SOME PROPERTIES OF CERTAIN CLASSES OF MEROMORPHICALLY p-VALENT FUNCTIONS DEFINED BY CONVOLUTION STRUCTURE

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Abstract. In this paper, we investigate interesting properties of certain subclasses of meromorphically multivalent functions associated with a linear operator $D^n_{\lambda,p}(f \ast g)$ defined by Hadamard product.

1. Introduction

Let $\Sigma_{p,m}$ denote the class of functions of the form:

$$f(z) = z^{-p} + \sum_{k=m}^{\infty} a_k z^k \quad (m > -p; \ p \in \mathbb{N} = \{1, 2, \ldots\}), \quad (1)$$

which are analytic and p-valent in the punctured unit disc $U^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = \mathbb{U}\{0\}$. For convenience, we write

$$\Sigma_{p,1-p} = \Sigma_p \quad (p \in \mathbb{N}).$$

For functions $f(z) \in \Sigma_{p,m}$ given by (1) and $g(z) \in \Sigma_{p,m}$ given by

$$g(z) = z^{-p} + \sum_{k=m}^{\infty} b_k z^k \quad (m > -p, \ p \in \mathbb{N}),$$

we define the Hadamard product (or convolution) of $f(z)$ and $g(z)$ by

$$(f \ast g)(z) = z^{-p} + \sum_{k=m}^{\infty} a_k b_k z^k = (g \ast f)(z). \quad (2)$$

For functions $f, g \in \Sigma_{p,m}$, we define the linear operator $D^n_{\lambda,p}(f \ast g)(z) : \Sigma_{p,m} \rightarrow \Sigma_{p,m} \ (m > -p, \lambda \geq 0; \ p \in \mathbb{N}; \ n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$ by

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\[
D_{\lambda,p}^0(f * g)(z) = (f * g)(z), \quad (3)
\]
\[
D_{\lambda,p}^1(f * g)(z) = D_{\lambda,p}(f * g)(z)
\]
and (in general)
\[
D_{\lambda,p}^n(f * g)(z) = D_{\lambda,p}(D_{\lambda,p}^{n-1}(f * g)(z))
\]
\[
= z^{-p} + \sum_{k=m}^{\infty} [1 + \lambda(k + p)]a_k b_k z^k \quad (\lambda \geq 0). \quad (5)
\]

From (4) it is easy to verify that:
\[
z(D_{\lambda,p}^n(f * g)(z))' = \frac{1}{\lambda} D_{\lambda,p}^{n+1}(f * g)(z) - (p + \frac{1}{\lambda}) D_{\lambda,p}^n(f * g)(z) \quad (\lambda > 0). \quad (6)
\]

Linear operator \(D_{\lambda,p}^n(f * g)(z)\) was introduced and studied by Aouf et al. [6], and Seoudy and Aouf [18].

We observe that the linear operator \(D_{\lambda,p}^n(f * g)(z)\) reduces to several interesting operators for different choices of \(n, \lambda, m, p\) and the function \(g(z)\):

(i) for \(\lambda = 1, m = 0\) and \(g(z) = \frac{1}{z^p (1 - z)}\) (or \(b_k = 1\)), \(D_{1,p}^n(f * g)(z) = D_p(f)(z)\)

(see Aouf and Hossen [3], Liu and Owa [13] and Srivastava and Patel [19]);

(ii) for \(m = 0\) and \(g(z) = \frac{1}{z^p (1 - z)}\) (or \(b_k = 1\)), we have

\[
D_{\lambda,p}^n(f * g)(z) = D_{\lambda,p}^n f(z) = z^{-p} + \sum_{k=0}^{\infty} [1 + \lambda(k + p)]a_k z^k \quad (\lambda \geq 0; p \in \mathbb{N}; n \in \mathbb{N}_0);
\]

(iii) for \(n = m = 0\) and

\[
g(z) = z^{-p} + \sum_{k=0}^{\infty} \left[ \frac{\ell + \lambda(k + p)}{\ell} \right]^r z^k \quad (\lambda \geq 0; p \in \mathbb{N}; \ell, r \in \mathbb{N}_0), \quad (7)
\]

we see that \(D_{\lambda,p}^0(f * g)(z) = (f * g)(z) = I_p^r(\lambda, \ell)f(z)\), where \(I_p^r(\lambda, \ell)\) is the generalized multiplier transformation which was introduced by El-Ashwah [10], the operator \(I_p^r(\lambda, \ell)\), contains as special cases, the multiplier transformation \(I(r, \ell)f(z)\)

(see Cho et al. [7],[8]) and for \(\ell = 1\) the operator \(I^r f(z)\) introduced by Uralegaddi and Somanatha [20];

