ERROR ANALYSIS OF AN EXPPLICIT FINITE DIFFERENCE APPROXIMATION FOR THE TWO-DIMENSIONAL SPACE FRACTIONAL DIFFUSION EQUATION

N. H. SWEILAM, A. M. NAGY, T. F. ALMAJBRI

ABSTRACT. In this paper, the space fractional diffusion equation (SFDE) is numerically studied, where the fractional derivative is defined in the sense of the right-shifted Grünwald. An explicit finite difference approximation (EFDA) for SFDE is presented. The stability and the error analysis of the EFDA are discussed. To demonstrate the effectiveness of the approximated method, some test examples are presented.

1. Introduction

It is well known that fractional derivatives in mathematics are natural extension of integer-order derivatives, where the order is non integer [9]. Fractional order differential equations have been the focus of many studies due to their frequent appearance in various applications especially in the fields of fluid mechanics, viscoelasticity, biology, physics and engineering, see ([2], [3], [5], [9], [10]) and the references cited in the thesis. When a fractional derivative of order \(1 < \alpha < 2\) replaces the second derivative in a diffusion or dispersion model, it leads to a superdiffusive flow model. Analytic closed-form solutions for these initial-boundary value problems are elusive. Difference methods and, in particular, explicit finite difference methods, are an important class of numerical methods for solving fractional differential equations ([4], [16], [17]). The usefulness of the explicit method and the reason why they are widely employed is based on their particularly attractive features ([18], [20]). In this paper, EFDA scheme is designed for solving a two-dimensional fractional order diffusion equation where the fractional derivative is in the right-shifted Grünwald sense. Moreover, since the explicit methods may be unstable, then, it is crucial to determine under which conditions, if any, these methods are stable. We will use here a kind of fractional von Neumann stability analysis to derive the stability conditions. Consider a two-dimensional fractional diffusion equation:

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\[ \frac{\partial u(x, y, t)}{\partial t} = d(x, y) \frac{\partial^\alpha u(x, y, t)}{\partial x^\alpha} + e(x, y) \frac{\partial^\beta u(x, y, t)}{\partial y^\beta} + q(x, y, t), \] (1)

on a finite domain \( x_L < x < x_H \) and \( y_L < y < y_H \), with fractional orders \( 1 < \alpha, \beta \leq 2 \), where the diffusion coefficients \( d(x, y) > 0 \) and \( e(x, y) > 0 \). The function \( q(x, y, t) \) can be used to represent sources and sinks. Assume the initial condition \( u(x, y, t = 0) = f(x, y) \) for \( x_L < x < x_H, y_L < y < y_H \), and Dirichlet boundary conditions \( u(x, y, t) = B(x, y, t) \) on the boundary (perimeter) of the rectangular region \( x_L < x < x_H, y_L < y < y_H \), with the additional restriction that \( B(x_L, y, t) = B(x, y_L, t) = 0 \). The classical dispersion equation in two dimensions is given by \( \alpha = \beta = 2 \), for more details on the model problem, see [15].

2. EFDA for SFDE

In this work, the spatial \( \alpha \)-order fractional derivative is discretized using the right-shifted Grünwald formula [4], for \( 1 < \alpha < 2 \):

\[ \frac{\partial^\alpha u(x, y, t)}{\partial x^\alpha} = \frac{1}{\Gamma(-\alpha)} \lim_{\Delta x \to 0} \frac{1}{h^\alpha} \sum_{k=0}^{N_x} \frac{\Gamma(k-\alpha)}{\Gamma(k+1)} u[x-(k-1)h, y, t], \] (2)

where \( N_x \) is a positive integer, \( h = (x - x_L)/N_x \) and \( \Gamma(\cdot) \) is the gamma function. We also define the normalized Grünwald weights by

\[ g_{\alpha, k} = \frac{\Gamma(k-\alpha)}{\Gamma(-\alpha)\Gamma(k+1)} = (-1)^k \left( \frac{\alpha}{k} \right), \] (3)

