ON THE ASYMPTOTIC BEHAVIOR OF THE RATIONAL
DIFFERENCE EQUATION $x_{n+1} = ax_n + \sum_{i=1}^{5} \alpha_i x_{n-i} \sum_{i=1}^{5} \beta_i x_{n-i}$

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Abstract. In this article, we study the periodicity, the boundedness and the global
stability of the positive solutions of the following nonlinear difference equation

$$x_{n+1} = ax_n + \sum_{i=1}^{5} \alpha_i x_{n-i} \sum_{i=1}^{5} \beta_i x_{n-i}, \quad n = 0, 1, 2, \ldots.$$

where the coefficients $a, \alpha_i, \beta_i \in (0, \infty)$, for $i = 1, 2, \ldots, 5$, while the initial conditions
$x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0$ are arbitrary positive real numbers. Some numerical examples
will be given to illustrate our results.

Keywords and Phrases: Difference equations, prime period two solution, boundedness character, locally asymptotically stable, global attractor, global stability.

1. Introduction

The qualitative study of difference equations is a fertile research area and increasingly
attracts many mathematicians. This topic draws its importance from the fact that many real
life phenomena are modeled using difference equations. Examples from economy, biology,
etc. can be found in [2, 3, 22, 25, 35]. It is known that nonlinear difference equations are
able to produce complicated behavior regardless of order.

There has been a great interest in studying the global attractivity, the boundedness character and the periodicity nature of nonlinear difference equations. For example, in the
articles [1, 9–21, 26–36] closely related global convergence results were obtained which can
be applied to nonlinear difference equations in proving that every solution of these equations
converges to a period two solution. For other closely related results, (see [4–8, 11, 22–24])
and the references cited therein. The study of these equations is challenging and rewarding
and is still in its infancy. We believe that the nonlinear rational difference equations are
of paramount importance in their own right. Furthermore the results about such equations
offer prototypes for the development of the basic theory of the global behavior of nonlinear
difference equations.

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Key words and phrases. Difference equations, prime period two solution, boundedness character, locally asymptotically stable, global attractor, global stability.
The objective of this article is to investigate some qualitative behavior of the solutions of the nonlinear difference equation

\[ x_{n+1} = ax_n + \frac{\sum_{i=1}^{5} \alpha_i x_{n-i}}{\sum_{i=1}^{5} \beta_i x_{n-i}}, \quad n = 0, 1, 2, \ldots \]  

(1)

where the coefficients \( a, \alpha_i, \beta_i \in (0, \infty) \), for \( i = 1, 2, \ldots, 5 \), while the initial conditions \( x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0 \) are arbitrary positive real numbers. Note that the special cases of Eq.(1) has been studied discussed in [11] when \( \alpha_3 = \alpha_4 = \alpha_5 = \beta_3 = \beta_4 = \beta_5 = 0 \) and Eq.(1) has been studied discussed in [37] in the special case when \( \alpha_4 = \alpha_5 = \beta_1 = \beta_5 = 0 \) and Eq.(1) has been studied discussed in [14] in the special case when \( \alpha_5 = \beta_5 = 0 \).

Aboutaleb et al. [1] studied the global attractivity of the positive equilibrium of the rational recursive sequence

\[ x_{n+1} = \frac{a - \beta x_n}{\gamma + x_n}, \quad n = 0, 1, 2, \ldots \]

where the coefficients \( a, b, A \) are nonnegative real numbers and \( k \in \{1, 2, \ldots\} \).

Li and Sun [34] investigated the periodic character and the global stability of all positive solutions of the equation

\[ x_{n+1} = \frac{px_n + x_{n-k}}{q + x_{n-k}}, \quad n = 0, 1, 2, \ldots, \]

where the parameters \( p \) and \( q \) and the initial conditions \( x_{-k}, \ldots, x_{-1}, x_0 \) are positive real numbers, \( k = \{1, 2, \ldots\} \).

M. Saleh et al. [38] investigated the periodic character and the global stability of all positive solutions of the equation

\[ x_{n+1} = \frac{\beta x_n + \gamma x_{n-k}}{Bx_n + Cx_{n-k}}, \quad n = 0, 1, 2, \ldots, \]

where the parameters \( \beta, \gamma \) and \( B, C \) and the initial conditions \( x_{-k}, \ldots, x_{-1}, x_0 \) are positive real numbers, \( k = \{1, 2, 3, \ldots\} \).

E. M. Elabbasy et al. [9] investigated the periodic character and the global stability of all positive solutions of the equation

\[ x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-1}}, \quad n = 0, 1, 2, \ldots, \]

where the parameters \( a, b, c \) and \( d \) and the initial conditions \( x_{-1}, x_0 \) are positive real numbers.

E. M. Elabbasy et al. [10] investigated the periodic character and the global stability of all positive solutions of the equation

\[ x_{n+1} = \frac{\alpha x_{n-l} + \beta x_{n-k}}{Ax_{n-l} + Bx_{n-k}}, \quad n = 0, 1, 2, \ldots, \]

where the parameters \( \alpha, \beta, A \) and \( B \) are positive real numbers and the initial conditions \( x_{-p}, x_{-p+1}, \ldots, x_{-1} \) and \( x_0 \in (0, \infty) \) where \( p = \max \{l, k\} \).

Our interest now is to study behavior of the solutions of Eq.(1) in its general form. For the related work, (see [39 – 54]). Let us now recall some well known results [21] which will be useful in the sequel.

**Definition 1.** Let

\[ F : I^{k+1} \rightarrow I \]
where $F$ is a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0 \in I$, the difference equation of order $(k+1)$ is an equation of the form

$$x_{n+1} = F(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, 2, \ldots$$

(2)

has a unique solution $\{x_n\}_{n=-k}^\infty$. An equilibrium point $\bar{x}$ of Eq.(2) is a point that satisfies the condition $\bar{x} = F(\bar{x}, \bar{x}, \ldots, \bar{x})$. That is, the constant sequence $\{x_n\}$ with $x_n = \bar{x}$ for all $n \geq 0$ is a solution of Eq.(2) or equivalently, $\bar{x}$ is a fixed point of $F$.

**Definition 2.** Let $\bar{x} \in I$ be an equilibrium point of Eq.(2). Then we have

(i) An equilibrium point $\bar{x}$ of Eq.(2) is called locally stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0 \in I$ with $|x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + \ldots + |x_{-1} - \bar{x}| + |x_0 - \bar{x}| < \delta$, then $|x_n - \bar{x}| < \varepsilon$ for all $n \geq -k$.

(ii) An equilibrium point $\bar{x}$ of Eq.(2) is called locally asymptotically stable if it is locally stable and there exists $\gamma > 0$ such that, if $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0 \in I$ with $|x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + \ldots + |x_{-1} - \bar{x}| + |x_0 - \bar{x}| < \gamma$, then

$$\lim_{n \to \infty} x_n = \bar{x}.$$

(iii) An equilibrium point $\bar{x}$ of Eq.(2) is called a global attractor if for every $x_{-l}, \ldots, x_{-k}, \ldots, x_{-1}, x_0 \in (0, \infty)$ we have

$$\lim_{n \to \infty} x_n = \bar{x}.$$

(iv) An equilibrium point $\bar{x}$ of Eq.(2) is called globally asymptotically stable if it is locally stable and a global attractor.

(v) An equilibrium point $\bar{x}$ of Eq.(2) is called unstable if it is not locally stable.

**Definition 3.** A sequence $\{x_n\}_{n=-k}^\infty$ is said to be periodic with period $r$ if $x_{n+r} = x_n$ for all $n \geq -p$. A sequence $\{x_n\}_{n=-k}^\infty$ is said to be periodic with prime period $r$ if $r$ is the smallest positive integer having this property.