(iv) for \(n = m = 0\) and

\[
g(z) = z^{-p} + \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \ldots (\alpha_q)_k z^{k-p}}{(\beta_1)_k \ldots (\beta_s)_k k!}
\]

\[
(\alpha_i \in \mathbb{C}; i = 1, \ldots, q; \beta_j \in \mathbb{C}\backslash \mathbb{Z}_0^- = \{0, -1, -2, \ldots\}; j = 1, \ldots, s;
q \leq s + 1; q, s \in \mathbb{N}_0, p \in \mathbb{N}; z \in U),
\]

\[
D_{\lambda,p}^0(f * g)(z) = (f * g)(z),
\]

and
We see that $D_{\lambda,p}^0(f \ast g)(z) = H_{p,q,s}(\alpha_1)$, where the operator $H_{p,q,s}(\alpha_1)$, was investigated recently by Liu and Srivastava [15] and Aouf [2]. The operator $H_{p,q,s}(\alpha_1)$ contains the operator $L_{p}(a,c)$ (see [14]) for $q = 2$, $s = 1$, $\alpha_1 = a > 0$, $\beta_1 = c (c \neq 0,-1,...)$ and $\alpha_2 = 1$ and also contains the operator $D^{\mu+p-1}$ (see [1] and [5]) for $q = 2$, $s = 1$, $\alpha_1 = \nu + p (\nu > -p, p \in \mathbb{N})$ and $\alpha_2 = \beta_1 = p (p \in \mathbb{N})$.

(v) for $n = 0$, $m = 1$ and

$$g(z) = z^{-p} + \frac{\Gamma(\alpha + \eta)}{\Gamma(\eta)} \sum_{k=1}^{\infty} \frac{\Gamma(k + \eta)}{\Gamma(k + \eta + \alpha)} z^{k-p} \quad (\alpha \geq 0, \eta > -1; p \in \mathbb{N}; f \in \Sigma_p),$$

we see that $D_{\lambda,p}^0(f \ast g)(z) = (f \ast g)(z) = Q_{\eta,p}^n f(z)$, where $Q_{\eta,p}^n$ was introduced by Aouf and Mostafa [4],

We denote by $\Sigma_{\lambda,p,m}^n(\alpha, \delta, \mu, \beta)$ the class of all functions $f(z) \in \Sigma_{p,m}$ such that

$$\text{Re} \left\{ (1 - \beta) \left( \frac{D_{\lambda,p}^n(f \ast g)(z)}{D_{\lambda,p}^n(h \ast g)(z)} \right)^\mu + \beta \frac{D_{\lambda,p}^{n+1}(f \ast g)(z)}{D_{\lambda,p}^{n+1}(h \ast g)(z)} \left( \frac{D_{\lambda,p}^n(f \ast g)(z)}{D_{\lambda,p}^n(h \ast g)(z)} \right)^\mu \right\} > \alpha,$$

where $(g \ast h)(z) \in \Sigma_{p,m}$ satisfies the following condition:

$$\text{Re} \left\{ \frac{D_{\lambda,p}^n(h \ast g)(z)}{D_{\lambda,p}^{n+1}(h \ast g)(z)} \right\} > \delta \quad (0 \leq \delta < 1; z \in U),$$

where $\alpha$ and $\mu$ are real numbers such that $0 \leq \alpha < 1, \mu > 0, p \in \mathbb{N}$ and $\beta \in \mathbb{C}$ with $\Re \{\beta\} > 0$.

We note that:

(i) For $\lambda = 1$ in (9), the class $\Sigma_{\lambda,p,m}^n(\alpha, \delta, \mu, \beta)$ satisfying the following condition:

$$\text{Re} \left\{ (1 - \beta) \left( \frac{D_{p}^n(f \ast g)(z)}{D_{p}^n(h \ast g)(z)} \right)^\mu + \beta \frac{D_{p}^{n+1}(f \ast g)(z)}{D_{p}^{n+1}(h \ast g)(z)} \left( \frac{D_{p}^n(f \ast g)(z)}{D_{p}^n(h \ast g)(z)} \right)^\mu \right\} > \alpha,$$

where $(g \ast h)(z) \in \Sigma_{p,m}$ satisfies the following condition:

$$\text{Re} \left\{ \frac{D_{p}^n(h \ast g)(z)}{D_{p}^{n+1}(h \ast g)(z)} \right\} > \delta \quad (0 \leq \delta < 1; z \in U),$$

(ii) If $b_k = 1 \left( \text{or } g(z) = \frac{1}{z^p (1 - z)} \right)$ in (9), the class $\Sigma_{\lambda,p,m}^n(\alpha, \delta, \mu, \beta)$ satisfying the following condition:
where \( h(z) \in \Sigma_{p,m} \) satisfies the following condition:

\[
\text{Re} \left\{ \left(1 - \beta \right) \left( \frac{D_{\lambda,p} h(z)}{D_{\lambda,p}^n h(z)} \right)^{\mu} + \beta \frac{D_{\lambda,p} h(z)}{D_{\lambda,p}^n h(z)} \right\} > \alpha, \tag{13}
\]

where \( 0 \leq \alpha < 1, \mu > 0, p \in \mathbb{N}, n \in \mathbb{N}_0 \) and \( \beta \in \mathbb{C} \) with \( \text{Re} \{ \beta \} > 0; \)