these normalized weights only depend on the order \( \alpha \) and the index \( k \). For example, the first four terms of this sequence are given by \( g_{\alpha, 0} = 1, g_{\alpha, 1} = -\alpha, g_{\alpha, 2} = \alpha(\alpha - 1)/2! \), \( g_{\alpha, 3} = -\alpha(\alpha - 1)(\alpha - 2)/3! \). For the numerical approximation scheme, define \( t_n = n\Delta t \), to be the integration time, \( 0< t_n, \Delta x = (x_H - x_L)/N_x = h_x > 0 \), is the grid size in \( x \)-direction, with \( x_i = x_L + i\Delta x \), for \( i = 0, 1, ..., N_x \); \( \Delta y = (y_H - y_L)/N_y = h_y > 0 \), is the grid size in \( y \)-direction, with \( y_j = y_L + j\Delta y \), for \( j = 0, 1, ..., N_y \). Define \( u^n_{i,j} \) as the numerical approximation to \( u(x_i, y_j, t_n) \). Similarly, define \( d_i,j = d(x_i, y_j), e_{i,j} = e(x_i, y_j) \), and \( q^n_{i,j} = q(x_i, y_j, t_n) \). The initial conditions are set by \( u^n_{i,j} = f_i,j = f(x_i, y_j) \). The Dirichlet boundary condition on the boundary of this rectangular region are at \( x = x_L, u^n_{i,0} = B^n_{i,0} = B(x_L, y_j, t_n) = 0 \); at \( x = x_H, u^n_{i,N_x,j} = B^n_{i,N_x,j} = B(x_H, y_j, t_n) \); at \( y = y_L, u^n_{i,0} = B^n_{i,0} = B(x_i, y_L, t_n) = 0 \); and at \( y = y_H, u^n_{i,N_y} = B^n_{i,N_y} = B(x_i, y_H, t_n) \). If the shifted Grünwald estimates are substituted into the two-dimensional diffusion problem [1] to get the explicit finite difference approximation, the resulting finite difference equations are:

\[ \frac{u^n_{i,j} - u^{n-1}_{i,j}}{\Delta t} = d_{i,j} \delta_{\alpha,x} u^n_{i,j} + e_{i,j} \delta_{\beta,y} u^n_{i,j} + q^n_{i,j} + T(x, y, t), \] (4)

where \( T(x, y, t) \) is the truncation term [4],

\[ \delta_{\alpha,x} u^n_{i,j} = \frac{1}{(\Delta x)^\alpha} \sum_{k=0}^{i+1} g_{\alpha, k} u^n_{i-k+1,j}, \]

\[ \delta_{\beta,y} u^n_{i,j} = \frac{1}{(\Delta y)^\beta} \sum_{k=0}^{j+1} g_{\beta, k} u^n_{i,j-k+1}. \]
Then

\[
\frac{u_{n+1}^{i,j} - u_n^{i,j}}{\Delta t} = \frac{d_{i,j}}{(\Delta x)^\alpha} \sum_{k=0}^{i+1} g_{\alpha,k} u_n^{i-k,j+1} + \frac{\epsilon_{i,j}}{(\Delta y)^\beta} \sum_{k=0}^{j+1} g_{\beta,k} u_n^{i,j-k+1} + q_n^{i,j},
\]

(5)

\[
u_{n+1}^{i,j} = u_n^{i,j} + s_1 \sum_{k=0}^{i+1} g_{\alpha,k} u_n^{i-k,j+1} + s_2 \sum_{k=0}^{j+1} g_{\beta,k} u_n^{i,j-k+1} + \Delta t q_n^{i,j},
\]

(6)

where \(s_1 = s_1 d_{i,j}, s_1 = \frac{\Delta t}{(\Delta x)^\alpha}; s_2 = s_2 \epsilon_{i,j}, s_2 = \frac{\Delta t}{(\Delta y)^\beta}\).

3. Stability Analysis of EFDA

In this section we will use here a kind of fractional von Neumann method to study the stability analysis of the explicit finite difference scheme (6).

