ON SOME NEW GRUSS-TYPE INEQUALITY USING HADAMARD FRACTIONAL INTEGRAL OPERATOR

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ABSTRACT. This paper deals with some new integral inequality of Gruss-type using the Hadamard fractional integral operator and related integral inequalities.

1. INTRODUCTION

In 1935, G. Gruss proved the following classical integral inequality, see [14]

\[
\frac{1}{b-a} \int_a^b f(x)g(x)dx - \left( \frac{1}{b-a} \int_a^b f(x)dx \right) \left( \frac{1}{b-a} \int_a^b g(x)dx \right) \leq \frac{(M-m)(P-p)}{4},
\]

provided that \( f \) and \( g \) are two integrable functions on \([a, b]\) and satisfy the condition

\[
m \leq f(x) \leq M, \quad p \leq g(x) \leq P; \quad m, M, p, P \in \mathbb{R}; x \in [a, b].
\]

Since the researchers has evoked the interest inequality (1), and various generalizations and extensions have appeared in the literature, to mention a few, see [11, 13-17] and the references cited therein.

In classical differential and integral equations mathematical inequalities plays very authoritative role. In the past several years, many author have studied on fractional differential and integral inequalities using Riemann-Liouville, Caputo fractional integral and q-fractional integral, see [2, 6-9, 11, 12, 18]. In, [11] Dahmani and et al. gave the following fractional integral inequality using Riemann-Liouville fractional integral as:

**Theorem 1.1.** [11] Let \( f \) and \( g \) be two integrable function on \([0, \infty]\), satisfying the condition

\[
m \leq f(x) \leq M, \quad p \leq g(x) \leq P; \quad m, M, p, P \in \mathbb{R}, x \in [0, \infty).
\]

Then for all \( t > 0, \alpha > 0 \), we have

\[
\left| \frac{t^\alpha}{\Gamma(\alpha + 1)} J^\alpha f g(t) - J^\alpha f(t) J^\alpha g(t) \right| \leq \left( \frac{t^\alpha}{2\Gamma(\alpha + 1)} \right)^2 (M-m)(P-p).
\]
Also, in [22] Chaowu Zhu and et al. established some new fractional q-integral inequality. In literature few results have been obtained on some fractional integral inequalities using Hadamard fractional integral in [3-5]. Motivated from [6, 10, 21], our purpose in this paper is to establish some new results using Hadamard fractional integral related to (1). The paper has been organized as follows, in Section 2, we define basic definitions and proposition related to Hadamard fractional derivatives and integrals. In Section 3, we give the results about Gurss-type fractional integral inequality using fractional Hadamard integral operator, In Section 4, we give some other inequalities using fractional Hadamard operator.

2. Preliminaries

Here, we present some definitions of Hadamard derivative and integral as given in [1, p.159-171].

Definition 2.1. [1] The Hadamard fractional integral of order $\alpha \in \mathbb{R}^+$ of function $f(x)$, for all $x > 1$ is defined as,

$$H_{D_1^+;x} \frac{\alpha}{\Gamma(\alpha)} \int_1^x \ln \left( \frac{t}{x} \right)^{\alpha-1} f(t) \frac{dt}{t}, \quad (5)$$

where $\Gamma(\alpha) = \int_0^\infty e^{-u} u^{\alpha-1} du$.

Definition 2.2. [1] The Hadamard fractional derivative of order $n \in \mathbb{Z}^+$ of function $f(x)$ is given as follows:

$$H_{D_1^+;x} \frac{\alpha}{\Gamma(n-\alpha)} \left( \frac{d}{dx} \right)^n \frac{\alpha}{\Gamma(\alpha)} \int_1^x \ln \left( \frac{t}{x} \right)^{n-\alpha-1} f(t) \frac{dt}{t}, \quad (6)$$

Proposition 2.1. [1] If $0 < \alpha < 1$, the following relation hold:

$$H_{D_1^+;x} (\ln x)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (\ln x)^{\beta+\alpha-1}, \quad (7)$$

$$H_{D_1^+;x} (\ln x)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (\ln x)^{\beta-\alpha-1}, \quad (8)$$

respectively.