(iii) for \( n = 0 \) and \( g(z) \) is defined by (6) in (9), the class \( \Sigma_{p,m}(\alpha, \delta, \mu, \beta) \) satisfying the following condition:

\[
\text{Re} \left\{ \left(1 - \beta \right) \left( \frac{I_p(\lambda, \ell) f(z)}{I_p(\lambda, \ell) h(z)} \right)^{\mu} + \beta \frac{I_p(\lambda, \ell) f(z)}{I_p(\lambda, \ell) h(z)} \right\} > \alpha, \tag{15}
\]

where \( h(z) \in \Sigma_{p,m} \) satisfies the following condition:

\[
\text{Re} \left\{ \frac{I_p(\lambda, \ell) h(z)}{I_p(\lambda, \ell) h(z)} \right\} > \delta \quad (0 \leq \delta < 1; z \in U), \tag{16}
\]

where \( 0 \leq \alpha < 1, \mu > 0, p \in \mathbb{N}, r \in \mathbb{N}_0 \) and \( \beta \in \mathbb{C} \) with \( \text{Re} \{ \beta \} > 0; \)

(iv) for \( n = 0 \) and \( g(z) \) is defined by (7) in (9) reduces to the class \( \Sigma_{p,m}(\alpha, \delta, \mu, \beta) \) which satisfying the following condition:

\[
\text{Re} \left\{ \left(1 - \beta \right) \left( \frac{H_{p,q,s}(\alpha_1) f(z)}{H_{p,q,s}(\alpha_1) h(z)} \right)^{\mu} + \beta \frac{H_{p,q,s}(\alpha_1) f(z)}{H_{p,q,s}(\alpha_1) h(z)} \right\} > \alpha, \tag{17}
\]

where \( h(z) \in \Sigma_{p,m} \) satisfies the following condition:

\[
\text{Re} \left\{ \frac{H_{p,q,s}(\alpha_1) h(z)}{H_{p,q,s}(\alpha_1) h(z)} \right\} > \delta \quad (0 \leq \delta < 1; z \in U), \tag{18}
\]

where \( 0 \leq \alpha < 1, \mu > 0, p \in \mathbb{N} \) and \( \beta \in \mathbb{C} \) with \( \text{Re} \{ \beta \} > 0. \)

(v) for \( n = 0 \) and \( g(z) \) is defined by (8) in (9) reduces to the class \( \Sigma_{p,m}(\alpha, \delta, \mu, \beta) \) which satisfying the following condition:

\[
\text{Re} \left\{ \left(1 - \beta \right) \left( \frac{Q_{q,p} f(z)}{Q_{q,p} h(z)} \right)^{\mu} + \beta \frac{Q_{q,p}^{-1} f(z)}{Q_{q,p}^{-1} h(z)} \right\} > \alpha, \tag{19}
\]

where \( h(z) \in \Sigma_{p,m} \) satisfies the following condition:

\[
\text{Re} \left\{ \frac{Q_{q,p}^{-1} h(z)}{Q_{q,p}^{-1} h(z)} \right\} > \delta \quad (0 \leq \delta < 1; z \in U), \tag{20}
\]

where \( 0 \leq \alpha < 1, \mu > 0, p \in \mathbb{N} \) and \( \beta \in \mathbb{C} \) with \( \text{Re} \{ \beta \} > 0. \)

To establish our main results we need the following lemmas.

**Lemma 1** [16]. Let \( \Omega \) be a set in the complex plane \( \mathbb{C} \) and let the function \( \Psi : \mathbb{C}^2 \to \mathbb{C} \) satisfy the function \( \Psi(ir_2, s_1) \notin \Omega \) for all real \( r_2, s_1 \leq -\frac{1+r_2^2}{2} \). If \( q(z) \)
is analytic in $U$ with $q(0) = 1$ and $\Psi(q(z), zq'(z)) \in \Omega, z \in U$, then $\Re \{q(z)\} > 0 \ (z \in U)$.

**Lemma 2** [17]. If $q(z)$ is analytic in $U$, with $q(0) = 1$, and If $\lambda \in \mathbb{C}^*$ with $\Re(\lambda) \geq 0$ then $\Re \left\{ q(z) + \lambda zq'(z) \right\} > \alpha \ (0 \leq \alpha < 1)$ implies

$$\Re q(z) > \alpha + (1 - \alpha)(2\gamma - 1),$$

where $\gamma$ is given by

$$\gamma = \gamma(\Re \lambda) = \int_0^1 \left( 1 + t^{\Re(\lambda)} \right)^{-1} dt,$$

which is an increasing function of $\Re \{\lambda\}$ and $\frac{1}{2} \leq \gamma < 1$. This estimate is sharp in the sense that the bound cannot be improved.

For real or complex numbers $a, b$ and $c \ (c \notin \mathbb{Z}^{-})$, the Gauss hypergeometric function is defined by

$$\left( a + 1 \right) \left( a + 2 \right) \cdots \left( a + n \right) \left( b \right) \left( c \right) \cdots \left( c + n \right) \frac{z^n}{n!}.$$

We note that the above series converges absolutely for $z \in U$ and hence represents an analytic function in $U$ (see, for details, [21, Chapter 14]).

