractional partial differential equations to classical partial differential equations which can be solved by many methods. They used in their method the relation

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}} \exp(wt + iax) = w^{\alpha} \exp(wt + iax) \tag{21}$$

which is an incorrect relation as seen below:
We prove for Caputo fractional derivative and by using the relations between the three definitions we deduce that the relation is not true in the three cases. From the definition of the Caputo fractional derivative we get

that

$$\begin{aligned}
\frac{d^\alpha}{dt^\alpha}e^{wt} &= \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\frac{d}{ds}(\sum_{k=0}^{\infty}\frac{(ws)^k}{k!})ds \\
&= \frac{w}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\frac{d}{ds}(\sum_{k=1}^{\infty}\frac{(ws)^{k-1}}{(k-1)!})ds \\
&= \sum_{k=1}^{\infty}\frac{w^k}{\Gamma(1-\alpha)\Gamma(k)}\int_0^t (t-s)^{-\alpha}s^{k-1}ds \\
&= \sum_{k=1}^{\infty}\frac{w^k t^{-\alpha+k}}{\Gamma(1-\alpha)\Gamma(k)}\beta(1-\alpha,k) \\
&= t^{-\alpha}\sum_{k=1}^{\infty}\frac{(wt)^k}{\Gamma(k+1-\alpha)} \neq w^\alpha e^{wt}.
\end{aligned}$$

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