EXISTENCE OF A POSITIVE SOLUTION FOR A BOUNDARY VALUE PROBLEM VIA A TOPOLOGICAL-VARIATIONAL THEOREM

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ABSTRACT. The aim of this paper is to prove the existence of a positive solution for a fractional nonlinear differential equation under homogeneous boundary conditions of Dirichlet type by using a method which combines between variational and topological methods.

1. INTRODUCTION

There are many papers which have studied mathematical problems by using fixed point theory or variational methods separately. In the paper [8], Y. Cui and J. Sun have combined the two methods to obtain a new fixed point theorem. In this paper, by using the above theorem, we study a boundary value problem associated to a fractional differential equation and we prove existence of a positive solution under growth conditions.

More precisely, we want to study the following fractional problem
\[
\begin{cases}
D^\alpha_{T-} (D^\alpha_0 u(t)) = f(t, u(t)), & t \in [0, T], \\
u(0) = u(T) = 0,
\end{cases}
\]
(1.1)

where $\frac{1}{2} < \alpha \leq 1$ and $f : [0, T] \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a continuous function.

2. PRELIMINARIES

Definition 2.1. ([4], [5], [6]) Let $u$ be a function defined on $[a, b]$. The left and right Riemann-Liouville fractional integrals of order $\alpha > 0$ for a function $u$ denoted by $I^\alpha_a u$ and $I^\alpha_b u$, respectively, are defined by
\[
I^\alpha_a u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} u(s) ds, \quad t > a,
\]
and
\[
I^\alpha_b u(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} u(s) ds, \quad t < b,
\]

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provided that the right-hand side is pointwise defined on \([a, b]\), where \(\Gamma(\alpha)\) is the gamma function.

**Definition 2.2.** ([3], [4], [5]) Let \(u\) be a function defined on \([a, b]\). For \(n - 1 < \alpha < n\) \((n \in \mathbb{N})\), the left and right Riemann-Liouville fractional derivatives of order \(\alpha\) for a function \(u\) denoted by \(D^\alpha_{a+} u\) and \(D^\alpha_{b-} u\), respectively, are defined by

\[
D^\alpha_{a+} u(t) = \frac{d^n}{dt^n} I_{a+}^{n-\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} u(s) ds, \quad t > a,
\]

and

\[
D^\alpha_{b-} u(t) = (-1)^n \frac{d^n}{dt^n} I_{b-}^{n-\alpha} u(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_t^b (s-t)^{n-\alpha-1} u(s) ds, \quad t < b,
\]

provided that the right-hand side is pointwise defined.

In particular, for \(\alpha = n\), \(D^n_{a+} u(t) = D^n u(t)\) and \(D^n_{b-} u(t) = (-1)^n D^n u(t)\), \(t \in [a, b]\).

**Proposition 2.3.** ([3], [7]) If \(D^\alpha_{a+} u, D^\alpha_{b-} u \in L^1([a, b])\) and \(n - 1 < \alpha < n\), then

\[
I^\alpha_{a+} D^\alpha_{a+} u(t) = u(t) + \sum_{j=1}^n c_j (t-a)^{\alpha-j}
\]

with \(c_j = \frac{D^{\alpha-j}_{a+} u(a)}{\Gamma(\alpha-j+1)} \in \mathbb{R}, \quad j = 1, 2, ..., n\) and

\[
I^\alpha_{b-} D^\alpha_{b-} u(t) = u(t) + \sum_{j=1}^n c'_j (b-t)^{\alpha-j}
\]

with \(c'_j = \frac{(-1)^{n-j} D^{\alpha-j}_{b-} u(b)}{\Gamma(\alpha-j+1)} \in \mathbb{R}, \quad j = 1, 2, ..., n\).

It is easy to prove the following lemma by using proposition 2.3.