Each of the identities (asserted by Lemma 3 below ) is fairly well known (cf., e.g., [21, Chapter 14]).

**Lemma 3.** For real or complex parameters $a, b$ and $c \ (c \notin \mathbb{Z}^{-})$,

\begin{equation}
\int_0^{1-b-1} (1-t)^{-b-1} (1-zt)^{-a} \, dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} 2F_1(a, b; c; z) \tag{21}
\end{equation}

$$(\Re(c) > \Re(b) > 0) ;$$

\begin{equation}
2F_1(a, b; c; z) = 2F_1(b, a; c; z) ; \tag{22}
\end{equation}

\begin{equation}
2F_1(a, b; c; z) = (1-z)^{-a} 2F_1(a, c-b; c; \frac{z}{z-1}) ; \tag{23}
\end{equation}

\begin{equation}
(a+1) 2F_1(1, a; a+1; z) = (a+1) + az \ 2F_1(1, a+1; a+2; z) \tag{24}
\end{equation}

and

\begin{equation}
2F_1(1, 1; 2; \frac{1}{2}) = 2 \ln 2. \tag{25}
\end{equation}

The methods used here to obtain our main results are similar to those of Kwon et al. [12], El-Ashwah [9], El-Ashwah and Aouf [11] and Aouf and Mostafa [4].
2. Main Result

Unless otherwise mentioned, we shall assume in the reminder of this paper that \(0 \leq \alpha < 1, 0 \leq \delta < 1, \mu > 0, \lambda \geq 0, m > -p, p \in \mathbb{N}, n \in \mathbb{N}_0\) and \(\beta \in \mathbb{C}\) with \(\Re \{\beta\} > 0\).

**Theorem 1.** Let \(f(z) \in \Sigma_n^{\alpha, \delta, \mu, \beta}\) with \((f \ast g)(z) \in \Sigma_{p,m}\). Then

\[
\Re \left( \frac{D_n^{\lambda}(f \ast g)(z)}{D_n^{\lambda}(h \ast g)(z)} \right) > \frac{2\alpha \mu + \delta \beta \lambda}{2\mu + \delta \beta \lambda} \quad (z \in U),
\]

where the function \((h \ast g)(z) \in \Sigma_{p,m}\) satisfies the condition (10).

**Proof.** Let \(\gamma = \frac{2\alpha \mu + \delta \beta \lambda}{2\mu + \delta \beta \lambda}\)

\[
q(z) = \frac{1}{(1 - \gamma)} \left\{ \left( \frac{D_n^{\lambda}(f \ast g)(z)}{D_n^{\lambda}(h \ast g)(z)} \right)^\mu - \gamma \right\}. \tag{27}
\]

Then \(q(z)\) is analytic in \(U\), \(q(0) = 1\). If we set

\[
\kappa(z) = \frac{D_n^{\lambda}(h \ast g)(z)}{D_n^{\lambda+1}(h \ast g)(z)}, \tag{28}
\]

then by the hypothesis \(\Re \{\kappa(z)\} > \delta\). Differentiating (27) and using the identity (5), we have

\[
(1 - \beta) \left( \frac{D_n^{\lambda}(f \ast g)(z)}{D_n^{\lambda}(h \ast g)(z)} \right)^\mu + \beta \frac{D_n^{\lambda+1}(f \ast g)(z)}{D_n^{\lambda+1}(h \ast g)(z)} \left( \frac{D_n^{\lambda}(f \ast g)(z)}{D_n^{\lambda}(h \ast g)(z)} \right)^{\mu-1} = [\gamma + (1 - \gamma) q(z)] + \frac{\lambda \beta (1 - \gamma)}{\mu} \kappa(z) q'(z). \tag{29}
\]

Let us define the function \(\Psi(r, s)\) by

\[
\Psi(r, s) = \gamma + (1 - \gamma) r + \frac{\lambda \beta (1 - \gamma)}{\mu} \kappa(z)s. \tag{30}
\]

Using (30) and the fact that \(f(z) \in \Sigma_n^{\alpha, \delta, \mu, \beta}\), we obtain

\[
\left\{ \Psi(q(z), zq'(z)); z \in U \right\} \subset \Omega = \left\{ w \in \mathbb{C} : \Re(w) > \alpha \right\}.
\]

Now for all real \(r_2, s_1 \leq -\frac{1 + r_2^2}{2}\), we have

\[
\Re \{\Psi(ir_2, s_1)\} = \gamma + \frac{\lambda \beta (1 - \gamma)}{\mu} \Re \{\kappa(z)\} \leq \gamma - \frac{\lambda \beta (1 - \gamma) \delta (1 + r_2^2)}{2\mu} \leq \gamma - \frac{\lambda \beta (1 - \gamma) \delta}{2\mu} = \alpha.
\]
Hence for each \( z \in U, \Psi(ir_2, s_1) \notin \Omega \). Thus by Lemma 1, we have \( \Re \{g(z)\} > 0(z \in U) \) and hence
\[
\Re \left( \frac{D^n_{\lambda, p}(f * g)(z)}{D^n_{\lambda, p}(h * g)(z)} \right)^\mu > \gamma \quad (z \in U).
\]
This proves Theorem 1.