For the convenience of establishing the result, we give the semigroup property is as follows,

$$(H_{D_1^+}^\alpha)(H_{D_1^+}^\beta) f(x) = H_{D_1^+}^{(\alpha+\beta)} f(x). \quad (9)$$

Also some details of fractional Hadamard calculus are given in the book A.A.Kilbas et al. [19], and in book of S.G.Samko et al. [20].

3. Gruss-type fractional integral inequality

To prove our main result we need the following lemma.

Lemma 3.1. Let $m, M \in \mathbb{R}$, and $v$ be an integrable function on $[0, \infty)$, Then for all $t > 0$, $\alpha > 0$, we have

$$\left( \frac{\ln(t)}{\Gamma(\alpha+1)} \right) H_{D_1^+;t}^\alpha v^2(t) - (H_{D_1^+}^\alpha v(t))^2 = \left( M \frac{(\ln(t))^{\alpha}}{\Gamma(\alpha+1)} - H_{D_1^+}^\alpha v(t) \right) \left( (H_{D_1^+}^\alpha v(t) - m \frac{(\ln(t))^{\alpha}}{\Gamma(\alpha+1)}) - (\ln(t))^{\alpha} \frac{1}{\Gamma(\alpha+1)} H_{D_1^+}^\alpha (M - v(t))(v(t) - m). \right) \quad (10)$$
Let $m, M \in R$, and $v$ be an integrable function on $[0, \infty)$. For all $\tau, \rho \in [0, \infty)$, we have

\[ (M - v(\rho))(v(\tau) - m) + (M - v(\tau))(v(\rho) - m) \]
\[ - (M - v(\tau))(v(\tau) - m) - (M - v(\rho))(v(\rho) - m) \]
\[ = v^2(\tau) + v^2(\rho) + 2v(\tau)v(\rho). \]  

Multiplying both sides of (11) by \( \frac{(\ln(t))^\alpha}{\Gamma(\alpha)} \), and is positive because $\tau \in (0, t)$, $t > 0$, then integrating resulting identity with respect to $\tau$ from 1 to $t$, we get

\[ (M - v(\rho)) \left( H^{1,\alpha} v(t) - m \frac{(\ln(t))^\alpha}{\Gamma(\alpha + 1)} \right) + \left( M \frac{(\ln(t))^\alpha}{\Gamma(\alpha + 1)} - H^{1,\alpha} v(t) \right) \]
\[ (v(\rho) - m) - H^{1,\alpha} ((M - v(\tau))(v(t) - m)) - ((M - v(\rho))(v(\rho) - m)) \times \]
\[ \frac{(\ln(t))^\alpha}{\Gamma(\alpha + 1)} = H^{1,\alpha} v^2(t) + v^2(\rho) \frac{(\ln(t))^\alpha}{\Gamma(\alpha + 1)} + 2v(\rho) H^{1,\alpha} v(t). \]

Now, again multiplying both side of (12) by \( \frac{(\ln(t))^\alpha}{\Gamma(\alpha)} \), which is positive because $\rho \in (0, t)$, $t > 0$, then integrating resulting identity with respect to $\rho$ from 1 to $t$, we get

\[ \left( H^{1,\alpha} v(t) - m \frac{(\ln(t))^\alpha}{\Gamma(\alpha + 1)} \right) \frac{1}{\Gamma(\alpha)} \int_1^t (\ln \frac{\rho}{t})^{\alpha-1}(M - v(\rho)) \frac{d\rho}{\rho} \]
\[ + \left( M \frac{(\ln(t))^\alpha}{\Gamma(\alpha + 1)} - H^{1,\alpha} v(t) \right) \frac{1}{\Gamma(\alpha)} \int_1^t (\ln \frac{\rho}{t})^{\alpha-1}(v(\rho) - m) \frac{d\rho}{\rho} \]
\[ - H^{1,\alpha} ((M - v(\tau))) \frac{1}{\Gamma(\alpha)} \int_1^t (\ln \frac{\rho}{t})^{\alpha-1}(1) \frac{d\rho}{\rho} \]
\[ - \frac{(\ln(t))^\alpha}{\Gamma(\alpha + 1)} \frac{1}{\Gamma(\alpha)} \int_1^t (\ln \frac{\rho}{t})^{\alpha-1}(M - v(\rho))(v(\rho) - m) \frac{d\rho}{\rho} \]
\[ = \frac{(\ln(t))^\alpha}{\Gamma(\alpha + 1)} H^{1,\alpha} v^2(t) + v^2(\rho) \frac{(\ln(t))^\alpha}{\Gamma(\alpha + 1)} + 2H^{1,\alpha} v(t) H^{1,\alpha} v(t). \]

Which gives (10), and proves the lemma.