**Definition 4.** Eq.(2) is called permanent and bounded if there exists numbers $m$ and $M$ with $0 < m < M < \infty$ such that for any initial conditions $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0 \in I$ there exists a positive integer $N$ which depends on these initial conditions such that

$$m \leq x_n \leq M \quad \text{for all} \quad n \geq N.$$

**Definition 5.** The linearized equation of Eq.(2) about the equilibrium point $\bar{x}$ is defined by the equation

$$z_{n+1} = \rho_0 z_n + \rho_1 z_{n-1} + \rho_2 z_{n-2} + \rho_3 z_{n-3} + \ldots = 0,$$

(3)

where

$$\rho_0 = \frac{\partial F(\bar{x}, \bar{x}, \ldots, \bar{x})}{\partial x_n}, \quad \rho_1 = \frac{\partial F(\bar{x}, \bar{x}, \ldots, \bar{x})}{\partial x_{n-1}}, \quad \rho_2 = \frac{\partial F(\bar{x}, \bar{x}, \ldots, \bar{x})}{\partial x_{n-2}}, \quad \rho_3 = \frac{\partial F(\bar{x}, \bar{x}, \ldots, \bar{x})}{\partial x_{n-3}}, \ldots$$

**Theorem 1** ([21]). Assume that $p_i \in R$, $i = 1, 2, \ldots, k$. Then

$$\sum_{i=1}^{k} |p_i| < 1,$$

(4)
is a sufficient condition for the asymptotic stability of the difference equation
\[ x_{n+k} + p_1 x_{n+k-1} + \ldots + p_k x_n = 0, \quad n = 0, 1, 2, \ldots \] (5)

**Theorem 2** ([21]). Let \( g : [a, b]^{k+1} \rightarrow [a, b] \) be a continuous function, where \( k \) is a positive integer, and where \([a, b]\) is an interval of real numbers. Consider the difference equation (2). Suppose that \( F \) satisfies the following conditions:

1. For each integer \( i \) with \( 1 \leq i \leq k+1 \); the function \( F(z_1, z_2, \ldots, z_{k+1}) \) is weakly monotonic in \( z_i \) for fixed \( z_1, z_2, \ldots, z_{i-1}, z_{i+1}, \ldots, z_{k+1} \).

2. If \((m, M)\) is a solution of the system
\[
\begin{align*}
    m &= F(m_1, m_2, \ldots, m_{k+1}) \quad \text{and} \quad M &= F(M_1, M_2, \ldots, M_{k+1}),
\end{align*}
\]
then \( m = M \), where for each \( i = 1, 2, \ldots, k+1 \), we set
\[
    m_i = \begin{cases}
    m & \text{if } F \text{ is non-decreasing in } z_i \\
    M & \text{if } F \text{ is non-increasing in } z_i
    \end{cases}
\]
and
\[
    M_i = \begin{cases}
    M & \text{if } F \text{ is non-decreasing in } z_i \\
    m & \text{if } F \text{ is non-increasing in } z_i
    \end{cases}
\]
Then there exists exactly one equilibrium \( x \) of Eq.(2), and every solution of Eq.(2) converges to \( x \).

2. THE LOCAL STABILITY OF THE SOLUTIONS

In this section, we study the local stability of the solutions of Eq.(1). The equilibrium point \( x \) of Eq.(1) is the positive solution of the equation
\[
    x = a x + \frac{\sum_{i=1}^{5} \alpha_i}{\sum_{i=1}^{5} \beta_i},
\] (6)
If \( a < 1 \), then the only positive equilibrium point \( x \) of Eq.(1) is given by
\[
    x = \frac{\sum_{i=1}^{5} \alpha_i}{(1 - a) \left( \sum_{i=1}^{5} \beta_i \right)}.
\] (7)
Let us now introduce a continuous function \( F : (0, \infty)^6 \rightarrow (0, \infty) \) which is defined by
\[
    F(u_0, \ldots, u_5) = u_0 + \frac{\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \alpha_4 u_4 + \alpha_5 u_5}{\beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3 + \beta_4 u_4 + \beta_5 u_5}.
\] (8)
Therefore it follows that

\[
\begin{align*}
\frac{\partial F(u_0, \ldots, u_5)}{\partial a_1} &= a_1 (\beta_2 u_2 + \beta_3 u_3 + \beta_4 u_4 + \beta_5 u_5) - \beta_1 (a_2 u_2 + a_3 u_3 + a_4 u_4 + a_5 u_5), \\
\frac{\partial F(u_0, \ldots, u_5)}{\partial a_2} &= a_2 (\beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3 + \beta_4 u_4 + \beta_5 u_5) - \beta_2 (a_1 u_1 + a_3 u_3 + a_4 u_4 + a_5 u_5), \\
\frac{\partial F(u_0, \ldots, u_5)}{\partial a_3} &= a_3 (\beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3 + \beta_4 u_4 + \beta_5 u_5) - \beta_3 (a_1 u_1 + a_2 u_2 + a_4 u_4 + a_5 u_5), \\
\frac{\partial F(u_0, \ldots, u_5)}{\partial a_4} &= a_4 (\beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3 + \beta_4 u_4 + \beta_5 u_5) - \beta_4 (a_1 u_1 + a_2 u_2 + a_3 u_3 + a_5 u_5), \\
\frac{\partial F(u_0, \ldots, u_5)}{\partial a_5} &= a_5 (\beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3 + \beta_4 u_4 + \beta_5 u_5) - \beta_5 (a_1 u_1 + a_2 u_2 + a_3 u_3 + a_4 u_4).
\end{align*}
\]

Consequently, we get

\[
\begin{align*}
\frac{\partial F(\bar{x}, \ldots, \bar{x})}{\partial a_0} &= a = -\rho_5, \\
\frac{\partial F(\bar{x}, \ldots, \bar{x})}{\partial a_1} &= (1-a)[a_1 (\beta_2 + \beta_3 + \beta_4 + \beta_5) - \beta_1 (a_2 + a_3 + a_4 + a_5)] \left(\sum_{i=1}^{5} \alpha_i \right) \left(\sum_{i=1}^{5} \beta_i \right) = -\rho_4, \\
\frac{\partial F(\bar{x}, \ldots, \bar{x})}{\partial a_2} &= (1-a)[a_2 (\beta_1 + \beta_3 + \beta_4 + \beta_5) - \beta_2 (a_1 + a_3 + a_4 + a_5)] \left(\sum_{i=1}^{5} \alpha_i \right) \left(\sum_{i=1}^{5} \beta_i \right) = -\rho_3, \\
\frac{\partial F(\bar{x}, \ldots, \bar{x})}{\partial a_3} &= (1-a)[a_3 (\beta_1 + \beta_2 + \beta_4 + \beta_5) - \beta_3 (a_1 + a_2 + a_4 + a_5)] \left(\sum_{i=1}^{5} \alpha_i \right) \left(\sum_{i=1}^{5} \beta_i \right) = -\rho_2, \\
\frac{\partial F(\bar{x}, \ldots, \bar{x})}{\partial a_4} &= (1-a)[a_4 (\beta_1 + \beta_2 + \beta_3 + \beta_5) - \beta_4 (a_1 + a_2 + a_3 + a_5)] \left(\sum_{i=1}^{5} \alpha_i \right) \left(\sum_{i=1}^{5} \beta_i \right) = -\rho_1, \\
\frac{\partial F(\bar{x}, \ldots, \bar{x})}{\partial a_5} &= (1-a)[a_5 (\beta_1 + \beta_2 + \beta_3 + \beta_4) - \beta_5 (a_1 + a_2 + a_3 + a_4)] \left(\sum_{i=1}^{5} \alpha_i \right) \left(\sum_{i=1}^{5} \beta_i \right) = -\rho_0.
\end{align*}
\]

Thus, the linearized equation of Eq.(1) about $\bar{x}$ takes the form

\[
y_{n+1} + \rho_5 y_n + \rho_4 y_{n-1} + \rho_3 y_{n-2} + \rho_2 y_{n-3} + \rho_1 y_{n-4} + \rho_0 y_{n-5} = 0,
\]

where $\rho_0, \rho_1, \rho_2, \rho_3, \rho_4$ and $\rho_5$ are given by (9).