**Theorem 1** The explicit finite-difference scheme (6) for SFDE is conditionally stable if

\[\Delta t \leq S,\]

where

\[S = \frac{-2A}{A^2 + B^2},\]

\[A = \frac{d_{i,j}}{(\Delta x)^\alpha} \sum_{k=0}^{i+1} g_{\alpha,k} [\cos(q_1(k-1)\Delta x)] + \frac{\epsilon_{i,j}}{(\Delta y)^\beta} \sum_{k=0}^{j+1} g_{\beta,k} [\cos(q_2(k-1)\Delta y)].\]

\[B = \frac{d_{i,j}}{(\Delta x)^\alpha} \sum_{k=0}^{i+1} g_{\alpha,k} [\sin(q_1(k-1)\Delta x)] + \frac{\epsilon_{i,j}}{(\Delta y)^\beta} \sum_{k=0}^{j+1} g_{\beta,k} [\sin(q_2(k-1)\Delta y)].\]

**Proof.** Let us analyze the stability of (6) by substituting in a separated solution

\[u_n^{i,j} = \zeta_n e^{mq_1i\Delta x + mq_2j\Delta y} = \zeta_n e^{mq_1i\Delta x + mq_2j\Delta y}\]

where \(m = \sqrt{-1}, q_1, q_2\) are real spatial wave-number.

Inserting this expression we get

\[\zeta_{n+1} e^{mq_1i\Delta x + mq_2j\Delta y} = \zeta_n e^{mq_1i\Delta x + mq_2j\Delta y}\]

+ \(s_1 \sum_{k=0}^{i+1} g_{\alpha,k} \zeta_n e^{mq_1(i-k+1)\Delta x + mq_2j\Delta y} + s_2 \sum_{k=0}^{j+1} g_{\beta,k} \zeta_n e^{mq_1i\Delta x + mq_2(j-k+1)\Delta y},\)

(7)

divided (7) by \(e^{mq_1i\Delta x + mq_2j\Delta y}\) then we get:

\[\zeta_{n+1} = \zeta_n + s_1 \sum_{k=0}^{i+1} g_{\alpha,k} \zeta_n e^{-mq_1(k-1)\Delta x} + s_2 \sum_{k=0}^{j+1} g_{\beta,k} \zeta_n e^{-mq_2(k-1)\Delta y}.\]

(8)

Using the known Euler’s formula \(e^{m\theta} = \cos \theta + m \sin \theta, m = \sqrt{-1},\) we have:

\[\zeta_{n+1} = \zeta_n + s_1 \sum_{k=0}^{i+1} g_{\alpha,k} \zeta_n [\cos(q_1(k-1)\Delta x) - m \sin(q_1(k-1)\Delta x)]
\]

+ \(s_2 \sum_{k=0}^{j+1} g_{\beta,k} \zeta_n [\cos(q_2(k-1)\Delta y) - m \sin(q_2(k-1)\Delta y)].\)

(9)
Where $\zeta(x)$ means the Riemann zeta function. The stability will be determined by the behaviour of $\zeta_n$. If we write $\zeta_{n+1} = \eta \zeta_n$ and assume that $\eta \equiv \eta(q)$ is independent of time, then we can obtain

$$
\eta \zeta_n = \frac{1}{n} \sum_{k=0}^{n} g_{\alpha,k} \zeta_{n}[\cos(q_1(k-1)\Delta x) - m \sin(q_1(k-1)\Delta x)]
\frac{1}{n} \sum_{k=0}^{n} g_{\beta,k} \zeta_n[\cos(q_2(k-1)\Delta y) - m \sin(q_2(k-1)\Delta y)],
$$

(10)

divided by $\zeta_n$ to obtain the following formula of $\eta$:

$$
\eta = 1 + \frac{1}{n} \sum_{k=0}^{n} g_{\alpha,k} \zeta_{n}[\cos(q_1(k-1)\Delta x) - m \sin(q_1(k-1)\Delta x)]
\frac{1}{n} \sum_{k=0}^{n} g_{\beta,k} \zeta_n[\cos(q_2(k-1)\Delta y) - m \sin(q_2(k-1)\Delta y)],
$$

(11)

$$
\eta = 1 + \frac{\Delta t.d_{i,j}}{\Delta x} \sum_{k=0}^{n} g_{\alpha,k} \zeta_{n}[\cos(q_1(k-1)\Delta x) - m \sin(q_1(k-1)\Delta x)]
\frac{\Delta t.d_{i,j}}{\Delta y} \sum_{k=0}^{n} g_{\alpha,k} \zeta_n[\cos(q_2(k-1)\Delta y) - m \sin(q_2(k-1)\Delta y)],
$$