**Theorem 3.2.** Let $f, g$ be two integrable function on $[0, \infty)$, satisfying the condition that

\[ m \leq f(t) \leq M, \ p \leq g(t) \leq P; \ m, M, p, P \in R, \ t \in [0, \infty). \]

we have

\[ \left| \frac{(\ln(t))^\alpha}{\Gamma(\alpha + 1)} H^{1,\alpha} fg(t) - H^{1,\alpha} f(t) H^{1,\alpha} g(t) \right| \]
\[ \leq \left( \frac{(\ln(t))^\alpha}{2\Gamma(\alpha + 1)} \right)^2 (M - m)(P - p). \]

**Proof.** Let $f$ and $g$ be two function satisfying the condition (14).

Define

\[ H(\tau, \rho) := (f(\tau) - f(\rho))(g(\tau) - g(\rho)); \ \tau, \rho \in (0, t), t > 0. \]
It follows that

\[ H(\tau, \rho) := f(\tau)g(\tau) - f(\tau)g(\rho) - f(\rho)g(\tau) + f(\rho)g(\rho). \]  

(17)

Then, multiplying (17) by \( \frac{(\ln\frac{1}{\tau})^{\alpha-1}}{\Gamma(\alpha)} \), which is positive because \( \tau \in (0, t) \), \( t > 0 \), then integrating resulting identity with respect to \( \tau \) from 1 to \( t \), we get

\[
\frac{1}{\Gamma(\alpha)} \int_{1}^{t} (\ln \frac{t}{\tau})^{\alpha-1} H(\tau, \rho) d\tau
\]

\[ = h D_{1,x}^{-\alpha} f(t) - f(\tau) h D_{1,x}^{-\alpha} f(t) - f(\rho) h D_{1,x}^{-\alpha} g(t) + f(\rho) h D_{1,x}^{-\alpha} g(t). \]  

(18)

Again, multiplying (18) by \( \frac{(\ln\frac{1}{\tau})^{\alpha-1}}{\rho^{\alpha}} \), which is positive because \( \rho \in (0, t) \), and integrate with respective \( \rho \) from 1 to \( t \), we have,

\[
\frac{1}{\Gamma(\alpha)} \int_{1}^{t} \int_{1}^{t} (\ln \frac{t}{\rho})^{\alpha-1} H(\tau, \rho) \frac{d\rho}{\rho} d\tau
\]

\[ = 2 \left( \frac{(\ln t)^{\alpha}}{\Gamma(\alpha + 1)} h D_{1,x}^{-\alpha} f(t) - h D_{1,x}^{-\alpha} f(t) h D_{1,x}^{-\alpha} g(t) \right). \]  

(19)

Applying the Cauchy-Schwarz inequality, we have,

\[
\left( \frac{(\ln t)^{\alpha}}{\Gamma(\alpha + 1)} h D_{1,x}^{-\alpha} f(t) - h D_{1,x}^{-\alpha} f(t) h D_{1,x}^{-\alpha} g(t) \right)^2 \leq
\]

\[
\left( \frac{(\ln t)^{\alpha}}{\Gamma(\alpha + 1)} h D_{1,x}^{-\alpha} f(t) - ( h D_{1,x}^{-\alpha} f(t) )^2 \right) \times
\]

\[
\left( \frac{(\ln t)^{\alpha}}{\Gamma(\alpha + 1)} h D_{1,x}^{-\alpha} g(t) - ( h D_{1,x}^{-\alpha} g(t) )^2 \right).
\]

(20)

Since \( (M - f(t))(g(t) - m) \geq 0 \) and \( (P - g(t))(g(t) - p) \geq 0 \), we have

\[
\frac{(\ln t)^{\alpha}}{\Gamma(\alpha + 1)} h D_{1,x}^{-\alpha} (M - f(t))(g(t) - m) \geq 0,
\]