**Remark 1.** Putting \( m = 0 \) and \( b_k = 1 \) \(( \text{or } g(z) = \frac{1}{z^p(1 - z)} \) in Theorem 1, we obtain the result obtained by El-Ashwah and Aouf [11], Corollary 1.

**Corollary 1.** Let the functions \((f * g)(z)\) and \((h * g)(z)\) be in \( \Sigma_{p,m} \) and let \((h * g)(z)\) satisfy the condition (10). If \( \lambda \geq 0, \gamma \geq 1, p \in \mathbb{N}, m \in \mathbb{N}_0 \) and
\[
\Re \left\{ (1 - \beta) \left( \frac{D^n_{\lambda, p}(f * g)(z)}{D^n_{\lambda, p}(h * g)(z)} \right) + \beta \frac{D^{n+1}_{\lambda, p}(f * g)(z)}{D^{n+1}_{\lambda, p}(h * g)(z)} \right\} > \alpha
\]
then
\[
\Re \left\{ \frac{D^{n+1}_{\lambda, p}(f * g)(z)}{D^{n+1}_{\lambda, p}(h * g)(z)} \right\} > \beta = \frac{\alpha(2 + \delta \lambda) + \delta \lambda (\beta - 1)}{2 + \delta \beta \lambda} \quad (z \in U).
\]

**Proof.** We have
\[
\beta \frac{D^{n+1}_{\lambda, p}(f * g)(z)}{D^{n+1}_{\lambda, p}(h * g)(z)} = \left\{ (1 - \beta) \left( \frac{D^n_{\lambda, p}(f * g)(z)}{D^n_{\lambda, p}(h * g)(z)} \right) + \beta \frac{D^{n+1}_{\lambda, p}(f * g)(z)}{D^{n+1}_{\lambda, p}(h * g)(z)} \right\}
\]
\[
+ (\beta - 1) \frac{D^n_{\lambda, p}(f * g)(z)}{D^n_{\lambda, p}(h * g)(z)} \quad (z \in U).
\]
Since \( \beta \geq 1 \), making use of (31) and (26) \(( \text{for } \mu = 1) \), we deduce that
\[
\Re \left\{ \frac{D^{n+1}_{\lambda, p}(f * g)(z)}{D^{n+1}_{\lambda, p}(h * g)(z)} \right\} > \beta = \frac{\alpha(2 + \delta \lambda) + \delta \lambda (\beta - 1)}{2 + \delta \beta \lambda}.
\]
Thus the proof of Corollary 1 is complete.

**Corollary 2.** Let \( \beta \in \mathbb{C}^* \) with \( \Re \{\beta\} \geq 0 \) and \( \lambda > 0 \). If \((f * g)(z)\) \(\in \Sigma_{p,m}\) satisfies the following condition:
\[
\Re \left\{ (1 - \beta)(z^p(D^n_{\lambda, p}(f * g)(z))^\mu + \beta z^pD^{n+1}_{\lambda, p}(f * g)(z)) \right\} > \alpha \quad (z \in U),
\]
then
\[
\Re \left\{ z^pD^n_{\lambda, p}(f * g)(z) \right\} \geq \frac{2\alpha + \lambda \Re(\beta)}{2\mu + \lambda \Re(\beta)} \quad (z \in U).
\]
Further, if \( \beta \geq 1, \lambda > 0 \) and \((f * g)(z)\) \(\in \Sigma_{p,m}\) satisfies
\[
\Re \left\{ (1 - \beta)z^pD^n_{\lambda, p}(f * g)(z) + \beta z^pD^{n+1}_{\lambda, p}(f * g)(z) \right\} > \alpha \quad (z \in U),
\]
then
\[
\Re \left\{ z^pD^{n+1}_{\lambda, p}(f * g)(z) \right\} > \frac{\alpha + \lambda (\beta - 1)}{2 + \beta \lambda} \quad (z \in U).
\]
Proof. The results (33) and (34) follows by putting $h(z) = \frac{1}{z^p}$ in Theorem 1 and Corollary 1, respectively.

Choosing $\beta, \delta, \mu, \lambda$ and $n$ appropriately in Corollary 2, we obtain the following results.