(12)

$$
\eta = 1 + \frac{\Delta t.e_{i,j}}{\Delta x} \sum_{k=0}^{n} g_{\alpha,k} \zeta_{n}[\cos(q_1(k-1)\Delta x) - m \sin(q_1(k-1)\Delta x)]
\frac{\Delta t.e_{i,j}}{\Delta y} \sum_{k=0}^{n} g_{\alpha,k} \zeta_n[\cos(q_2(k-1)\Delta y) - m \sin(q_2(k-1)\Delta y)],
$$

(13)

Let

$$
A = \frac{d_{i,j}}{\Delta x} \sum_{k=0}^{n} g_{\alpha,k} \zeta_{n}[\cos(q_1(k-1)\Delta x) - m \sin(q_1(k-1)\Delta x)]
\frac{e_{i,j}}{\Delta y} \sum_{k=0}^{n} g_{\beta,k} \zeta_n[\cos(q_2(k-1)\Delta y) - m \sin(q_2(k-1)\Delta y)].
$$

B = \frac{d_{i,j}}{\Delta x} \sum_{k=0}^{n} g_{\alpha,k} \zeta_{n}[\cos(q_1(k-1)\Delta x) - m \sin(q_1(k-1)\Delta x)]
\frac{e_{i,j}}{\Delta y} \sum_{k=0}^{n} g_{\beta,k} \zeta_n[\cos(q_2(k-1)\Delta y) - m \sin(q_2(k-1)\Delta y)].
$$

(14)

The mode will be stable as long as $|\eta| \leq 1$, i.e.,

$$
|1 + \Delta tA - m\Delta tB| \leq 1,
$$

(15)

that mean:

$$
(1 + \Delta tA)^2 + (\Delta tB)^2 \leq 1.
$$

(16)

$$
1 + 2\Delta tA + (\Delta t)^2(A)^2 + (\Delta t)^2(B)^2 \leq 1.
$$

(17)

So,

$$
2A + (\Delta t)(A)^2 + (\Delta t)(B)^2 \leq 0.
$$

(18)
\[ \Delta t \leq S, \quad (19) \]

since

\[ S = \frac{-2A}{A^2 + B^2}. \quad (20) \]

And

\[ A^2 + B^2 > 0. \]

**Theorem 2** The truncation error of SFDE is:

\[ T(x, y, t) = O(\Delta t) + O(\Delta x) + O(\Delta y). \]

**Proof.** Evaluating (1) at the point \((x_i, y_j, t_n)\), gives

\[ \frac{\partial u}{\partial t}(x_i, y_j, t_n) = \Delta u_i^n + d \Delta x u_i^{n+1,j} - e \Delta y u_j^{n,i+1} = T(x_i, y_j, t_n), \quad (21) \]

Neglecting the truncation error term \(T(x_i, y_j, t_n)\), we get the explicit difference scheme (6). From (1)-(6) and (22), we get

\[ \frac{\partial u}{\partial t}(x_i, y_j, t_n) - \Delta t u_i^n = \Delta u_i^n + d \Delta x u_i^{n+1,j} - e \Delta y u_j^{n,i+1} = T(x_i, y_j, t_n), \quad (23) \]

\[ \frac{\partial}{\partial t} u(x_i, y_j, t_n) = \Delta u(x_i, y_j, t_n) + O(\Delta t), \quad (24) \]

\[ \Delta x u_i^{n+1,j} = \frac{\partial^\alpha u}{\partial x^\alpha}(x_i, y_j, t_n) + O(\Delta x)^2, \quad (25) \]

\[ \frac{\partial^\alpha u}{\partial x^\alpha}(x_{i+1}, y_j, t_n) = \frac{\partial^\alpha u}{\partial x^\alpha}(x_i, y_j, t_n) + \Delta x \frac{d\partial^\alpha u}{d\Delta x^\alpha}(x, y_j, t_n) + O(\Delta x)^2, \quad (26) \]

so that

\[ \Delta x u_i^{n+1,j} = \Delta x u_i^n + O(\Delta x) + O(\Delta x)^2. \quad (27) \]