**Lemma 2.4.** If \(u\) is a solution of the integral equation

\[
u(t) = \int_0^T G_1(t, s) \left[ \int_0^T G_2(s, \tau) f(\tau, u(\tau)) d\tau \right] ds
\]

where

\[
G_1(t, s) = \begin{cases} 
\frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1}, & s \leq t, \\
0, & s > t,
\end{cases}
\]

and

\[
G_2(s, \tau) = \begin{cases} 
\frac{1}{\Gamma(\alpha)} (\tau-s)^{\alpha-1}, & s \leq \tau, \\
0, & s > \tau,
\end{cases}
\]

then it is a solution of the problem [4, 7].

We set

\[
Au(t) = \int_0^T G_1(t, s) \left[ \int_0^T G_2(s, \tau) f(\tau, u(\tau)) d\tau \right] ds.
\]

Then, the operator \(A\) satisfies the problem

\[
\left\{ \begin{array}{l}
D^\alpha_{a+} (D^\alpha_{b-} Au(t)) = f(t, u(t)), \quad t \in [0, T], \\
Au(0) = Au(T) = 0.
\end{array} \right.
\]
Definition 2.5. \(^{[2, \underline{3}]}\) Let \(\varphi \in C^1(X, \mathbb{R})\). If any sequence \((u_n) \subset X\) for which \((\varphi(u_n))\) is bounded in \(\mathbb{R}\) and \(\varphi'(u_n) \rightarrow 0\) when \(n \rightarrow +\infty\) in \(X'\), possesses a convergent subsequence, then we say that \(\varphi\) satisfies the Palais-Smale condition (denoted by (P.S) condition).

Theorem 2.6. \(^{[3]}\) Let \(H\) be an ordered real Hilbert space and \(P\) a cone in \(H\) such that \((x, y) \geq 0\) for \(x, y \in P\). Suppose that \(\varphi \in C^1(H, \mathbb{R})\) satisfies the following hypotheses:

(i) \(\varphi\) satisfies the (PS) condition on \(P\) and its gradient \(\varphi'\) admits the decomposition \(I - A\) such that \(A(P) \subset P\).

(ii) There exist a positive linear operator \(B_1\) with \(r(B_1) > 1, x^*, \phi_1 \in P\) with \(\|\phi_1\| = 1\) such that

\[Ax \geq B_1x - x^*, \ x \in P,\]

and

\[B_1\phi_1 = r(B_1)\phi_1,\]

that is, \(\phi_1\) is the normalized first eigenfunction.

(iii) There exist a positive linear operator \(B_2\) with \(r(B_2) < 1\) and a positive number \(r\) such that

\[Ax \leq B_2x, \ x \in P \cap B_r(\theta), \ B_r(\theta) = \{x : \|x\| \leq r\}.\]

Then \(A\) has a positive fixed point.

3. Variational structure

Definition 3.1. \(^{[1]}\) The fractional derivative space \(E_0^\alpha\) is defined by the closure of \(C_0^\infty([0, T], \mathbb{R})\) with respect to the norm

\[
\|u\|_\alpha = \left( \int_0^T |D_0^\alpha u(t)|^2 dt \right)^{\frac{1}{2}}, \quad \text{for} \quad u \in E_0^\alpha.
\]  

(3.1)

We put

\[E_0^\alpha = \left\{ u \in L^2([0, T]), D_0^\alpha u \in L^2([0, T]) \quad \text{and} \quad u(0) = u(T) = 0 \right\}.\]

Proposition 3.2. \(^{[1]}\) The space \(E_0^\alpha\) is a reflexive and separable Banach space.

The set \(E_0^\alpha\) is endowed with the structure of Hilbert space together with the inner product

\[(u, v)_{E_0^\alpha} : E_0^\alpha \times E_0^\alpha \rightarrow \mathbb{R},\]

\[(u, v)_{E_0^\alpha} = (D_0^\alpha u, D_0^\alpha v)_{L^2} = \int_0^T D_0^\alpha u(t) D_0^\alpha v(t) dt.\]

Let the standard cone in the space \(E_0^\alpha\), \(P = \{u \in E_0^\alpha, u(t) \geq 0, t \in [0, T]\}\).