**Remark 2.** (i) For $\beta = \lambda = 1$ and $n = 0$ in Corollary 2, we have:

$$\text{Re} \left\{ \left( 1 + \frac{\beta}{p} \frac{z((f * g)(z))'}{f * g(z)} \right) \left( z^p(f * g)(z) \right)^\mu \right\} > \alpha$$  (35)

implies

$$\text{Re} \{ z^p(f * g)(z) \} > \frac{2\mu \alpha + 1}{2\mu + 1} \quad (z \in U).$$

(ii) For $\beta \in \mathbb{C}^*$ with $\text{Re}\{\beta\} \geq 0$, $\mu = \lambda = 1$ and $n = 0$, in Corollary 2, we have:

$$\text{Re} \left\{ \left( 1 + \beta p \right) z^p(f * g)(z) + \frac{\beta}{p} z^{p+1}((f * g)(z))' \right\} > \alpha$$

implies

$$\text{Re} \{ z^p(f * g)(z) \} > \frac{2\alpha + \text{Re}\{\beta\}}{2 + \text{Re}\{\beta\}} \quad (z \in U).$$

(iii) Replacing $(f * g)(z)$ by $-\frac{z((f * g)(z))'}{p}$ in the result (ii), we have:

$$-\text{Re} \left\{ \left( 1 + \beta p + \frac{\beta}{p} \frac{z^{p+1}((f * g)(z))'}{p} + \frac{\beta}{p} z^{p+1}((f * g)(z))'' \right) \right\} > \alpha$$

implies

$$-\text{Re} \left\{ \frac{z^{p+1}((f * g)(z))'}{p} \right\} > \frac{2\alpha + \text{Re}\{\beta\}}{2 + \text{Re}\{\beta\}} \quad (z \in U).$$

(iv) For $\beta \in \mathbb{R}$ with $\beta \geq 1$, $\mu = \lambda = 1$ and $n = 0$ in Corollary 2, we have:

$$\text{Re} \left\{ \left( 1 + \beta p \right) z^p(f * g)(z) + \frac{\beta}{p} z^{p+1}((f * g)(z))' \right\} > \alpha$$

implies

$$\text{Re} \{ z^p(f * g)(z) \} > \frac{3\alpha + \beta - 1}{2 + \beta} \quad (z \in U).$$

(v) Putting $\beta = \lambda = 1$, in Corollary 2, we have:

$$\text{Re} \left\{ z^p D^{n+1}_p \left( f * g \right)(z) \left( z^p D^n_p(f * g)(z) \right)^\mu \right\} > \alpha$$

implies

$$\text{Re} \left\{ z^p D^n_p(f * g)(z) \right\}^\mu > \frac{2\mu \alpha + 1}{2\mu + 1} \quad (z \in U).$$

(vi) For $\beta \in \mathbb{C}^*$ with $\text{Re}\{\beta\} \geq 0$, $\mu = \lambda = 1$ in Corollary 2, we have:

$$\text{Re} \left\{ \left( 1 - \beta \right) z^p D^n_p(f * g)(z) + \beta z^{p+1}(f * g)(z) \right\} > \alpha$$

implies

$$\text{Re} \left\{ z^p D^n_p(f * g)(z) \right\} > \frac{2\alpha + \text{Re}\{\beta\}}{2 + \text{Re}\{\beta\}} \quad (z \in U).$$
Theorem 2. Let $\beta \in C$ with $\text{Re}\{\beta\} > 0$ and $\lambda > 0$. Let $(f \ast g)(z) \in \Sigma_{p,m}$ satisfy the following condition:

$$\text{Re}\left\{ (1 - \beta)(z^p D_{\lambda,p}^n(f \ast g)(z))^\mu + \beta z^p D_{\lambda,p}^{n+1}(f \ast g)(z)(z^p D_{\lambda,p}^n(f \ast g)(z))^{\mu-1} \right\} > \alpha.$$  \hfill (36)

Then

$$\text{Re}\left\{ z^p D_{\lambda,p}^n(f \ast g)(z) \right\}^\mu > \alpha + (1 - \alpha)(2\rho - 1), \hfill (37)$$

where

$$\rho = \frac{1}{2} \left[ 2 F_1(1, 1; \frac{\mu}{\lambda \text{Re}\{\beta\}} + 1; \frac{1}{2}) \right]. \hfill (38)$$

Proof. Let

$$q(z) = (z^p D_{\lambda,p}^n(f \ast g)(z))^\mu. \hfill (39)$$

Then $q(z)$ is analytic in $U$ with $q(0) = 1$. Differentiating (39) with respect to $z$ and using the identity (5), we obtain

$$(1 - \beta)(z^p D_{\lambda,p}^n(f \ast g)(z))^\mu + \beta z^p D_{\lambda,p}^{n+1}(f \ast g)(z)(z^p D_{\lambda,p}^n(f \ast g)(z))^{\mu-1} = q(z) + \frac{\beta \lambda z q'(z)}{\mu},$$

so that by the hypothesis (36), we have

$$\text{Re}\left\{ q(z) + \frac{\beta \lambda z q'(z)}{\mu} \right\} > \alpha \quad (z \in U).$$

In view of Lemma 2, this implies that

$$\text{Re}\{q(z)\} > \alpha + (1 - \alpha)(2\rho - 1),$$

where

$$\rho = \rho(\text{Re}\{\beta\}) = \int_0^1 \left( 1 + t \frac{\lambda \text{Re}\{\beta\}}{\mu} \right)^{-1} dt.$$

Putting $\text{Re}\{\beta\} = \beta_1 > 0$, we have

$$\rho = \int_0^1 \left( 1 + t \frac{\lambda \beta_1}{\mu} \right)^{-1} dt = \frac{\mu}{\lambda \beta_1} \int_0^{\frac{\mu}{\lambda \beta_1}} (1 + u)^{-1} du.$$

Using (21), (22), (23) and (25), we obtain

$$\rho = \frac{1}{2} 2 F_1(1, 1; \frac{\mu}{\lambda \beta_1} + 1; 1)$$.  