(21)

and

\[
\frac{(\ln t)^{\alpha}}{\Gamma(\alpha + 1)} h D_{1,x}^{-\alpha} (P - g(t))(g(t) - p) \geq 0.
\]

(22)

Thus

\[
\frac{(\ln t)^{\alpha}}{\Gamma(\alpha + 1)} h D_{1,x}^{-\alpha} f^2(t) - ( h D_{1,x}^{-\alpha} f(t) )^2
\]

\[ \leq \left( M \frac{(\ln t)^{\alpha}}{\Gamma(\alpha + 1)} - h D_{1,x}^{-\alpha} f(t) \right) \left( h D_{1,x}^{-\alpha} f(t) - m \frac{(\ln t)^{\alpha}}{\Gamma(\alpha + 1)} \right). \]  

(23)

And

\[
\frac{(\ln t)^{\alpha}}{\Gamma(\alpha + 1)} h D_{1,x}^{-\alpha} g^2(t) - ( h D_{1,x}^{-\alpha} g(t) )^2
\]

\[ \leq \left( M \frac{(\ln t)^{\alpha}}{\Gamma(\alpha + 1)} - h D_{1,x}^{-\alpha} f(t) \right) \left( h D_{1,x}^{-\alpha} f(t) - m \frac{(\ln t)^{\alpha}}{\Gamma(\alpha + 1)} \right). \]  

(24)
Combining (20), (23) and (24), using lemma 3.1, we conclude that

\[
\left( \frac{(\ln t)^\alpha}{\Gamma(\alpha+1)} H D_{1,t}^{-\alpha} f g(t) - H D_{1,t}^{-\alpha} f(t) H D_{1,t}^{-\alpha} g(t) \right)^2 
\leq \left( P \frac{(\ln t)^\alpha}{\Gamma(\alpha+1)} - H D_{1,t}^{-\alpha} g(t) \right) \left( H D_{1,t}^{-\alpha} g(t) - p \frac{(\ln t)^\alpha}{\Gamma(\alpha+1)} \right) \times \left( P \frac{(\ln t)^\alpha}{\Gamma(\alpha+1)} - H D_{1,t}^{-\alpha} g(t) \right) \left( H D_{1,t}^{-\alpha} g(t) - p \frac{(\ln t)^\alpha}{\Gamma(\alpha+1)} \right). 
\]

Now using the elementary inequality \(4ab \leq (a + b)^2\), \(a, b \in \mathbb{R}\), we can show that

\[
4 \left( M \frac{(\ln t)^\alpha}{\Gamma(\alpha+1)} - H D_{1,t}^{-\alpha} f(t) \right) \left( H D_{1,t}^{-\alpha} f(t) - m \frac{(\ln t)^\alpha}{\Gamma(\alpha+1)} \right) 
\leq \left( \frac{(\ln t)^\alpha}{\Gamma(\alpha+1)} (M - m) \right)^2,
\]

and

\[
4 \left( P \frac{(\ln t)^\alpha}{\Gamma(\alpha+1)} - H D_{1,t}^{-\alpha} g(t) \right) \left( H D_{1,t}^{-\alpha} g(t) - m \frac{(\ln t)^\alpha}{\Gamma(\alpha+1)} \right) 
\leq \left( \frac{(\ln t)^\alpha}{\Gamma(\alpha+1)} (P - p) \right)^2.
\]

From (25), (26) and (27), we get the result (15).

**Lemma 3.3.** Let \(f\) and \(g\) be two integrable function on \([0, \infty)\). then for all \(t > 0, \alpha > 0, \beta > 0\), we have:

\[
\left( \frac{(\ln t)^\alpha}{\Gamma(\alpha+1)} H D_{1,t}^{-\beta} f g(t) + \frac{(\ln t)^\beta}{\Gamma(\beta+1)} H D_{1,t}^{-\alpha} f g(t) \right)^2 
\leq \left( \frac{(\ln t)^\alpha}{\Gamma(\alpha+1)} H D_{1,t}^{-\beta} f(t) + \frac{(\ln t)^\beta}{\Gamma(\beta+1)} H D_{1,t}^{-\alpha} f(t) \right) \left( \frac{(\ln t)^\alpha}{\Gamma(\alpha+1)} H D_{1,t}^{-\beta} f(t) + \frac{(\ln t)^\beta}{\Gamma(\beta+1)} H D_{1,t}^{-\alpha} f(t) \right) \times \left( \frac{(\ln t)^\alpha}{\Gamma(\alpha+1)} H D_{1,t}^{-\beta} f(t) + \frac{(\ln t)^\beta}{\Gamma(\beta+1)} H D_{1,t}^{-\alpha} f(t) \right) \left( \frac{(\ln t)^\alpha}{\Gamma(\alpha+1)} H D_{1,t}^{-\beta} f(t) + \frac{(\ln t)^\beta}{\Gamma(\beta+1)} H D_{1,t}^{-\alpha} f(t) \right).
\]