And

\[ \Delta y u_j^{n+1,i} = \frac{\partial^\beta u}{\partial y^\beta}(x_i, y_j, t_n) + O(\Delta y)^2, \quad (28) \]

\[ \frac{\partial^\beta u}{\partial y^\beta}(x_i, y_{j+1}, t_n) = \frac{\partial^\beta u}{\partial y^\beta}(x_i, y_j, t_n) + \Delta y \frac{d\partial^\beta u}{d\Delta y^\beta}(x_i, y_j, t_n) + O(\Delta y)^2, \quad (29) \]

so that

\[ \Delta y u_j^{n+1,i} = \Delta y u_j^n + O(\Delta y) + O(\Delta y)^2. \quad (30) \]

From this results and from (24), we claim that

\[ T(x, y, t) = O(\Delta t) + O(\Delta x) + O(\Delta y). \quad (31) \]
4. Numerical Results

Example 1 Consider the space fractional diffusion equation:
\[
\frac{\partial u(x, y, t)}{\partial t} = d(x, y) \frac{\partial^{1.7} u(x, y, t)}{\partial x^{1.7}} + e(x, y) \frac{\partial^{1.7} u(x, y, t)}{\partial y^{1.7}} + q(x, y, t),
\]
(32)
on a finite rectangular domain \(0 < x < 1, 0 < y < 1\), for \(0 \leq t \leq T_{end}\). The diffusion coefficients are
\[
d(x, y) = \Gamma(2.2)x^{2.7}y/6,
\]
and
\[
e(x, y) = 2xy^{2.7}/\Gamma(4.7),
\]
and the forcing function is
\[
q(x, y, t) = -(1 + 2xy)e^{-t}x^{3}y^{3.7},
\]
with the initial conditions
\[
u(x, y, 0) = x^{3}y^{3.7},
\]
and Dirichlet boundary conditions on the rectangle in the form \(u(0, y, t) = u(x, 0, t) = 0, u(1, y, t) = e^{-t}y^{3.7},\) and \(u(x, 1, t) = e^{-t}x^{3}\) for all \(t \geq 0\).
The exact solution to this two-dimensional fractional diffusion equation is given by
\[
u(x, y, t) = e^{-t}x^{3}y^{3.7}.
\]
(33)

Table 1. The maximum errors at \(T_{end} = 1\).

<table>
<thead>
<tr>
<th>(\Delta t)</th>
<th>(\Delta x = \Delta y)</th>
<th>maximum error</th>
<th>(S)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/100</td>
<td>1/100</td>
<td>1.1050310E-3</td>
<td>3.9213168E-1</td>
</tr>
<tr>
<td>1/200</td>
<td>1/200</td>
<td>1.2548736E-3</td>
<td>3.8923329E-1</td>
</tr>
<tr>
<td>1/500</td>
<td>1/500</td>
<td>1.5300256E-3</td>
<td>3.8398275E-1</td>
</tr>
<tr>
<td>1/1000</td>
<td>1/1000</td>
<td>1.7921654E-3</td>
<td>3.7835699E-1</td>
</tr>
</tbody>
</table>

Figure 1. EFDA solution when \(\Delta x = \Delta y = 0.02\), \(\Delta t = 0.002\) and \(S = 0.378\).

Figure 2. Exact solution
Figure 3. EFDA solution when $\Delta x = \Delta y = 0.03$, $\Delta t = 0.1$ and $S = 0.046$.

Figure 4. Exact solution

The numerical studies are given as follows: Table 1 shows the maximum absolute numerical error, at time $T_{end} = 1$, between the exact solution and the numerical solution of the EFDA. In order to test the numerical scheme, Figure 1 shows the approximate solution where $\alpha = \beta = 1.7$, at $T_{end} = 1$, $\Delta x = \Delta y = 0.02$, $\Delta t = 0.002$, and $S = 0.378$, while Figure 2 shows the exact solution in this case. Figure 3 shows the unstable solution behaviour when $\Delta x = \Delta y = 0.03$, $\Delta t = 0.1$ and $S = 0.046$, where the value of $\Delta t$ is larger than the stability bound $S$, while Figure 4 shows the exact solution in this case, for more details on the the stability conditions see Theorem 1.

