INCLUSION PROPERTIES FOR CERTAIN $k$–UNIFORMLY SUBCLASSES OF $p$–VALENT FUNCTIONS DEFINED BY CERTAIN INTEGRAL OPERATOR

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Abstract. We introduce several $k$–uniformly subclasses of $p$–valent functions defined by certain integral operator and investigate various inclusion relationships for these subclasses. Some interesting applications involving certain classes of integral operators are also considered.

1. Introduction

Let $\mathcal{A}_p$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \ (p \in \mathbb{N} = \{1, 2, 3, \ldots\})$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. If $f$ and $g$ are analytic in $U$, we say that $f$ is subordinate to $g$, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function $\omega$, analytic in $U$ with $\omega(0) = 0$ and $|\omega(z)| < 1 \ (z \in U)$, such that $f(z) = g(\omega(z)) \ (z \in U)$. In particular, if the function $g$ is univalent in $U$ the above subordination is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$ (see [9] and [10]).

For $0 \leq \gamma, \eta < p$, $k \geq 0$ and $z \in U$, we define $US_p^*(k; \gamma)$, $UC_p(k; \gamma)$, $UK_p(k; \gamma, \eta)$ and $UK^*_p(k; \gamma, \eta)$ the $k$–uniformly subclasses of $\mathcal{A}_p$ consisting of all analytic functions which are, respectively, $p$–valent starlike of order $\gamma$, $p$–valent convex of order $\gamma$, $p$–valent close-to-convex of order $\gamma$, and type $\eta$ and $p$–valent quasi-convex of order $\gamma$, and type $\eta$ as follows:

$$US_p^*(k; \gamma) = \left\{ f \in \mathcal{A}_p : \Re \left( \frac{zf'(z)}{f(z)} - \gamma \right) > k \left| \frac{zf'(z)}{f(z)} - p \right| \right\},$$

$$UC_p(k; \gamma) = \left\{ f \in \mathcal{A}_p : \Re \left( 1 + \frac{zf''(z)}{f'(z)} - \gamma \right) > k \left| 1 + \frac{zf''(z)}{f'(z)} - p \right| \right\}.$$
where we define the function $q_{p;k;\cdot}$ as the following:

\[ q_{p;k;\cdot}(z) = \begin{cases} 
\frac{p + (p - 2\gamma)z}{1 - z} & (k = 0), \\
\frac{p - \gamma}{1 - k^2} \cos \left\{ \frac{\pi}{2} \left( \cos^{-1} k \right) i \log \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right\} - \frac{k^2 p - \gamma}{1 - k^2} & (0 < k < 1), \\
p + \frac{2(p - \gamma)}{\pi^2} \left( \log \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right)^2 & (k = 1), \\
p - \gamma \sin \left\{ \frac{\pi}{2} \zeta(k) \right\} \int_0^{\frac{u(z)}{\sqrt{1 - x^2} \sqrt{1 - k^2}}} \frac{dt}{\sqrt{1 - t^2}} + \frac{k^2 p - \gamma}{k^2 - 1} & (k > 1), 
\end{cases} \]

where $u(z) = \frac{\sqrt{1 - z^2}}{1 - \sqrt{1 - z^2}}$, $x \in (0, 1)$ and $\zeta(k)$ is such that $k = \cosh \frac{\pi \zeta'(z)}{\pi \zeta(z)}$. By virtue of the properties of the conic domain $\Omega_{p,k,\gamma}$, we have

\[ \Re \{q_{p,k,\gamma}(z)\} > \frac{kp + \gamma}{k + 1}. \]  

Making use of the principal of subordination between analytic functions and the definition of $q_{p,k,\gamma}(z)$, we may rewrite the subclasses $US_{p}^*(k;\cdot)$, $UC_{p}(k;\cdot)$, $UK_{p}(k;\cdot,\gamma)$ and $UK_{p}^*(k;\cdot,\beta)$ as the following:

\[ US_{p}^*(k;\gamma) = \left\{ f \in A_p : \frac{zf'(z)}{f(z)} < q_{p,k,\gamma}(z) \right\}, \]

\[ UC_{p}(k;\cdot) = \left\{ f \in A_p : 1 + \frac{zf''(z)}{f'(z)} < q_{p,k,\gamma}(z) \right\}, \]

\[ UK_{p}(k;\cdot,\gamma) = \left\{ f \in A_p : \frac{zf'(z)}{g(z)} > k \left| \frac{zf'(z)}{g(z)} - p \right| \right\}, \]

\[ UK_{p}^*(k;\cdot,\beta) = \left\{ f \in A_p : \frac{zf'(z)}{g'(z)} > k \left| \frac{zf'(z)}{g'(z)} - p \right| \right\}. \]
We note that the one-parameter family of integral operator

\[ UK_p (k; \gamma, \eta) = \left\{ f \in A_p : \exists g \in US_p^* (k; \gamma), \frac{zf'(z)}{g(z)} \prec q_{p,k,\gamma} (z) \right\}, \]