**Proof.** Multiplying (18) by \(\frac{(\ln t)^{\beta - 1}}{\rho(t)}\), which is positive because \(\rho \in (0, t)\), and integrate with respective \(\rho\) from 1 to \(t\), then applying the Cauchy-Schwarz inequality for double integral, we obtain (28).
Lemma 3.4. Let $v$ be an integrable function on $[0, \infty)$ and $m, M \in \mathbb{R}$, then for all $t > 0, \alpha > 0, \beta > 0$, we have:

\[
\begin{align*}
&\frac{(\ln t)^\alpha}{\Gamma(\alpha+1)} H_{1,1}^{-\alpha} v^2(t) + \frac{(\ln t)^\beta}{\Gamma(\beta+1)} H_{1,1}^{-\alpha} v^2(t) - 2 H_{1,1}^{-\alpha} v(t) H_{1,1}^{-\beta} v(t) = \\
& \left( M \frac{(\ln t)^\alpha}{\Gamma(\alpha+1)} - H_{1,1}^{-\alpha} v(t) \right) \left( H_{1,1}^{-\alpha} v(t) - m \frac{(\ln t)^\beta}{\Gamma(\beta+1)} \right) \\
& + \left( M \frac{(\ln t)^\beta}{\Gamma(\beta+1)} - H_{1,1}^{-\beta} v(t) \right) \left( H_{1,1}^{-\beta} v(t) - m \frac{(\ln t)^\alpha}{\Gamma(\alpha+1)} \right) \\
& - \frac{(\ln t)^\alpha}{\Gamma(\alpha+1)} H_{1,1}^{-\alpha} (M - v(t))(v(t) - m) \\
& - \frac{(\ln t)^\beta}{\Gamma(\beta+1)} H_{1,1}^{-\beta} (M - v(t))(v(t) - m).
\end{align*}
\]

Proof. Multiplying (12) by $\frac{(\ln t)^{\beta-1}}{\rho^\beta(\beta)}$, which is positive because $\rho \in (0, t)$, $t > 0$, then integrating resulting identity with respect to $\rho$ from 1 to $t$, we get

\[
\begin{align*}
& \frac{H_{1,1}^{-\alpha} v(t) - m \frac{(\ln t)^\alpha}{\Gamma(\alpha+1)}}{\Gamma(\beta)} \int_1^t \frac{(\ln \rho)^{\beta-1}(M - v(\rho))}{\rho} \, d\rho \\
& \left( M \frac{(\ln t)^\alpha}{\Gamma(\alpha+1)} - H_{1,1}^{-\alpha} v(t) \right) \frac{1}{\Gamma(\beta)} \int_1^t \frac{(\ln \rho)^{\beta-1}(v(\rho) - m)}{\rho} \, d\rho \\
& - H_{1,1}^{-\alpha} ((M - v(t))(v(t) - m)) \frac{1}{\Gamma(\beta)} \int_1^t \frac{(\ln \rho)^{\beta-1}(1)}{\rho} \, d\rho \\
& - \frac{(\ln t)^\alpha}{\Gamma(\alpha+1)} \frac{1}{\Gamma(\beta)} \int_1^t \frac{(\ln \rho)^{\beta-1}(M - v(\rho))(v(\rho) - m)}{\rho} \, d\rho \\
& = \frac{(\ln t)^\beta}{\Gamma(\beta+1)} H_{1,1}^{-\beta} v^2(t) + H_{1,1}^{-\beta} v^2(t) \frac{(\ln t)^\alpha}{\Gamma(\alpha+1)} - 2 H_{1,1}^{-\beta} v(t) H_{1,1}^{-\alpha} v(t).
\end{align*}
\]

This gives the proof lemma 3.4.