The characteristic equation associated with Eq.(10) is

\[
\lambda^6 + \rho_5 \lambda^5 + \rho_4 \lambda^4 + \rho_3 \lambda^3 + \rho_2 \lambda^2 + \rho_1 \lambda + \rho_0 = 0,
\]

(11)
Theorem 3. Assume that $a < 1$ and

$$
|a_1 (\beta_2 + \beta_3 + \beta_4 + \beta_5) - \beta_1 (\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)| \\
+ |\alpha_2 (\beta_1 + \beta_3 + \beta_4 + \beta_5) - \beta_2 (\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5)| \\
+ |\alpha_3 (\beta_1 + \beta_2 + \beta_4 + \beta_5) - \beta_3 (\alpha_1 + \alpha_2 + \alpha_4 + \alpha_5)| \\
+ |\alpha_4 (\beta_1 + \beta_2 + \beta_3 + \beta_5) - \beta_4 (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5)| \\
+ |\alpha_5 (\beta_1 + \beta_2 + \beta_3 + \beta_4) - \beta_5 (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)| \\
< \left( \sum_{i=1}^{5} \alpha_i \right) \left( \sum_{i=1}^{5} \beta_i \right),
$$

(12)

then the positive equilibrium point (7) of Eq.(1) is locally asymptotically stable.

Proof. It follows by Theorem 1 that Eq.(10) is asymptotically stable if all roots of Eq.(11) lie in the open disk is $|\lambda| < 1$ that is if $|\rho_5| + |\rho_4| + |\rho_3| + |\rho_2| + |\rho_1| + |\rho_0| < 1$,
and so

\[
\left(1 - a\right)\left[\alpha_1 (\beta_2 + \beta_3 + \beta_4 + \beta_5) - \beta_1 (\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)\right] \\
\left(\sum_{i=1}^{5} \alpha_i\right) \left(\sum_{i=1}^{5} \beta_i\right)
\]

\[
+ \left(1 - a\right)\left[\alpha_2 (\beta_1 + \beta_3 + \beta_4 + \beta_5) - \beta_2 (\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5)\right] \\
\left(\sum_{i=1}^{5} \alpha_i\right) \left(\sum_{i=1}^{5} \beta_i\right)
\]

\[
+ \left(1 - a\right)\left[\alpha_3 (\beta_1 + \beta_2 + \beta_4 + \beta_5) - \beta_3 (\alpha_1 + \alpha_2 + \alpha_4 + \alpha_5)\right] \\
\left(\sum_{i=1}^{5} \alpha_i\right) \left(\sum_{i=1}^{5} \beta_i\right)
\]

\[
+ \left(1 - a\right)\left[\alpha_4 (\beta_1 + \beta_2 + \beta_3 + \beta_5) - \beta_4 (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5)\right] \\
\left(\sum_{i=1}^{5} \alpha_i\right) \left(\sum_{i=1}^{5} \beta_i\right)
\]

\[
+ \left(1 - a\right)\left[\alpha_5 (\beta_1 + \beta_2 + \beta_3 + \beta_4) - \beta_5 (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)\right] \\
\left(\sum_{i=1}^{5} \alpha_i\right) \left(\sum_{i=1}^{5} \beta_i\right)
\]

\[
< (1 - a), \quad a < 1,
\]

or

\[
\begin{align*}
&\left|\alpha_1 (\beta_2 + \beta_3 + \beta_4 + \beta_5) - \beta_1 (\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)\right| \\
&\quad + \left|\alpha_2 (\beta_1 + \beta_3 + \beta_4 + \beta_5) - \beta_2 (\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5)\right| \\
&\quad + \left|\alpha_3 (\beta_1 + \beta_2 + \beta_4 + \beta_5) - \beta_3 (\alpha_1 + \alpha_2 + \alpha_4 + \alpha_5)\right| \\
&\quad + \left|\alpha_4 (\beta_1 + \beta_2 + \beta_3 + \beta_5) - \beta_4 (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5)\right| \\
&\quad + \left|\alpha_5 (\beta_1 + \beta_2 + \beta_3 + \beta_4) - \beta_5 (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)\right| \\
&< \left(\sum_{i=1}^{5} \alpha_i\right) \left(\sum_{i=1}^{5} \beta_i\right),
\end{align*}
\]

Thus, the proof is now completed.

\[\square\]

3. Boundedness of the solutions

In this section, we investigate the boundedness of the positive solutions of Eq.(1).

**Theorem 4.** Every solution of Eq.(1) is bounded if \(a < 1\).

**Proof.** Let \(\{x_n\}_{n=-5}^{\infty}\) be a solution of Eq.(1). It follows from Eq.(1) that

\[
x_{n+1} = ax_n + \sum_{i=1}^{5} \alpha_i x_{n-i} - \sum_{i=1}^{5} \beta_i x_{n-i}
\]

\[
= ax_n + \frac{\alpha_1 x_{n-1}}{\sum_{i=1}^{5} \beta_i x_{n-i}} + \frac{\alpha_2 x_{n-2}}{\sum_{i=1}^{5} \beta_i x_{n-i}} + \frac{\alpha_3 x_{n-3}}{\sum_{i=1}^{5} \beta_i x_{n-i}} + \frac{\alpha_4 x_{n-4}}{\sum_{i=1}^{5} \beta_i x_{n-i}} + \frac{\alpha_5 x_{n-5}}{\sum_{i=1}^{5} \beta_i x_{n-i}}.
\]
Then
\[ x_{n+1} \leq ax_n + \frac{\alpha_1 x_{n-1}}{\beta_1 x_n} + \frac{\alpha_2 x_{n-2}}{\beta_2 x_n} + \frac{\alpha_3 x_{n-3}}{\beta_3 x_n} + \frac{\alpha_4 x_{n-4}}{\beta_4 x_n} + \frac{\alpha_5 x_{n-5}}{\beta_5 x_n} \]
\[ = ax_n + \frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2} + \frac{\alpha_3}{\beta_3} + \frac{\alpha_4}{\beta_4} + \frac{\alpha_5}{\beta_5} \quad \text{for all } n \geq 1. \]

By using a comparison, we can write the right hand side as follows
\[ y_{n+1} = ay_n + \frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2} + \frac{\alpha_3}{\beta_3} + \frac{\alpha_4}{\beta_4} + \frac{\alpha_5}{\beta_5}. \]

then
\[ y_n = a^n y_0 + \text{constant}, \]
and this equation is locally asymptotically stable because \( a < 1 \), and converges to the equilibrium point
\[ \bar{y} = \frac{\alpha_1 \beta_2 \beta_3 \beta_4 \beta_5 + \alpha_2 \beta_1 \beta_3 \beta_4 \beta_5 + \alpha_3 \beta_1 \beta_2 \beta_4 \beta_5 + \alpha_4 \beta_1 \beta_2 \beta_3 \beta_5 + \alpha_5 \beta_1 \beta_2 \beta_3 \beta_4}{\beta_1 \beta_2 \beta_3 \beta_4 \beta_5 (1 - a)}. \]

Therefore
\[ \lim_{n \to \infty} \sup x_n \leq \frac{\alpha_1 \beta_2 \beta_3 \beta_4 \beta_5 + \alpha_2 \beta_1 \beta_3 \beta_4 \beta_5 + \alpha_3 \beta_1 \beta_2 \beta_4 \beta_5 + \alpha_4 \beta_1 \beta_2 \beta_3 \beta_5 + \alpha_5 \beta_1 \beta_2 \beta_3 \beta_4}{\beta_1 \beta_2 \beta_3 \beta_4 \beta_5 (1 - a)}. \]

Thus, the solution of Eq.(1) is bounded and the proof is now completed.