Example 2 Consider the space fractional diffusion equation \cite{1}:

$$\frac{\partial u(x,y,t)}{\partial t} = d(x,y) \frac{\partial^{1.9} u(x,y,t)}{\partial x^{1.9}} + e(x,y) \frac{\partial^{1.6} u(x,y,t)}{\partial y^{1.6}} + q(x,y,t) \quad (34)$$

on a finite rectangular domain $0 < x < 1$, $0 < y < 1$, for $0 \leq T_{end}$. The diffusion coefficients are

$$d(x,y) = x^3 y^{1.4}/\Gamma(3.9),$$

and

$$e(x,y) = x^{1.1} y^{3}/\Gamma(3.6),$$

and the forcing function is

$$q(x, y, t) = -(1 + 2x^{1.1} y^{1.4}) e^{-t} x^{2.9} y^{2.6},$$

with the initial conditions

$$u(x, y, 0) = x^{2.9} y^{2.6},$$

and Dirichlet boundary conditions on the rectangle in the form $u(0, y, t) = u(x, 0, t) = 0$, $u(1, y, t) = e^{-t} y^{2.6}$, and $u(x, 1, t) = e^{-t} x^{2.9}$ for all $t_0$.

The exact solution to this two-dimensional fractional diffusion equation is given by

$$u(x, y, t) = e^{-t} x^{2.9} y^{2.6}. \quad (35)$$
Table 2. The maximum errors at $T_{end} = 5$.

<table>
<thead>
<tr>
<th>$\frac{\Delta t}{\Delta x}$</th>
<th>$\Delta x = \Delta y$</th>
<th>maximum error</th>
<th>$S$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{5}$</td>
<td>5</td>
<td>$1.4263080E-3$</td>
<td>$2.2769429E-1$</td>
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<td>$\frac{1}{5}$</td>
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<td>$\frac{1}{5}$</td>
<td>$2.1672336E-4$</td>
<td>$2.7955559E-1$</td>
</tr>
</tbody>
</table>

The numerical studies are given as follows: Table 2 shows the maximum absolute numerical error, at time $T_{end} = 5$, between the exact solution and the numerical solution of the EFDA. In order to test the numerical scheme, Figure 5 shows the approximate solution where $\alpha = 1.9$, $\beta = 1.6$, at $T_{end} = 5$, $\Delta x = \Delta y = 0.05$, $\Delta t = 0.01$ and $S = 0.280$. Figure 6 shows the exact solution.

Figure 5. EFDA solution when $\Delta x = \Delta y = 0.05$, $\Delta t = 0.01$ and $S = 0.280$.

Figure 6. Exact solution

Figure 7. EFDA solution when $\Delta x = \Delta y = 0.04$, $\Delta t = 0.1$ and $S = 0.073$.

Figure 8. Exact solution

The numerical studies are given as follows: Table 2 shows the maximum absolute numerical error, at time $T_{end} = 5$, between the exact solution and the numerical solution of the EFDA. In order to test the numerical scheme, Figure 5 shows the approximate solution where $\alpha = 1.9$, $\beta = 1.6$, at $T_{end} = 5$, $\Delta x = \Delta y = 0.05$, $\Delta t = 0.01$ and $S = 0.280$. Figure 6 shows the exact solution.
\( \Delta t = 0.01 \) and \( S = 0.280 \), while Figure 6 shows the exact solution in this case. Figure 7 shows the unstable solution behaviour when \( \Delta x = \Delta y = 0.04, \Delta t = 0.1 \) and \( S = 0.073 \), where the value of \( \Delta t \) is larger than the stability bound \( S \), while Figure 8 shows the exact solution in this case, for more details on the the stability conditions see Theorem 1.

5. Conclusions

In this paper two-dimensional space fractional order diffusion equation is studied using EFDA, where the fractional derivative is defined in the right-shifted Grünwald sense. Error analysis and stability of the explicit numerical method for SFDE were discussed by means of a fractional version of the von Neumann stability analysis. Some numerical results example are presented. These numerical results demonstrate that the EFDA is a computationally simple and efficient method for SFDE.

References


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