Here we prove the our second main result, in which we use two real positive parameters.

Theorem 3.5. Let $f$, $g$ be two integrable function on $[0, \infty)$, satisfying the condition that

\[
m \leq f(t) \leq M, \ p \leq g(t) \leq P; \ m, M, p, P \in \mathbb{R}, \ t \in [0, \infty),
\]

\[(31)\]
then for all $T > 0$, $\alpha > 0$, $\beta > 0$, we have:

\[
\left(\frac{\ln t}{\Gamma(\alpha + 1)} H D_{1, t}^{-\beta} f(t) + \frac{\ln t^\beta}{\Gamma(\beta + 1)} H D_{1, t}^\alpha f(t) - H D_{1, t}^{-\beta} f(t) H D_{1, t}^\alpha g(t)\right)^2 \leq \left(\left(M \frac{\ln t}{\Gamma(\alpha + 1)} - H D_{1, t}^{-\beta} f(t)ight) + \left(M \frac{\ln t}{\Gamma(\alpha + 1)} - H D_{1, t}^\alpha f(t)\right) + \left(M \frac{\ln t^\beta}{\Gamma(\beta + 1)} - H D_{1, t}^{-\beta} f(t)\right)\right] \times \left(\left(M \frac{\ln t}{\Gamma(\alpha + 1)} - H D_{1, t}^\alpha g(t)\right) + \left(M \frac{\ln t^\beta}{\Gamma(\beta + 1)} - H D_{1, t}^{-\beta} g(t)\right)\right)
\]

\[
(32)
\]

**Proof.** Since $(M - f(x))(f(x) - m) \geq 0$ and $(P - g(x))(g(x) - p) \geq 0$, then we can write

\[
- \frac{\ln t}{\Gamma(\alpha + 1)} H D_{1, t}^{-\beta}(M - f(t))(f(t) - m) \leq 0,
\]

and

\[
- \frac{\ln t}{\Gamma(\alpha + 1)} H D_{1, t}^{-\beta}(P - g(t))(g(t) - p) \leq 0.
\]

Applying lemma 3.4 to $f$ and $g$, then using lemma 3.3 and the equation (33) and (34) we get (32).

### 4. Other fractional Hadamard integral inequalities

In this section, we give some new fractional Hadamard integral inequalities.

**Theorem 4.1.** Let $f$ and $g$ be two positive function defined on $[0, \infty[$, and $p, q > 1$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Then the following inequalities hold:

(a) $\frac{1}{p} H D_{1, t}^\alpha (f)^p + \frac{1}{q} H D_{1, t}^\alpha (g)^q \geq \frac{\Gamma(\alpha + 1)}{(\ln t)^p} H D_{1, t}^\alpha (f) H D_{1, t}^\alpha (g)$.

(b) $\frac{1}{p} H D_{1, t}^{-\alpha} (f)^p H D_{1, t}^\alpha (g)^q + \frac{1}{q} H D_{1, t}^{-\alpha} (f)^q H D_{1, t}^\alpha (g)^q \geq (H D_{1, t}^{-\alpha} (fg))^2$.

(c) $\frac{1}{p} H D_{1, t}^\alpha (f)^p H D_{1, t}^{-\alpha} (g)^q + \frac{1}{q} H D_{1, t}^\alpha (f)^q H D_{1, t}^{-\alpha} (g)^q \geq H D_{1, t}^{-\alpha} (fg)^{p-1} H D_{1, t}^-\alpha (fg)^{q-1}$.

(d) $H D_{1, t}^{-\alpha} (f)^p H D_{1, t}^-\alpha (g)^q \geq H D_{1, t}^-\alpha (fg) H D_{1, t}^-\alpha (f^{p-1} g^{q-1})$.