**Theorem 5.** Every solution of Eq.(1) is unbounded if \( a > 1 \).

Proof. Let \( \{x_n\}_{n=-5}^{\infty} \) be a solution of Eq.(1). Then from Eq.(1) we see that
\[ x_{n+1} = ax_n + \frac{\sum_{i=1}^{5} \alpha_i x_{n-i}}{\sum_{i=1}^{5} \beta_i x_{n-i}} > ax_n \quad \text{for all } n \geq 1. \]

We can see that the right hand side can be written as follows
\[ y_{n+1} = ay_n \Rightarrow y_n = a^n y_0, \]
and this equation is unstable because \( a > 1 \), and
\[ \lim_{n \to \infty} y_n = \infty. \]

Then, by using the ratio test \( \{x_n\}_{n=-5}^{\infty} \) is unbounded from above. Thus, the proof is now completed.

### 4. Periodic solutions

In this section, we study the existence of periodic solutions of Eq.(1). The following theorem states the necessary and sufficient conditions that the equation has periodic solutions of prime period two.
Theorem 6. If \(( \beta_1 + \beta_3 + \beta_5 ) > ( \beta_2 + \beta_4 )\) and \(( \alpha_1 + \alpha_3 + \alpha_5 ) > ( \alpha_2 + \alpha_4 )\), then the necessary and sufficient condition for Eq.(1) to have positive solutions of prime period two is that the inequality

\[
(a + 1) \left[ (\alpha_1 + \alpha_3 + \alpha_5) - (\alpha_2 + \alpha_4) \right] \left[ (\alpha_1 + \alpha_3 + \alpha_5) - (\alpha_2 + \alpha_4) \right]^2 \\
+ 4 \left[ (\alpha_1 + \alpha_3 + \alpha_5) - (\alpha_2 + \alpha_4) \right] \left[ (\alpha_2 + \alpha_4) (\beta_1 + \beta_3 + \beta_5) + a (\beta_2 + \beta_4) (\alpha_1 + \alpha_3 + \alpha_5) \right] > 0.
\]

(13)

is valid.

Proof. Suppose that there exist positive distinctive solutions of prime period two

\[ 
\ldots, P, Q, P, Q, \ldots \ldots
\]

of Eq.(1). From Eq.(1) we have

\[
x_{n+1} = ax_n + \sum_{i=1}^{5} \alpha_i x_{n-i} + bx_{n-1} + cx_{n-2} + f x_{n-3} + g x_{n-4} + s x_{n-4}
\]

\[
P = aQ + \frac{(\alpha_1 + \alpha_3 + \alpha_5) P + (\alpha_2 + \alpha_4) Q}{(\beta_1 + \beta_3 + \beta_5) P + (\beta_2 + \beta_4) Q}, \quad Q = aP + \frac{(\alpha_1 + \alpha_3 + \alpha_5) Q + (\alpha_2 + \alpha_4) P}{(\beta_1 + \beta_3 + \beta_5) Q + (e + s) P}.
\]

Consequently, we obtain

\[
(\beta_1 + \beta_3 + \beta_5) P^2 + (\beta_2 + \beta_4) PQ = a (\beta_1 + \beta_3 + \beta_5) P Q + a (\beta_2 + \beta_4) Q^2 + (\alpha_1 + \alpha_3 + \alpha_5) P + (\alpha_2 + \alpha_4) Q,
\]

(14)

and

\[
(\beta_1 + \beta_3 + \beta_5) Q^2 + (\beta_2 + \beta_4) PQ = a (\beta_1 + \beta_3 + \beta_5) P Q + a (\beta_2 + \beta_4) P^2 + (\alpha_1 + \alpha_3 + \alpha_5) Q + (\alpha_2 + \alpha_4) P.
\]

(15)

By subtracting (14) from (15), we have

\[
[a (\beta_2 + \beta_4) + (\beta_1 + \beta_3 + \beta_5)] (P^2 - Q^2) = [(\alpha_1 + \alpha_3 + \alpha_5) - (\alpha_2 + \alpha_4)] (P - Q).
\]

Since \( P \neq Q \), it follows that

\[
P + Q = \frac{[(\alpha_1 + \alpha_3 + \alpha_5) - (\alpha_2 + \alpha_4)]}{[a (\beta_2 + \beta_4) + (\beta_1 + \beta_3 + \beta_5)]}, \tag{16}
\]

while, by adding (14) and (15) and by using the relation

\[
P^2 + Q^2 = (P + Q)^2 - 2PQ \quad \text{for all} \quad P, Q \in R,
\]

we obtain

\[
PQ = \frac{[(\alpha_1 + \alpha_3 + \alpha_5) - (\alpha_2 + \alpha_4)] [(\alpha_2 + \alpha_4) (\beta_1 + \beta_3 + \beta_5) + a (\beta_2 + \beta_4) (\alpha_1 + \alpha_3 + \alpha_5)]}{(a + 1) [(\beta_2 + \beta_4) - (\beta_1 + \beta_3 + \beta_5)] [a (\beta_2 + \beta_4) + (\beta_1 + \beta_3 + \beta_5)]^2} \tag{17}
\]

Assume that \( P \) and \( Q \) are two distinct real roots of the quadratic equation

\[ t^2 - (P + Q) t + PQ = 0. \]
Now, we are going to show that

\[
t^2 = \left(\frac{[(\alpha_1 + \alpha_3 + \alpha_5) - (\alpha_2 + \alpha_4)]}{a (\beta_2 + \beta_4) + (\beta_1 + \beta_3 + \beta_5)}\right) t + \frac{[(\alpha_1 + \alpha_3 + \alpha_5) - (\alpha_2 + \alpha_4)] [(\alpha_2 + \alpha_4) (\beta_1 + \beta_3 + \beta_5) + a (\beta_2 + \beta_4) (\alpha_1 + \alpha_3 + \alpha_5)]}{(a + 1) [(\beta_2 + \beta_4) - (\beta_1 + \beta_3 + \beta_5)] [a (\beta_2 + \beta_4) + (\beta_1 + \beta_3 + \beta_5)]^2}
\]

\[= 0,
\] (18)

and so

\[
& \quad 4 \frac{[(\alpha_1 + \alpha_3 + \alpha_5) - (\alpha_2 + \alpha_4)] [(\alpha_2 + \alpha_4) (\beta_1 + \beta_3 + \beta_5) + a (\beta_2 + \beta_4) (\alpha_1 + \alpha_3 + \alpha_5)]}{(a + 1) [(\beta_2 + \beta_4) - (\beta_1 + \beta_3 + \beta_5)]} > 0,
\]

or

\[
\frac{[(\alpha_1 + \alpha_3 + \alpha_5) - (\alpha_2 + \alpha_4)]^2}{(a + 1) [(\beta_1 + \beta_3 + \beta_5) - (\beta_2 + \beta_4)]} > 0.
\]