Thus the proof of Theorem 2 is complete.

Remark 3. Putting $m = 0$ and $b_k = 1 \left( \text{or } g(z) = \frac{1}{z^p (1 - z)} \right)$ in Theorem 2, we obtain the result obtained by El-Ashwah and Aouf [11, Corollary 4].

Corollary 3. Let $\beta \in R$ with $\beta \geq 1$. If $(f \ast g)(z) \in \Sigma_{p,m}$ satisfies

$$\text{Re}\left\{ (1 - \beta)z^p D_{\lambda,p}^n(f \ast g)(z) + \beta z^p D_{\lambda,p}^{n+1}(f \ast g)(z) \right\} > \alpha$$ \hfill (40)
then
\[ \text{Re}\{z^p D_{\lambda,p}^{n+1}(f \ast g)(z)\} > \alpha + (1 - \alpha)(2\rho^* - 1)(1 - \beta^{-1}) \quad (z \in U), \]

where
\[ \rho^* = \frac{1}{2} {}_2F_1(1, 1; \frac{1}{\beta\lambda} + 1; \frac{1}{2}). \]

\textbf{Proof.} The result follows by using the identity
\[ \beta z D_{n+1}^p(f \ast g)(z) = [(1 - \beta)z D_{n}^p(f \ast g)(z) + \beta z D_{n}^p(f \ast g)(z)] \]
\[ + (\beta - 1)z D_{n}^p(f \ast g)(z). \quad (41) \]

\textbf{Remark 4.} (i) We note that if \( \beta = \mu > 0, \lambda = 1 \) and \( n = 0 \) in Corollary 2, that is,
\[ \text{Re}\{z p D_{n+1}^p(f \ast g)(z)\} > \alpha \quad (42) \]
then (33) implies that
\[ \text{Re}\{z^p (f \ast g)(z)\} > \frac{2\alpha + 1}{3} \quad (z \in U), \quad (43) \]
whereas, if \( (f \ast g)(z) \in \Sigma_{p,m} \) satisfies the condition (42) then by using Theorem 2, we have
\[ \text{Re}\{z^p (f \ast g)(z)\} > 2(1 - \ln 2)\alpha + (2 \ln 2 - 1) \quad (z \in U), \]
which is better than (43).

(ii) We observe that if \( \beta \in \mathbb{R}, \) satisfying \( \beta > 0 \) and
\[ k(z) = \frac{D_{n+1}^p(f \ast g)(z)}{D_{n}^p(f \ast g)(z)} + \frac{1}{\beta - 1} \frac{D_{n}^p(f \ast g)(z)}{D_{n}^p(h \ast g)(z)} \quad (z \in U), \]
then from Theorem 1 (for \( \mu = 1 \)), we have
\[ \text{Re}\{k(z)\} > \frac{\alpha}{\beta} \]
implies
\[ \text{Re}\left\{ \frac{D_{n}^p(f \ast g)(z)}{D_{n}^p(h \ast g)(z)} \right\} > \frac{2\alpha + \delta\beta\lambda}{2 + \delta\beta\lambda} \quad (44) \]
whenever
\[ \text{Re}\left\{ \frac{D_{n}^p(h \ast g)(z)}{D_{n+1}^{n+1}(h \ast g)(z)} \right\} > \delta. \]

Let \( \beta \to +\infty, \) then from (44), we have:
\[ \text{Re}\{k(z)\} \geq 0 \quad (z \in U) \text{ implies} \quad \text{Re}\left\{ \frac{D_{n}^p(f \ast g)(z)}{D_{n}^p(h \ast g)(z)} \right\} \geq 1 \quad (z \in U), \]
whenever \( \text{Re}\left\{ \frac{D_{n}^p(h \ast g)(z)}{D_{n+1}^{n+1}(h \ast g)(z)} \right\} > \delta \quad (z \in U). \)

In the following theorem, we shall extend the above results as follows:
Theorem 3. Suppose the functions \((f \ast g)(z)\) and \((h \ast g)(z)\) are in \(\Sigma_{p,m}\) and suppose \((h \ast g)(z)\) satisfies the condition (10). If

\[
\text{Re} \left\{ \frac{D_{n+1}^n(f \ast g)(z)}{D_{n+1}^n(h \ast g)(z)} - \frac{D_n^m(f \ast g)(z)}{D_n^m(h \ast g)(z)} \right\} > -\frac{(1-\alpha)\delta \lambda}{2},
\]

then

\[
\text{Re} \left\{ \frac{D_n^m(h \ast g)(z)}{D_n^m(h \ast g)(z)} \right\} > \alpha \quad (z \in U)
\]

and

\[
\text{Re} \left\{ \frac{D_n^m(f \ast g)(z)}{D_n^m(h \ast g)(z)} \right\} > \frac{(2+\lambda \delta)\alpha - \lambda \delta}{2}.
\]