(11)

\[ UK^*_p (k; \gamma, \eta) = \left\{ f \in A_p : \exists g \in UC_p (k; \eta), \frac{zf'(z)}{g(z)} \prec q_{p,k,\gamma} (z) \right\}. \]

(12)

Motivated essentially by Jung et al. [8], Shams et al. [15] introduced the integral operator \( I^\alpha_p : A_p \to A_p \) as follows (see also Aouf et al. [3]):

\[
I^\alpha_p f (z) = \begin{cases} 
\frac{(p + 1)^\alpha}{z \Gamma (\alpha)} \int_0^z \left( \log \frac{t}{z} \right)^{\alpha - 1} f (t) \, dt & (\alpha > 0; p \in \mathbb{N}), \\
0 & (\alpha = 0; p \in \mathbb{N}).
\end{cases}
\]

(13)

For \( f \in A_p \) given by (1), then from (13), we deduce that

\[ I^\alpha_p f (z) = z^p + \sum_{n=p+1}^{\infty} \left( \frac{p + 1}{n + 1} \right)^\alpha a_n z^n, \quad (\alpha \geq 0; p \in \mathbb{N}). \]

(14)

Using the above relation, it is easy to verify the identity:

\[ z (I^{\alpha+1}_p f (z))' = (p + 1) I^\alpha_p f (z) - I^{\alpha+1}_p f (z). \]

(15)

We note that the one-parameter family of integral operator \( I^\alpha_1 = I^\alpha \) was defined by Jung et al. [8].

Next, using the operator \( I^\alpha_p \), we introduce the following \( k \)--uniformly subclasses of \( p \)--valent functions for \( \alpha \geq 0, p \in \mathbb{N}, k \geq 0 \) and \( 0 \leq \gamma, \eta < p \):

\[ US_p^* (\alpha; k; \gamma) = \left\{ f \in A_p : I^\alpha_p f (z) \in US_p^* (k; \gamma) (z \in \mathbb{U}) \right\}, \]

(16)

\[ UC_p (\alpha; k; \gamma) = \left\{ f \in A_p : I^\alpha_p f (z) \in UC_p (k; \gamma) (z \in \mathbb{U}) \right\}, \]

(17)

\[ UK_p (\alpha; k; \gamma, \eta) = \left\{ f \in A_p : I^\alpha_p f (z) \in UK_p (k; \gamma, \eta) (z \in \mathbb{U}) \right\}, \]

(18)

\[ UK^*_p (\alpha; k; \gamma, \eta) = \left\{ f \in A_p : I^\alpha_p f (z) \in UK^*_p (k; \gamma, \eta) (z \in \mathbb{U}) \right\}. \]

(19)

We also note that

\[ f \in US_p^* (\alpha; k; \gamma) \Leftrightarrow \frac{zf'(z)}{p} \in UC_p (\alpha; k; \gamma), \]

(20)

and

\[ f \in UK_p (\alpha; k; \gamma, \eta) \Leftrightarrow \frac{zf'(z)}{p} \in UK^*_p (\alpha; k; \gamma, \eta). \]

(21)

In this paper, we investigate several inclusion properties of the classes \( US_p^* (\alpha; k; \gamma), \)

\( UC_p (\alpha; k; \gamma), \)

\( UK_p (\alpha; k; \gamma, \eta), \) and \( UK^*_p (\alpha; k; \gamma, \eta) \) associated with the operator \( I^\alpha_p \).

Some applications involving integral operators are also considered.
2. Inclusion properties involving the operator $I_p^\alpha$

In order to prove the main results, we shall need the following lemmas.