From (19), we obtain

\[
(a + 1) [(\beta_1 + \beta_3 + \beta_5) - (\beta_2 + \beta_4)] [(\alpha_1 + \alpha_3 + \alpha_5) - (\alpha_2 + \alpha_4)]^2
\]

\[+ 4 [(\alpha_1 + \alpha_3 + \alpha_5) - (\alpha_2 + \alpha_4)] [(\alpha_2 + \alpha_4) (\beta_1 + \beta_3 + \beta_5) + a (\beta_2 + \beta_4) (\alpha_1 + \alpha_3 + \alpha_5)] > 0.
\]

Thus, the condition (13) is valid. Conversely, suppose that the condition (13) is valid where \((\beta_1 + \beta_3 + \beta_5) > (\beta_2 + \beta_4)\) and \((\alpha_1 + \alpha_3 + \alpha_5) > (\alpha_2 + \alpha_4)\). Then, we deduce immediately from (13) that the inequality (19) holds. There exist two positive distinctive real numbers \(P\) and \(Q\) representing two positive roots of Eq.(18) such that

\[
P = \frac{[(\alpha_1 + \alpha_3 + \alpha_5) - (\alpha_2 + \alpha_4)] + \delta}{2 [a (\beta_2 + \beta_4) + (\beta_1 + \beta_3 + \beta_5)]}
\]

(20)

and

\[
Q = \frac{[(\alpha_1 + \alpha_3 + \alpha_5) - (\alpha_2 + \alpha_4)] - \delta}{2 [a (\beta_2 + \beta_4) + (\beta_1 + \beta_3 + \beta_5)]}
\]

(21)

where \(\delta = \sqrt{[(\alpha_1 + \alpha_3 + \alpha_5) - (\alpha_2 + \alpha_4)]^2 - \frac{4 [(\alpha_1 + \alpha_3 + \alpha_5) - (\alpha_2 + \alpha_4)] [(\alpha_2 + \alpha_4) (\beta_1 + \beta_3 + \beta_5) + a (\beta_2 + \beta_4) (\alpha_1 + \alpha_3 + \alpha_5)]}{(a + 1) [(\beta_2 + \beta_4) - (\beta_1 + \beta_3 + \beta_5)]}}\).

Now, we are going to prove that \(P\) and \(Q\) are positive solutions of prime period two of Eq.(1).

To this end, we assume that \(x_{-5} = P,\ x_{-4} = Q,\ x_{-3} = P,\ x_{-2} = Q,\ x_{-1} = P,\ x_{0} = Q\).

Now, we are going to show that \(x_1 = P\) and \(x_2 = Q\). From Eq.(1) we deduce that

\[
x_1 = ax_0 + \frac{\alpha_1 x_{-1} + \alpha_2 x_{-2} + \alpha_3 x_{-3} + \alpha_4 x_{-4} + \alpha_5 x_{-5}}{\beta_1 x_{-1} + \beta_2 x_{-2} + \beta_3 x_{-3} + \beta_4 x_{-4} + \beta_5 x_{-5}} = aQ + \frac{(\alpha_1 + \alpha_3 + \alpha_5) P + (\alpha_2 + \alpha_4) Q}{(\beta_1 + \beta_3 + \beta_5) P + (\beta_2 + \beta_4) Q}.
\]

(22)

Substituting (20) and (21) into (22) we deduce that

\[
x_1 - P = aQ + \frac{(\alpha_1 + \alpha_3 + \alpha_5) P + (\alpha_2 + \alpha_4) Q}{(\beta_1 + \beta_3 + \beta_5) P + (\beta_2 + \beta_4) Q} - P
\]

\[= \frac{[a (\beta_1 + \beta_3 + \beta_5) - (\beta_2 + \beta_4)] PQ + a (\beta_2 + \beta_4) Q^2 - (\beta_1 + \beta_3 + \beta_5) P^2}{(\beta_1 + \beta_3 + \beta_5) P + (\beta_2 + \beta_4) Q}
\]

\[+ \frac{(\alpha_1 + \alpha_3 + \alpha_5) P + (\alpha_2 + \alpha_4) Q}{(\beta_1 + \beta_3 + \beta_5) P + (\beta_2 + \beta_4) Q}.
\]
\[
\begin{align*}
&= \frac{a(\beta_1 + \beta_3 + \beta_5) - (\beta_2 + \beta_4)(a(\alpha_1 + \alpha_3 + \alpha_5) - (a_2 + a_4)) [a(\alpha_2 + a_4) (\beta_1 + \beta_3 + \beta_5) + a(\beta_2 + \beta_4)(\alpha_1 + \alpha_3 + \alpha_5)]}{(\alpha + 1)(a(\beta_2 + \beta_4) - (\beta_2 + \beta_4) (\beta_1 + \beta_3 + \beta_5))} \\
&\quad + \frac{a(\beta_2 + \beta_4) \left( (\alpha_1 + \alpha_3 + \alpha_5) - (a_2 + a_4) \right) - \delta}{K} \\
&\quad - \frac{\left( (\alpha_1 + \alpha_3 + \alpha_5) - (a_2 + a_4) \right) + \delta}{K} \\
&\quad + \frac{2 \left[ a(\beta_2 + \beta_4) + (\beta_1 + \beta_3 + \beta_5) \right] (\alpha_1 + \alpha_3 + \alpha_5) \left( (\alpha_1 + \alpha_3 + \alpha_5) - (a_2 + a_4) \right) + \delta}{K} \\
&\quad + \frac{2 \left[ a(\beta_2 + \beta_4) + (\beta_1 + \beta_3 + \beta_5) \right] (\alpha_2 + a_4) \left( (\alpha_1 + \alpha_3 + \alpha_5) - (a_2 + a_4) \right) - \delta}{K} \\
&\quad = \frac{4 \left[ a(\beta_2 + \beta_4) - (\beta_2 + \beta_4) (\beta_1 + \beta_3 + \beta_5) \right] (\alpha_1 + \alpha_3 + \alpha_5) \left( (\alpha_1 + \alpha_3 + \alpha_5) - (a_2 + a_4) \right) [a(\alpha_2 + a_4) (\beta_1 + \beta_3 + \beta_5) + a(\beta_2 + \beta_4)(\alpha_1 + \alpha_3 + \alpha_5)]}{(\alpha + 1)(a(\beta_2 + \beta_4) - (\beta_2 + \beta_4) (\beta_1 + \beta_3 + \beta_5))} \\
&\quad + \frac{a(\beta_2 + \beta_4) \left( (\alpha_1 + \alpha_3 + \alpha_5) - (a_2 + a_4) \right)^2 - (\beta_1 + \beta_3 + \beta_5) \left( (\alpha_1 + \alpha_3 + \alpha_5) - (a_2 + a_4) \right)^2}{K} \\
&\quad + \frac{2 (\alpha_1 + \alpha_3 + \alpha_5) \left[ a(\beta_2 + \beta_4) + (\beta_1 + \beta_3 + \beta_5) \right] [(\alpha_1 + \alpha_3 + \alpha_5) - (a_2 + a_4)]}{K}
\end{align*}
\]
By differentiating the function $u$

By using the mathematical induction, we have

where

Similarly, we can show that

By using the mathematical induction, we have $x_n = P$ and $x_{n+1} = Q$, $n \geq -5$. 

5. Global stability

In this section we study the global asymptotic stability of the positive solutions of Eq. (1).

**Lemma 7.** For any values of the quotient $\frac{a_1}{a_2}, \frac{a_2}{a_3}, \frac{a_3}{a_4}$ and $\frac{a_4}{a_5}$, the function $F(u_0, \ldots, u_5)$ defined by Eq. (8) is monotonic in its arguments.