**Proof.** According to well-known Young inequality,

\[
\frac{1}{p} a^p + \frac{1}{q} b^q \geq ab \text{ for all } a, b \geq 0, p, q > 1 \frac{1}{p} + \frac{1}{q} = 1,
\]

putting $a = f(\tau)$ and $b = g(\rho)$, $\tau, \rho > 0$, we have

\[
\frac{1}{p} f(\tau)^p + \frac{1}{q} g(\rho)^q \geq f(\tau)g(\rho) \text{ for all } f(\tau), g(\rho) \geq 0.
\]
Multiplying both side of (36) by \( \frac{(\ln t)^{\alpha-1}}{\tau^\alpha} \), which is positive because \( \tau \in (0,t) \), \( t > 0 \), then integrating resulting identity with respect to \( \tau \) from 1 to \( t \), we get

\[
\frac{1}{p} \frac{(\ln t)^{\alpha}}{\Gamma(\alpha)} \int_1^t \ln\left(\frac{t}{\tau}\right)^{\alpha-1} f(\tau)^p d\tau + \frac{g(\rho)^q}{q} \frac{(\ln t)^{\alpha-1}}{\Gamma(\alpha)} \int_1^t \ln\left(\frac{t}{\tau}\right)^{\alpha-1} d\tau \\
\geq g(\rho) \frac{1}{\Gamma(\alpha)} \int_1^t \ln\left(\frac{t}{\tau}\right)^{\alpha-1} f(\tau)^{\frac{\alpha}{\alpha-1}} d\tau,
\]

we get

\[
\frac{1}{p} H D_{1,t}^\alpha f(t)^p + \frac{1}{q} \frac{(\ln t)^{\alpha}}{\Gamma(\alpha + 1)} g(\rho)^q \geq g(\rho) H D_{1,t}^\alpha f(t),
\]

Again multiplying (38) by \( \frac{(\ln t)^{\alpha-1}}{\tau^\alpha} \), which is positive because \( \rho \in (0,t) \), and integrate with respective \( \rho \) from 1 to \( t \), we have

\[
\frac{1}{p} \frac{(\ln t)^{\alpha}}{\Gamma(\alpha + 1)} H D_{1,t}^\alpha f(t)^p + \frac{1}{q} \frac{(\ln t)^{\alpha}}{\Gamma(\alpha + 1)} H D_{1,t}^\alpha g(\rho)^q \geq \frac{1}{p} H D_{1,t}^\alpha g(\rho) H D_{1,t}^\alpha f(t),
\]

which implies that,

\[
\frac{(\ln t)^{\alpha}}{\Gamma(\alpha + 1)} \left[ \frac{1}{p} H D_{1,t}^\alpha f(t)^p + \frac{1}{q} H D_{1,t}^\alpha g(\rho)^q \right] \geq \frac{1}{p} H D_{1,t}^\alpha g(\rho) H D_{1,t}^\alpha f(t),
\]

we get

\[
\frac{1}{p} H D_{1,t}^\alpha f(t)^p + \frac{1}{q} H D_{1,t}^\alpha g(\rho)^q \geq \frac{1}{p} H D_{1,t}^\alpha g(\rho) H D_{1,t}^\alpha f(t),
\]

which implies (a). The rest of inequalities can be shown in similar way by the following choice of parameters in the Young inequality.

(b) \( a = f(\tau) g(\rho), \ b = f(\rho) g(\tau) \).

(c) \( a = f(\tau), \ b = F(\rho) g(\tau), g(\tau), g(\rho) \neq 0 \).

Repeating the foregoing argument, we obtain (b)-(d).

**Theorem 4.2.** Let \( f \) and \( g \) be positive functions on \([0, \infty], \) and \( p, q > 1 \) satisfying \( \frac{1}{p} + \frac{1}{q} = 1 \). Then the following inequalities hold:

(a) \( \frac{1}{p} H D_{1,t}^\alpha (f)^p + \frac{1}{q} H D_{1,t}^\alpha (g)^q \geq \frac{1}{p} H D_{1,t}^\alpha (f)^q + \frac{1}{q} H D_{1,t}^\alpha (g)^q \).

(b) \( \frac{1}{p} H D_{1,t}^\alpha (f)^q + \frac{1}{q} H D_{1,t}^\alpha (g)^q \geq \frac{1}{p} H D_{1,t}^\alpha (f)^{q-1} + \frac{1}{q} H D_{1,t}^\alpha (g)^{q-1} \).

(c) \( H D_{1,t}^\alpha (f)^q \geq H D_{1,t}^\alpha (f)^{q-1} \).