Proposition 3.3. \(^{[1, \underline{4}]}\) If \(u \in E_0^\alpha\) and \(v \in C_0^\infty([0, T])\), then

\[\int_0^T D_0^\alpha u(t)v(t) dt = \int_0^T u(t)D_0^\alpha v(t) dt.\]

By using proposition 3.3, we can define a weak solution of problem (1.1).
Definition 3.4. A weak solution of the fractional boundary value problem (1.1) is given by a solution of the following variational formula

\[ \int_{0}^{T} \left[ D_{0+}^{\alpha} u(t) D_{0+}^{\alpha} v(t) - f(t, u(t)) v(t) \right] dt = 0, \quad \text{for all } v \in E_{0}^{\alpha}. \]

Proposition 3.5. [1] For any \( u \in E_{0}^{\alpha} \), we have

\[ \| u \|_{L^2} \leq \frac{T^{\alpha}}{\Gamma(\alpha + 1)} \| D_{0+}^{\alpha} u \|_{L^2}. \] (3.2)

Moreover, if \( \alpha > \frac{1}{2} \)

\[ \| u \|_{\infty} \leq \frac{T^{\alpha - \frac{1}{2}}}{\Gamma(\alpha)(2(\alpha - 1) + 1)^{\frac{1}{2}}} \| D_{0+}^{\alpha} u \|_{L^2}, \] (3.3)

where \( \| u \|_{\infty} = \sup_{t \in [0, T]} | u(t) | \).

Proposition 3.6. [1] The space \( E_{0}^{\alpha} \) is compactly embedded in \( C([0, T], \mathbb{R}) \).

Proposition 3.7. For any \( u \in E_{0}^{\alpha}, a_1, a_2 \in L^1([0, T]) \), we have

\[ \| u \|_{L^2_{a_i}}^2 \leq \| a_i \|_{L^1} T^{2\alpha - 1} \frac{\| F \|_{L^2}}{\Gamma^2(\alpha)(2(\alpha - 1) + 1)} \| u \|_{\alpha}^2, \quad i = 1, 2, \]

where \( \| u \|_{L^2_{a_i}}^2 = \int_{0}^{T} a_i(t) u^2(t) dt \).

Proof. By the formula (3.3), we have

\[ \| u \|_{L^2_{a_i}}^2 = \int_{0}^{T} a_i(t) u^2(t) dt \leq \| a_i \|_{L^1} \| u \|_{\infty}^2 \leq \| a_i \|_{L^1} T^{2\alpha - 1} \frac{\| F \|_{L^2}}{\Gamma^2(\alpha)(2(\alpha - 1) + 1)} \| u \|_{\alpha}^2. \]

We define the functional

\[ \varphi : E_{0}^{\alpha} \rightarrow \mathbb{R}, \quad u \mapsto \varphi(u) = \frac{1}{2} \| u \|_{\alpha}^2 - \int_{0}^{T} F(t, u(t)) dt. \]

with \( F(t, u) = \int_{0}^{u} f(t, s) ds \). We have then

\[ \varphi'(u)(v) = \int_{0}^{T} D_{0+}^{\alpha} u(t) D_{0+}^{\alpha} v(t) dt - \int_{0}^{T} f(t, u(t)) v(t) dt, \forall u, v \in E_{0}^{\alpha}. \]

We can see that a critical point of the functional \( \varphi \) is a weak solution of the problem (1.1).

Proposition 3.8. Let

\[ \lambda_1(a_i) = \inf_{u \in E_{0}^{\alpha} \setminus \{0\}} \frac{\| u \|_{\alpha}^2}{\| u \|_{L^2_{a_i}}^2} = \inf_{u \in E_{0}^{\alpha} \setminus \{0\}} \frac{\int_{0}^{T} \left| D_{0+}^{\alpha} u(t) \right|^2 dt}{\int_{0}^{T} a_i(t) u^2(t) dt}. \]

Then, \( \lambda_1(a_i) \) is positive and is achieved by some \( u_1 \in E_{0}^{\alpha} \setminus \{0\}, i = 1, 2. \).
Proof. For \( u \in E_0^\alpha \), we put \( \varphi_1(u) = \|u\|_1^2, \varphi_2(u) = \|u\|_{L^2}^2 \) and define the quotient functional

\[
\psi : E_0^\alpha \rightarrow \mathbb{R}
\]

\[
u \rightarrow \psi(u) = \frac{\varphi_1(u)}{\varphi_2(u)}
\]

then

\[
\lambda_1(a_i) = \inf_{u \in E_0^\alpha - \{0\}} \psi(u).
\]