**Proof.** By differentiating the function $F(u_0, \ldots, u_5)$ given by the formula (8) with respect to $u_i$ ($i = 0, \ldots, 5$) we obtain

$$F(u_0, \ldots, u_5) = au_0 + \frac{\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \alpha_4 u_4 + \alpha_5 u_5}{\beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3 + \beta_4 u_4 + \beta_5 u_5}$$

$$F_{u_0} = a,$$  

$$F_{u_1} = \frac{(\alpha_1 \beta_2 - \alpha_2 \beta_1) u_2 + (\alpha_1 \beta_3 - \alpha_3 \beta_1) u_3 + (\alpha_1 \beta_4 - \alpha_4 \beta_1) u_4 + (\alpha_1 \beta_5 - \alpha_5 \beta_1) u_5}{(\beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3 + \beta_4 u_4 + \beta_5 u_5)^2},$$

$$F_{u_2} = -\frac{(\alpha_1 \beta_2 - \alpha_2 \beta_1) u_1 + (\alpha_2 \beta_3 - \alpha_3 \beta_2) u_3 + (\alpha_2 \beta_4 - \alpha_4 \beta_2) u_4 + (\alpha_2 \beta_5 - \alpha_5 \beta_2) u_5}{(\beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3 + \beta_4 u_4 + \beta_5 u_5)^2},$$

$$F_{u_3} = -\frac{(\alpha_1 \beta_3 - \alpha_3 \beta_1) u_1 + (\alpha_2 \beta_3 - \alpha_3 \beta_2) u_2 + (\alpha_3 \beta_4 - \alpha_4 \beta_3) u_4 + (\alpha_3 \beta_5 - \alpha_5 \beta_3) u_5}{(\beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3 + \beta_4 u_4 + \beta_5 u_5)^2},$$

$$F_{u_4} = -\frac{(\alpha_1 \beta_4 - \alpha_4 \beta_1) u_1 + (\alpha_2 \beta_4 - \alpha_4 \beta_2) u_2 + (\alpha_3 \beta_4 - \alpha_4 \beta_3) u_3 + (\alpha_4 \beta_5 - \alpha_5 \beta_4) u_5}{(\beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3 + \beta_4 u_4 + \beta_5 u_5)^2},$$

$$F_{u_5}.$$
Let $u$ be non-increasing in $x$ and it is not clear what is going on with $u$ prove that

$$M(1 - a) Mm = M(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 M - (1 - a)(\beta_1 + \beta_2 + \beta_3 + \beta_4) M^2$$  \hspace{1cm} (31)

and

$$\beta_5 (1 - a) Mm = m(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 M - (1 - a)(\beta_1 + \beta_2 + \beta_3 + \beta_4) m^2.$$  \hspace{1cm} (32)

From (31) and (32), we obtain

$$\{[(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) - \alpha_5] - (1 - a)(\beta_1 + \beta_2 + \beta_3 + \beta_4) (m + M)\} = 0.$$  \hspace{1cm} (33)

Since $a < 1$ and $\alpha_5 \geq (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)$, we deduce from (33) that $M = m$. It follows by Theorem 2, that $\bar{x}$ of Eq.(1) is a global attractor and the proof is now completed.
Case 2. Assume that the function $F(u_0, ..., u_5)$ is non-decreasing in $u_0, u_1$ and non-increasing in $u_2, u_3, u_4, u_5$.

Suppose that $(m, M)$ is a solution of the system

$$M = F(M, M, m, m, m, m) \quad \text{and} \quad m = F(m, m, M, M, M).$$

Then we get

$$M = aM + \frac{\alpha_1 M + \alpha_2 M + \alpha_3 m + \alpha_4 m + \alpha_5 m}{\beta_1 M + \beta_2 M + \beta_3 m + \beta_4 m + \beta_5 m} \quad \text{and} \quad m = am + \frac{\alpha_1 m + \alpha_2 M + \alpha_3 M + \alpha_4 M + \alpha_5 M}{\beta_1 m + \beta_2 M + \beta_3 M + \beta_4 M + \beta_5 M},$$

or

$$M(1 - a) = \frac{\alpha_1 M + (\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5) m}{\beta_1 M + (\beta_2 + \beta_3 + \beta_4 + \beta_5) m} \quad \text{and} \quad m(1 - a) = \frac{\alpha_1 m + (\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5) M}{\beta_1 m + (\beta_2 + \beta_3 + \beta_4 + \beta_5) M}.$$

From which we have

$$(\beta_2 + \beta_3 + \beta_4 + \beta_5) (1 - a) Mm = \alpha_1 M + (\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5) m - \beta_1 (1 - a) M^2 \quad (34)$$

and

$$(\beta_2 + \beta_3 + \beta_4 + \beta_5) (1 - a) Mm = \alpha_1 m + (\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5) M - \beta_1 (1 - a) m^2. \quad (35)$$

From (34) and (35), we obtain

$$(m - M) \{[\alpha_1 - (\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)] - \beta_1 (1 - a) (m + M)\} = 0. \quad (36)$$

Since $a < 1$ and $(\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5) \geq \alpha_1$, we deduce from (36) that $M = m$. It follows by Theorem 2, that $\bar{x}$ of Eq.(1) is a global attractor and the proof is now completed.

Case 3. Assume that the function $F(u_0, ..., u_5)$ is non-decreasing in $u_0, u_1, u_2$ and non-increasing in $u_3, u_4, u_5$. Suppose that $(m, M)$ is a solution of the system

$$M = F(M, M, M, m, m, m) \quad \text{and} \quad m = F(m, m, M, M, M, M).$$

Then we get

$$M = aM + \frac{\alpha_1 M + \alpha_2 M + \alpha_3 m + \alpha_4 m + \alpha_5 m}{\beta_1 M + \beta_2 M + \beta_3 m + \beta_4 m + \beta_5 m} \quad \text{and} \quad m = am + \frac{\alpha_1 m + \alpha_2 M + \alpha_3 M + \alpha_4 M + \alpha_5 M}{\beta_1 m + \beta_2 M + \beta_3 M + \beta_4 M + \beta_5 M},$$

or

$$M(1 - a) = \frac{M (\alpha_1 + \alpha_2) + (\alpha_3 + \alpha_4 + \alpha_5) m}{M (\beta_1 + \beta_2) + (\beta_3 + \beta_4 + \beta_5) m} \quad \text{and} \quad m(1 - a) = \frac{m (\alpha_1 + \alpha_2) + (\alpha_3 + \alpha_4 + \alpha_5) M}{m (\beta_1 + \beta_2) + (\beta_3 + \beta_4 + \beta_5) M}.$$

From which we have

$$(\beta_3 + \beta_4 + \beta_5) (1 - a) Mm = M (\alpha_1 + \alpha_2) + (\alpha_3 + \alpha_4 + \alpha_5) m - (1 - a) (\beta_1 + \beta_2) M^2 \quad (37)$$

and

$$(\beta_3 + \beta_4 + \beta_5) (1 - a) Mm = m (\alpha_1 + \alpha_2) + (\alpha_3 + \alpha_4 + \alpha_5) M - (1 - a) (\beta_1 + \beta_2) m^2. \quad (38)$$

From (37) and (38), we obtain

$$(m - M) \{[(\alpha_1 + \alpha_2) - (\alpha_3 + \alpha_4 + \alpha_5)] - (1 - a) (\beta_1 + \beta_2) (m + M)\} = 0. \quad (39)$$

Since $a < 1$ and $(\alpha_3 + \alpha_4 + \alpha_5) \geq (\alpha_1 + \alpha_2)$, we deduce from (39) that $M = m$. It follows by Theorem 2, that $\bar{x}$ of Eq.(1) is a global attractor and the proof is now completed.