**Proof.** The proof is based on the Young inequality with the appropriate choice of parameter, as in previous Theorem 4.2,

(a) \( a = f(\tau) g(\rho), \ b = f(\rho) g(\tau) \).

(b) \( a = \frac{f(\tau)}{g(\rho)}, \ b = \frac{g(\rho)}{g(\tau)}, g(\rho), f(\rho) \neq 0 \).

(c) \( a = \frac{f(\tau)}{g(\rho)}, \ b = \frac{g(\rho)}{g(\tau)} g(\tau), g(\rho) \neq 0 \).
Theorem 4.3. Suppose that \( f \) and \( g \) are two positive function defined on \([0, \infty]\) such that for all \( t > 0 \),
\[
m = \min_{0 \leq \tau \leq t} \frac{f(\tau)}{g(\tau)}, \quad M = \max_{0 \leq \tau \leq t} \frac{f(\tau)}{g(\tau)}.
\]
Then following inequalities hold:
(a) \( 0 \leq H D_{1,t}^{-\alpha} (f)^2 H D_{1,t}^{-\alpha} (g)^2 \leq \frac{(m+M)}{4mM} (H D_{1,t}^{-\alpha} (fg))^2 H D_{1,t}^{-\alpha} (g).
\)
(b) \( 0 \leq \sqrt{H D_{1,t}^{-\alpha} (f)^2 H D_{1,t}^{-\alpha} (g)^2} - H D_{1,t}^{-\alpha} (fg) \leq \frac{\sqrt{m-M}}{2\sqrt{mM}} H D_{1,t}^{-\alpha} (fg).
\)
(c) \( 0 \leq H D_{1,t}^{-\alpha} (f)^2 H D_{1,t}^{-\alpha} (g)^2 - (H D_{1,t}^{-\alpha} (fg))^2 \leq \frac{(M-m)}{4mM} (H D_{1,t}^{-\alpha} (fg))^2.
\)

Proof. It follows from (42) and
\[
\left( \frac{f(\tau)}{g(\tau)} - m \right) \left( M - \frac{f(\tau)}{g(\tau)} \right) g^2(\tau) \geq 0, \quad 0 \leq \tau \leq t,
\]
then we can write as,
\[
f^2(\tau) + mMg^2(\tau) \leq (m+M)f(\tau)g(\tau).
\]
Multiplying equation (44) by \( \frac{\ln(\tau)}{\Gamma(\alpha)} \), which is positive because \( \tau \in (0,t). \) Then by integrating with respect to \( \tau \), over \((1,t)\) we get,
\[
\frac{1}{\Gamma(\alpha)} \int_{1}^{t} \ln\left( \frac{t}{\tau} \right)^{\alpha-1} f^2(\tau) \frac{d\tau}{\tau} + \frac{mM}{\Gamma(\alpha)} \int_{1}^{t} \ln\left( \frac{t}{\tau} \right)^{\alpha-1} g^2(\tau) \frac{d\tau}{\tau} \leq \frac{(m+M)}{\Gamma(\alpha)} \int_{1}^{t} \ln\left( \frac{t}{\tau} \right)^{\alpha-1} f(\tau) g(\tau) \frac{d\tau}{\tau},
\]
implies that
\[
H D_{1,t}^{-\alpha} f^2(t) + mM H D_{1,t}^{-\alpha} g^2(t) \leq (m+M) H D_{1,t}^{-\alpha} fg(t).
\]
On other hand, it follows from \( mM > 0 \) and
\[
\left( \sqrt{H D_{1,t}^{-\alpha} f^2 - mM H D_{1,t}^{-\alpha} g^2} \right)^2 \geq 0,
\]
that
\[
2\sqrt{H D_{1,t}^{-\alpha} f^2} \sqrt{mM H D_{1,t}^{-\alpha} g^2} \leq H D_{1,t}^{-\alpha} f^2 + mM H D_{1,t}^{-\alpha} g^2,
\]
then from equation (46) and (47), we obtain,
\[
4mM H D_{1,t}^{-\alpha} f^2 H D_{1,t}^{-\alpha} g^2 \leq (m+M)^2 (H D_{1,t}^{-\alpha} fg)^2.
\]
Which implies (a). By some transformation of (a), similarly, we obtain (b) and (c).

References
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