The Poincaré type inequality (see proposition 3.7) immediately gives \( \lambda_1(a_i) > 0 \). Let \( (u_n) \) be a minimizing sequence of \( \psi \). It is easy that \( \{|u_n|\} \) is also a minimizing sequence for \( \psi \), so we can assume that \( u_n(t) \geq 0 \) for \( t \in [0, T] \). As the functional \( \psi \) is homogeneous of degree zero, i.e., \( \psi(su) = \psi(u) \) for every \( s \in \mathbb{R} \), we can normalize \( u_n \) by setting \( \|u_n\|_{L^2}^2 = 1 \), for every \( n \). We remark that \( \varphi_1(u_n) = \|u_n\|_\alpha^2 \) must be bounded, because \( (u_n) \) is a minimizing sequence, i.e.

\[
\lim_{n \rightarrow +\infty} \varphi(u_n) = \inf_{u \in E_0^\alpha - \{0\}} \varphi(u_n) = \lambda_1(a_i),
\]

which prove that \( (u_n) \) is a bounded sequence.

By the Sobolev embedding, we deduce that, up to subsequences

\[
u_n \rightarrow \text{for some } u_1 \in E_0^\alpha,
\]

\[
u_n(t) \rightarrow u_1(t) \text{ in } [0, T].
\]

In particular, \( \psi(u_1) = 1 \) and \( u_1(t) \geq 0 \). Then, by weak lower semi-continuity of the norm

\[
\lambda_1(a_i) \leq \psi(u_1) = \varphi_1(u_1) \leq \lim inf_{n \rightarrow +\infty} \varphi_1(u_n) = \lim inf_{n \rightarrow +\infty} \psi(u_n) = \lambda_1(a_i);
\]

so, \( u_1 \in E_0^\alpha \) and \( \psi(u_1) = \lambda_1(a_i) \).

\[
\square
\]

**Corollary 3.9.** For any \( u \in E_0^\alpha \), we have

\[
\|u\|_{L^2}^2 \leq \frac{1}{\lambda_1(a_i)} \|u\|_\alpha^2, \quad i = 1, 2.
\]

4. **Main result**

**Theorem 4.1.** Assume that \( \lambda_1(a_1) < 1, \lambda_1(a_2) > 1 \) and the following conditions hold:

\( (H) \) there exist functions \( a_1, a_2 \in L^1([0, T]), b_1 : [0, T] \rightarrow \mathbb{R}^+ \) such that

\[
a_1(t)u(t) - b_1(t) \leq f(t, u(t)) \leq a_2(t)u(t)
\]

with \( \frac{1}{2} - \frac{T^{2\alpha-1} \|a_2\|_{L^1}}{\Gamma^2(\alpha)(2(\alpha-1)+1)} > 0 \) and \( \int_0^T G_1(t, s) \left[ \int_0^T G_2(s, \tau)b_1(\tau)d\tau \right] ds < +\infty. \) Then the problem (1.1) has a positive solution.
Proof. By the definition of the inner product in the space $E_0^\alpha$, we can obtain for all $u, v \in E_0^\alpha$,

$$
\varphi'(u)(v) = \int_0^T D_0^\alpha u(t) D_0^\alpha v(t) dt - \int_0^T f(t, u(t))v(t) dt
$$

$$
= \int_0^T D_0^\alpha u(t) D_0^\alpha v(t) dt - \int_0^T D_T^\alpha D_0^\alpha (Au)(t)v(t) dt
$$

$$
= \int_0^T D_0^\alpha u(t) D_0^\alpha v(t) dt - \int_0^T D_0^\alpha (Au)(t) D_0^\alpha v(t) dt
$$

$$
= (u, v) - (Au, v) = (u - Au, v) = ((I - A)u, v).
$$

Thus

$$
\varphi' = I - A.
$$

We put

$$
B_1 u(t) = \int_0^T G_1(t, s) \left[ \int_0^T G_2(\tau, s)a_1(\tau)u(\tau) d\tau \right] ds
$$

and

$$
B_2 u(t) = \int_0^T G_1(t, s) \left[ \int_0^T G_2(\tau, s)a_2(\tau)u(\tau) d\tau \right] ds.
$$

One can verify easily that $A$ is a compact operator and $A(P) \subset P$. Now, we prove that the operator $A$ verify the conditions of theorem 2.6.