Case 4. Assume that the function $F(u_0, ..., u_5)$ is non-decreasing in $u_0, u_1, u_3$ and non-increasing in $u_2, u_4, u_5$. Suppose that $(m, M)$ is a solution of the system

$$M = F(M, M, m, M, m, m) \quad \text{and} \quad m = F(m, m, M, m, M, M).$$
Then we get
\[ M = aM + \frac{(a_1 + a_3) M + (a_2 + a_4 + a_5) m}{(\beta_1 + \beta_3) M + (\beta_2 + \beta_4 + \beta_5) m} \quad \text{and} \quad m = am + \frac{(a_1 + a_3) m + (a_2 + a_4 + a_5) M}{(\beta_1 + \beta_3) m + (\beta_2 + \beta_4 + \beta_5) M}, \]
or
\[ M (1 - a) = \frac{(a_1 + a_3) M + (a_2 + a_4 + a_5) m}{(\beta_1 + \beta_3) M + (\beta_2 + \beta_4 + \beta_5) m} \quad \text{and} \quad m (1 - a) = \frac{(a_1 + a_3) m + (a_2 + a_4 + a_5) M}{(\beta_1 + \beta_3) m + (\beta_2 + \beta_4 + \beta_5) M}. \]

From which we have
\[ (\beta_2 + \beta_4 + \beta_5) (1 - a) M m = M (a_1 + a_3) + (a_2 + a_4 + a_5) m - (1 - a) (\beta_1 + \beta_3) M^2 \]
and
\[ (\beta_2 + \beta_4 + \beta_5) (1 - a) M m = m (a_1 + a_3) + (a_2 + a_4 + a_5) M - (1 - a) (\beta_1 + \beta_3) m^2. \]

From (40) and (41), we obtain
\[ (m - M) \{[(a_1 + a_3) - (a_2 + a_4 + a_5)] - (1 - a) (\beta_1 + \beta_3) (m + M)\} = 0. \]

Since \( a < 1 \) and \( (a_2 + a_4 + a_5) \geq (a_1 + a_3) \), we deduce from (42) that \( M = m \). It follows by Theorem 2, that \( \bar{x} \) of Eq.(1) is a global attractor and the proof is now completed.

Case 5. Assume that the function \( F(u_0,...,u_5) \) is non-decreasing in \( u_0,u_1,u_2,u_3 \) and non-increasing in \( u_4,u_5 \). Suppose that \( (m,M) \) is a solution of the system
\[ M = F(M, M, M, M, m, m) \quad \text{and} \quad m = F(m, m, m, m, M, M). \]

Then we get
\[ M = aM + \frac{(a_1 + a_2 + a_3) M + (a_4 + a_5) m}{(\beta_1 + \beta_2 + \beta_3) M + (\beta_4 + \beta_5) m} \quad \text{and} \quad m = am + \frac{(a_1 + a_2 + a_3) m + (a_4 + a_5) M}{(\beta_1 + \beta_2 + \beta_3) m + (\beta_4 + \beta_5) M}, \]

or
\[ M (1 - a) = \frac{(a_1 + a_2 + a_3) M + (a_4 + a_5) m}{(\beta_1 + \beta_2 + \beta_3) M + (\beta_4 + \beta_5) m} \quad \text{and} \quad m (1 - a) = \frac{(a_1 + a_2 + a_3) m + (a_4 + a_5) M}{(\beta_1 + \beta_2 + \beta_3) m + (\beta_4 + \beta_5) M}. \]

From which we have
\[ (\beta_4 + \beta_5) (1 - a) M m = M (a_1 + a_2 + a_3) + (a_4 + a_5) m - (1 - a) (\beta_1 + \beta_2 + \beta_3) M^2 \]
and
\[ (\beta_4 + \beta_5) (1 - a) M m = m (a_1 + a_2 + a_3) + (a_4 + a_5) M - (1 - a) (\beta_1 + \beta_2 + \beta_3) m^2. \]

From (43) and (44), we obtain
\[ (m - M) \{[(a_1 + a_2 + a_3) - (a_4 + a_5)] - (1 - a) (\beta_1 + \beta_2 + \beta_3) (m + M)\} = 0. \]

Since \( a < 1 \) and \( (a_4 + a_5) \geq (a_1 + a_2 + a_3) \), we deduce from (45) that \( M = m \). It follows by Theorem 2, that \( \bar{x} \) of Eq.(1) is a global attractor and the proof is now completed.

Case 6. Assume that the function \( F(u_0,...,u_5) \) is non-decreasing in \( u_0,u_1,u_3,u_4 \) and non-increasing in \( u_2,u_5 \). Suppose that \( (m,M) \) is a solution of the system
\[ M = F(M, M, m, M, m, m) \quad \text{and} \quad m = F(m, m, m, m, M, m). \]

Then we get
\[ M = aM + \frac{(a_1 + a_3 + a_4) M + (a_2 + a_5) m}{(\beta_1 + \beta_3 + \beta_4) M + (\beta_2 + \beta_5) m} \quad \text{and} \quad m = am + \frac{(a_1 + a_3 + a_4) m + (a_2 + a_5) M}{(\beta_1 + \beta_3 + \beta_4) m + (\beta_2 + \beta_5) M}, \]
or

\[ M(1 - a) = \frac{(\alpha_1 + \alpha_3 + \alpha_4) M + (\alpha_2 + \alpha_5) m}{(\beta_1 + \beta_3 + \beta_4) M + (\beta_2 + \beta_5) m} \quad \text{and} \quad m(1 - a) = \frac{(\alpha_1 + \alpha_3 + \alpha_4) m + (\alpha_2 + \alpha_5) M}{(\beta_1 + \beta_3 + \beta_4) m + (\beta_2 + \beta_5) M}. \]

From which we have

\[ (\beta_2 + \beta_5)(1 - a) M m = M(\alpha_1 + \alpha_3 + \alpha_4) + (\alpha_2 + \alpha_5) m - (1 - a)(\beta_1 + \beta_3 + \beta_4) M^2 \quad (46) \]

and

\[ (\beta_2 + \beta_5)(1 - a) M m = m(\alpha_1 + \alpha_3 + \alpha_4) + (\alpha_2 + \alpha_5) M - (1 - a)(\beta_1 + \beta_3 + \beta_4) m^2. \quad (47) \]

From (46) and (47), we obtain

\[ (m - M) \left\{\left[(\alpha_1 + \alpha_3 + \alpha_4) - (\alpha_2 + \alpha_5)\right] - (1 - a)(\beta_1 + \beta_3 + \beta_4)(m + M)\right\} = 0. \quad (48) \]

Since \( a < 1 \) and \( (\alpha_2 + \alpha_5) \geq (\alpha_1 + \alpha_3 + \alpha_4) \), we deduce from (48) that \( M = m \). It follows by Theorem 2, that \( \bar{x} \) of Eq. (1) is a global attractor and the proof is now completed.