Proof. Let

\[ q(z) = \frac{1}{(1-\alpha)} \left\{ \frac{D_n^m(h \ast g)(z)}{D_n^m(h \ast g)(z)} \alpha \right\}. \]

Then \(q(z)\) is analytic in \(U\) with \(q(0) = 1\). Putting

\[
\phi(z) = \frac{D_n^m(h \ast g)(z)}{D_n^m(h \ast g)(z)} \quad (z \in U)
\]

we observe that by hypothesis, \(\text{Re}\{\phi(z)\} > \delta \quad (0 \leq \delta < 1)\) in \(U\). A simple computation shows that

\[
\lambda(1-\alpha)zq'(z)\phi(z) = \frac{D_n^m(h \ast g)(z)}{D_n^m(h \ast g)(z)} - \frac{D_n^m(h \ast g)(z)}{D_n^m(h \ast g)(z)}
\]

\[
= \Psi(q(z), zq'(z)),
\]

where

\[
\Psi(r, s) = \lambda(1-\alpha)\phi(z)s.
\]

Using the hypothesis (45), we obtain

\[
\left\{ \Psi(q(z), zq'(z); z \in U \right\} \subset \Omega = \left\{ w \in \mathbb{C} : \text{Re} \ w > -\frac{\lambda \delta(1-\alpha)}{2} \right\}.
\]

Now, for all real \(r_2, s_1 \leq -\frac{(1+r_2^2)}{2}\), we have

\[
\text{Re} \left\{ \Psi(ir_2, s_1) \right\} = \lambda s_1 (1-\alpha) \text{Re} \left\{ \phi(z) \right\}
\]

\[
\leq -\frac{\lambda \delta(1-\alpha)(1+r_2^2)}{2}
\]

\[
\leq -\frac{\lambda \delta(1-\alpha)}{2}.
\]
This shows that $\Psi(\alpha_2, s_1) \notin \Omega$ for each $z \in U$. Hence by Lemma 1, we have $Re\{q(z)\} > 0$ ($z \in U$). This proves (46). The proof of (47) follows by using (46) and (47) in the identity:

$$
Re \left\{ \frac{D_{\lambda_p}^{n+1}(f \ast g)(z)}{D_{\lambda_p}^{n+1}(h \ast g)(z)} \right\} = Re \left\{ \frac{D_{\lambda_p}^{n+1}(f \ast g)(z)}{D_{\lambda_p}^{n+1}(h \ast g)(z)} - \frac{D_{\lambda_p}^{n}(f \ast g)(z)}{D_{\lambda_p}^{n}(h \ast g)(z)} \right\} + Re \left\{ \frac{D_{\lambda_p}^{n}(f \ast g)(z)}{D_{\lambda_p}^{n}(h \ast g)(z)} \right\}.
$$

This completes the proof of Theorem 3.

Putting $\lambda = 1$ in Theorem 3, we obtain

**Corollary 4.** Suppose the functions $(f \ast g)(z)$ and $(h \ast g)(z)$ are in $\Sigma_{p,m}$ and suppose $(h \ast g)(z)$ satisfies

$$
Re \left\{ \frac{D_{p}^{n}(h \ast g)(z)}{D_{p}^{n+1}(h \ast g)(z)} \right\} > \delta \quad (z \in U).
$$

If

$$
Re \left\{ \frac{D_{p}^{n+1}(f \ast g)(z)}{D_{p}^{n+1}(h \ast g)(z)} - \frac{D_{p}^{n}(f \ast g)(z)}{D_{p}^{n}(h \ast g)(z)} \right\} > \frac{(1 - \alpha)\delta}{2}
$$

then

$$
Re \left\{ \frac{D_{p}^{n}(f \ast g)(z)}{D_{p}^{n+1}(h \ast g)(z)} \right\} > \alpha \quad (z \in U)
$$

and

$$
Re \left\{ \frac{D_{p}^{n+1}(f \ast g)(z)}{D_{p}^{n+1}(h \ast g)(z)} \right\} > \frac{(2 + \delta)\alpha - \delta}{2}.
$$

**Remark 5.** Putting $\delta = \lambda = 1, n = 0$ and $(h \ast g)(z) = \frac{1}{z^p}$ in Theorem 3, we obtain:

$$
Re\{z^p(f \ast g)(z) + \frac{z^{p+1}}{p}((f \ast g)(z))'\} > \frac{(1 - \alpha)}{2p},
$$

implies

$$
Re\{z^p(f \ast g)(z)\} > \alpha
$$

and

$$
Re\{(1 + p)z^p(f \ast g)(z) + z^{p+1}((f \ast g)(z))'\} > \frac{3\alpha - 1}{2}.
$$

**REFERENCES**


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