Step 1: $\varphi$ satisfies the (PS) condition. Let $(u_n)$ be a sequence in $E_0^\alpha$ such that $\lim_{n \to +\infty} \varphi'(u_n) = 0$ and $\varphi(u_n)$ is bounded, i.e., $|\varphi'(u_n)| \leq K$, $\forall n$, for some $K > 0$.

In view of the condition $(H_1)$, we have

$$
K \geq \varphi(u_n) = \frac{1}{2} \int_0^T |D_0^\alpha u_n(t)|^2 dt - \int_0^T F(t, u_n(t)) dt
$$

$$
\geq \frac{1}{2} \| u_n \|_\alpha^2 - \int_0^T a_2(t) u_n^2(t) dt
$$

$$
\geq \frac{1}{2} \| u_n \|_\alpha^2 - \| u_n \|_\infty^2 \| a_2 \|_{L_1}
$$

$$
\geq \frac{1}{2} \| u_n \|_\alpha^2 - \frac{T^{2\alpha-1} \| a_2 \|_{L_1}}{\Gamma^2(\alpha)(2(\alpha - 1) + 1)} \| u_n \|_\alpha^2
$$

$$
\geq \left( \frac{1}{2} - \frac{T^{2\alpha-1} \| a_2 \|_{L_1}}{\Gamma^2(\alpha)(2(\alpha - 1) + 1)} \right) \| u_n \|_\alpha^2
$$

which implies that $(u_n)$ is bounded in $E_0^\alpha$.

We note that, $\varphi'(u_n) = u_n - A(u_n)$ with $\lim_{n \to +\infty} \varphi'(u_n) = 0$. Since the sequence $(u_n)$ is bounded and the operator $A$ is compact, this implies that $(A(u_n))$ is relatively compact, then there exists a subsequence $(u_{n_k}) \subset (u_n)$ such that $A(u_{n_k}) \longrightarrow v$ which implies that $u_{n_k} \longrightarrow v$ in $E_0^\alpha$. In fact, we have

$$
\| u_{n_k} - v \| \leq \| u_{n_k} - A(u_{n_k}) \| + \| A(u_{n_k}) - v \| \longrightarrow 0 \text{ when } k \longrightarrow +\infty.
$$

Thus, the (PS) condition is satisfied.

Step 2: the operator $A$ satisfies the hypothesis $(ii)$. 

By the condition \((H)\), we have for all \(u \in P\),

\[
Au(t) - B_1 u(t) = \int_0^T G_1(t, s) \left[ \int_0^T G_2(s, \tau) f(\tau, u(\tau)) d\tau \right] ds 
- \int_0^T G_1(t, s) \left[ \int_0^T G_2(s, \tau) a_1(\tau) u(\tau) d\tau \right] ds 
\geq \int_0^T G_1(t, s) \left[ \int_0^T G_2(s, \tau) (a_1(\tau) u(\tau) - b_1(\tau)) d\tau \right] ds 
- \int_0^T G_1(t, s) \left[ \int_0^T G_2(s, \tau) a_1(\tau) u(\tau) d\tau \right] ds 
\geq - \int_0^T G_1(t, s) \left[ \int_0^T G_2(s, \tau) b_1(\tau) d\tau \right] ds.
\]

Then

\[
Au(t) - B_1 u(t) + u^*(t) \geq 0, \quad \forall u \in P
\]

with

\[
u^*(t) = \int_0^T G_1(t, s) \left[ \int_0^T G_2(s, \tau) b_1(\tau) d\tau \right] ds.
\]

By the definition of \(B_1\) and the Poincaré type inequality in corollary 3.9, we have \(r(B_1) = \frac{1}{\lambda_{1}(a_1)} > 1\) and \(B_1 u_1 = r(B_1) u_1\). Therefore the condition \((ii)\) is satisfied.