Case 7. Assume that the function \( F(u_0, \ldots, u_5) \) is non-decreasing in \( u_0, u_1, u_4 \) and non-increasing in \( u_2, u_3, u_5 \). Suppose that \( (m, M) \) is a solution of the system

\[ M = F(M, M, m, m, M, m) \quad \text{and} \quad m = F(m, m, M, M, m, M). \]

Then we get

\[ M = am + \frac{(\alpha_1 + \alpha_4) M + (\alpha_2 + \alpha_3 + \alpha_5) m}{(\beta_1 + \beta_4) M + (\beta_2 + \beta_3 + \beta_5) m} \quad \text{and} \quad m = am + \frac{(\alpha_1 + \alpha_4) m + (\alpha_2 + \alpha_3 + \alpha_5) M}{(\beta_1 + \beta_4) m + (\beta_2 + \beta_3 + \beta_5) M}. \]

or

\[ M(1 - a) = \frac{(\alpha_1 + \alpha_4) M + (\alpha_2 + \alpha_3 + \alpha_5) m}{(\beta_1 + \beta_4) M + (\beta_2 + \beta_3 + \beta_5) m} \quad \text{and} \quad m(1 - a) = \frac{(\alpha_1 + \alpha_4) m + (\alpha_2 + \alpha_3 + \alpha_5) M}{(\beta_1 + \beta_4) m + (\beta_2 + \beta_3 + \beta_5) M}. \]

From which we have

\[ (\beta_2 + \beta_3 + \beta_5)(1 - a) M m = M(\alpha_1 + \alpha_4) + (\alpha_2 + \alpha_3 + \alpha_5) m - (1 - a)(\beta_1 + \beta_4) M^2 \quad (49) \]

and

\[ (\beta_2 + \beta_3 + \beta_5)(1 - a) M m = m(\alpha_1 + \alpha_4) + (\alpha_2 + \alpha_3 + \alpha_5) M - (1 - a)(\beta_1 + \beta_4) m^2. \quad (50) \]

From (49) and (50), we obtain

\[ (m - M) \left\{\left[(\alpha_1 + \alpha_4) - (\alpha_2 + \alpha_3 + \alpha_5)\right] - (1 - a)(\beta_1 + \beta_4)(m + M)\right\} = 0. \quad (51) \]

Since \( a < 1 \) and \( (\alpha_2 + \alpha_3 + \alpha_5) \geq (\alpha_1 + \alpha_4) \), we deduce from (51) that \( M = m \). It follows by Theorem 2, that \( \bar{x} \) of Eq. (1) is a global attractor and the proof is now completed.

Case 8. Assume that the function \( F(u_0, \ldots, u_5) \) is non-decreasing in \( u_0, u_1, u_2, u_4 \) and non-increasing in \( u_3, u_5 \). Suppose that \( (m, M) \) is a solution of the system

\[ M = F(M, M, m, m, M, m) \quad \text{and} \quad m = F(m, m, M, M, m, M). \]

Then we get

\[ M = am + \frac{(\alpha_1 + \alpha_2 + \alpha_4) M + (\alpha_3 + \alpha_5) m}{(\beta_1 + \beta_2 + \beta_4) M + (\beta_3 + \beta_5) m} \quad \text{and} \quad m = am + \frac{(\alpha_1 + \alpha_2 + \alpha_4) m + (\alpha_3 + \alpha_5) M}{(\beta_1 + \beta_2 + \beta_4) m + (\beta_3 + \beta_5) M}. \]

or

\[ M(1 - a) = \frac{(\alpha_1 + \alpha_2 + \alpha_4) M + (\alpha_3 + \alpha_5) m}{(\beta_1 + \beta_2 + \beta_4) M + (\beta_3 + \beta_5) m} \quad \text{and} \quad m(1 - a) = \frac{(\alpha_1 + \alpha_2 + \alpha_4) m + (\alpha_3 + \alpha_5) M}{(\beta_1 + \beta_2 + \beta_4) m + (\beta_3 + \beta_5) M}. \]
From which we have
\[(\beta_3 + \beta_5) (1 - a) M m = M (\alpha_1 + \alpha_2 + \alpha_4) + (\alpha_3 + \alpha_5) m - (1 - a) (\beta_1 + \beta_2 + \beta_4) M^2\] (52)
and
\[(\beta_3 + \beta_5) (1 - a) M m = m (\alpha_1 + \alpha_2 + \alpha_4) + (\alpha_3 + \alpha_5) M - (1 - a) (\beta_1 + \beta_2 + \beta_4) m^2.\] (53)
From (52) and (53), we obtain
\[(m - M) \{[(\alpha_1 + \alpha_2 + \alpha_4) - (\alpha_3 + \alpha_5)] - (1 - a) (\beta_1 + \beta_2 + \beta_4) (m + M)\} = 0.\] (54)
Since \(a < 1\) and \((\alpha_3 + \alpha_5) \geq (\alpha_1 + \alpha_2 + \alpha_4)\), we deduce from (54) that \(M = m\). It follows by Theorem 2, that \(\bar{x}\) of Eq.(1) is a global attractor and the proof is now completed.

From Theorems 3 and 8, we arrive at the following result:

**Theorem 9.** The positive equilibrium point \(\bar{x}\) given by (7) of Eq.(1) is globally asymptotic stable.

### 6. Numerical examples

In order to illustrate the results of the previous section and to support our theoretical discussions, we consider some numerical examples in this section. These examples represent different types of qualitative behavior of solutions of Eq.(1).

**Example 1.** Figure 1, shows that the solution of Eq.(1) is unbounded if \(x_{-5} = 1, x_{-4} = 2, x_{-3} = 3, x_{-2} = 4, x_{-1} = 5, x_0 = 6, a = 1.1, \alpha_1 = 10, \alpha_2 = 1, \alpha_3 = 12, \alpha_4 = 4, \alpha_5 = 5, \beta_1 = 2, \beta_2 = 3, \beta_3 = 40, \beta_4 = 50, \beta_5 = 60.\)
Example 2. Figure 2, shows that Eq.(1) has prime period two solutions if $x_{-5} = x_{-3} = x_{-1} \simeq 0.84$, $x_{-4} = x_{-2} = x_0 \simeq -0.84$, $a = 1$, $\alpha_1 = 10$, $\alpha_2 = 1$, $\alpha_3 = 30$, $\alpha_4 = 8$, $\alpha_5 = 5$, $\beta_1 = 20$, $\beta_2 = 5$, $\beta_3 = 40$, $\beta_4 = 9$, $\beta_5 = 60$.

Example 3. Figure 3, shows that Eq.(1) is globally asymptotically stable if $x_{-5} = 1$, $x_{-4} = 2$, $x_{-3} = 3$, $x_{-2} = 4$, $x_{-1} = 5$, $x_0 = 6$, $a = 0.5$, $\alpha_1 = 10$, $\alpha_2 = 1$, $\alpha_3 = 12$, $\alpha_4 = 4$, $\alpha_5 = 5$, $\beta_1 = 2$, $\beta_2 = 3$, $\beta_3 = 40$, $\beta_4 = 50$, $\beta_5 = 60$. 
Example 4. Figure 4, shows that Eq.(1) is not globally asymptotically stable if $x_{-5} = 1, x_{-4} = 2, x_{-3} = 3, x_{-2} = 4, x_{-1} = 5, x_{0} = 6, a = 100, \alpha_{1} = 10, \alpha_{2} = 1, \alpha_{3} = 12, \alpha_{4} = 4, \alpha_{5} = 5, \beta_{1} = 2, \beta_{2} = 3, \beta_{3} = 40, \beta_{4} = 50, \beta_{5} = 60$. 
7. Conclusions

We have discussed some properties of the nonlinear rational difference equation (1), namely the periodicity, the boundedness and the global stability of the positive solutions of this equation. We gave some figures to illustrate the behavior of these solutions. Our results in this article can be considered as a more generalization than the results obtained in Refs.[11,14,37].

Note that example 1 verifies Theorem 5 which shows that the solution of Eq.(1) is unbounded. While example 2 verifies Theorem 6 which shows that Eq.(1) has prime period two solutions, while example 3 verifies Theorem 9 which shows that Eq.(1) is globally asymptotically stable. But example 4 shows that Eq.(1) is not globally asymptotically stable if \( a > 1 \).

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