**Step 3**: The operator \(A\) satisfies the hypothesis \((iii)\).

By the condition \((H)\), for all \(u \in P \cap B(0, r)\) with \(r > 0\), we have

\[
Au(t) - B_2 u(t) = \int_0^T G_1(t, s) \left[ \int_0^T G_2(s, \tau) f(\tau, u(\tau)) d\tau \right] ds 
- \int_0^T G_1(t, s) \left[ \int_0^T G_2(s, \tau) a_2(\tau) u(\tau) d\tau \right] ds 
\leq \int_0^T G_1(t, s) \left[ \int_0^T G_2(s, \tau) a_2(\tau) u(\tau) d\tau \right] ds 
- \int_0^T G_1(t, s) \left[ \int_0^T G_2(s, \tau) a_2(\tau) u(\tau) d\tau \right] ds 
\leq 0.
\]

Then

\[
Au(t) \leq B_2 u(t) \quad \forall u \in P \cap B(0, r).
\]

By the definition of \(B_2\) and the Poincaré type inequality in corollary 3.9, we have \(r(B_2) = \frac{1}{\lambda_{1}(a_2)} < 1\). Therefore the condition \((iii)\) is satisfied.

Thus, by theorem 2.6 the operator \(A\) has a positive fixed point, which implies that the problem (1.1) has a positive solution. \(\square\)

**Example 4.2.** We consider the following problem

\[
\begin{cases} 
D_{0+}^\frac{2}{3} (D_{0+}^\frac{2}{3} u(t)) = f(t, u(t)), & t \in [0, 2], \\
u(0) = u(2) = 0,
\end{cases}
\]

with

\[
f(t, y) = \begin{cases} 
y^2, & \text{if } y \in [0, a_2] \\
a_2 y, & \text{if } y \in [a_2, +\infty],
\end{cases}
\]
where $\frac{35}{100} = a_2 > a_1 = \frac{33}{100}$, $\alpha = \frac{2}{3}$ and $T = 2$. By the condition $(H)$ of theorem 4.1, we have
\[
\frac{33}{100} u - \frac{3}{100} \leq f(t, u) \leq \frac{35}{100} u \quad \text{with} \quad \frac{1}{2} - \frac{(0.35)(2^{\frac{4}{3}})}{\Gamma^2(\frac{2}{3})(2(\frac{2}{3} - 1) + 1)} = \frac{1}{2} - \frac{(0.8)(0.35)}{0.60} > 0.
\]
Since $\frac{r^2(\frac{2}{3} + 1)}{2^{\frac{4}{3}}} \leq a_1 = 0.33$, we have $\lambda_1(a_1) < 1$. Indeed, we have
\[
u(t) = \frac{33}{100} \lambda_1(a_1) \int_0^2 G_1(t, s) \left[ \int_0^2 G_2(s, \tau)u(\tau)d\tau \right]ds
\]
is the solution of the following boundary value problem
\[
\begin{cases}
D^{\frac{4}{3}}_2 (D_0^{\frac{2}{3}} u(t)) = \frac{33}{100} \lambda_1(a_1) u(t), & t \in [0, 2],
\quad u(0) = u(2) = 0.
\end{cases}
\]
After a simple calculation, we find
\[
\|u\|_\infty \leq \frac{33}{100} \cdot \lambda_1(a_1) \|u\|_\infty \int_0^T G_1(t, s) \left[ \int_0^T G_2(s, \tau)d\tau \right]ds
\]
\[
\leq \frac{33}{100} \cdot \frac{2^{\frac{4}{3}}}{\Gamma^2(\frac{2}{3} + 1)} \cdot \lambda_1(a_1)
\]
and if $\lambda_1(a_2) > 1$, then the problem (4.1) has a positive solution.

References