**Lemma 1** [6] Let $h(z)$ be convex univalent in $\mathbb{U}$ with $\Re\{\eta h(z) + \gamma\} > 0 (\eta, \gamma \in \mathbb{C})$.
If $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = h(0)$, then

$$p(z) + \frac{zp'(z)}{\eta p(z) + \gamma} \prec h(z)$$

implies

$$p(z) \prec h(z).$$

**Lemma 2** [9] Let $h(z)$ be convex univalent in $\mathbb{U}$ and let $w$ be analytic in $\mathbb{U}$ with $\Re\{w(z)\} \geq 0$. If $p(z)$ is analytic in $\mathbb{U}$ and $p(0) = h(0)$, then

$$p(z) + w(z)zp'(z) \prec h(z)$$

implies

$$p(z) \prec h(z).$$

**Theorem 1** Let $k \geq 0$ and $0 \leq \gamma < p$. Then,

$$\text{US}^*_{p}(\alpha; k; \gamma) \subset \text{US}^*_{p}(\alpha + 1; k; \gamma).$$

*Proof.* Let $f \in \text{US}^*_{p}(\alpha; k; \gamma)$ and set

$$p(z) = \frac{z(I_p^\alpha f(z))'}{I_p^\alpha f(z)} \quad (z \in \mathbb{U}),$$

where the function $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = p$. Using (15), (26) and (27), we have

$$\frac{z(I_p^\alpha f(z))'}{I_p^\alpha f(z)} = p(z) + \frac{zp'(z)}{p(z) + 1} \prec q_{p,k,\gamma}(z).$$

Since $k \geq 0$ and $0 \leq \gamma < p$, we see that

$$\Re\{q_{p,k,\gamma}(z) + 1\} > 0 \quad (z \in \mathbb{U}).$$

Applying Lemma 1 to (28), it follows that $p(z) \prec q_{p,k,\gamma}(z)$, that is, $f \in \text{US}^*_{p}(\alpha + 1; k; \gamma)$. Therefore, we complete the proof of Theorem 1. □

**Theorem 2** Let $k \geq 0$ and $0 \leq \gamma < p$. Then,

$$\text{UC}_{p}(\alpha; k; \gamma) \subset \text{UC}_{p}(\alpha + 1; k; \gamma).$$

*Proof.* Applying (21) and Theorem 1, we observe that

$$f \in \text{UC}_{p}(\alpha; k; \gamma) \iff \frac{zf'}{p} \in \text{US}^*_{p}(\alpha; k; \gamma)$$

$$\iff \frac{zf'}{p} \in \text{US}^*_{p}(\alpha + 1; k; \gamma) \quad \text{(by Theorem 1)},$$

which evidently proves Theorem 2. □

Next, by using Lemma 2, we obtain the following inclusion relation for the class $\text{UK}_{p}(\alpha; k; \gamma, \eta)$.

**Theorem 3** Let $k \geq 0$ and $0 \leq \gamma, \eta < p$. Then,

$$\text{UK}_{p}(\alpha; k; \gamma, \eta) \subset \text{UK}_{p}(\alpha + 1; k; \gamma, \eta).$$
Proof. Let \( f \in UK_p (\alpha; k; \gamma, \eta) \). Then, from the definition of \( UK_p (\alpha; k; \gamma, \eta) \), there exists a function \( r(z) \in US_p^* (k; \eta) \) such that
\[
\frac{z (I_p^{\alpha} f (z))'}{r (z)} < q_{p,k,\gamma} (z).
\] (32)

Choose the function \( g \) such that \( I_p^{\alpha} g (z) = r (z) \). Then, \( g \in US_p^* (\alpha; k; \eta) \) and
\[
\frac{z (I_p^{\alpha} f (z))'}{I_p^{\alpha} g (z)} < q_{p,k,\gamma} (z).
\] (33)

Now let
\[
p(z) = \frac{z (I_p^{\alpha+1} f(z))'}{I_p^{\alpha+1} g(z)} \quad (z \in U),
\] (34)
where \( p(z) \) is analytic in \( U \) with \( p(0) = p \). Since \( g \in US_p^* (\alpha; k; \eta) \), by Theorem 1, we know that \( g \in US_p^* (\alpha + 1; k; \eta) \). Let
\[
t(z) = \frac{z (I_p^{\alpha+1} g(z))'}{I_p^{\alpha+1} g(z)} \quad (z \in U),
\] (35)
where \( t(z) \) is analytic in \( U \) with \( \Re \{ t(z) \} > \frac{k p + \eta}{k + 1} \). Also, from (34), we note that
\[
I_p^{\alpha+1} z f' (z) = I_p^{\alpha+1} g (z) \ p(z).
\] (36)
Differentiating both sides of (36) with respect to \( z \), we obtain
\[
\frac{z (I_p^{\alpha+1} z f' (z))'}{I_p^{\alpha+1} g(z)} = \frac{z (I_p^{\alpha+1} g(z))'}{I_p^{\alpha+1} g(z)} p(z) + z p' (z)
\] \[= t(z) p(z) + z p' (z).
\] (37)

Now using the identity (15) and (35), we obtain
\[
\frac{z (I_p^{\alpha} f (z))'}{I_p^{\alpha} g (z)} = \frac{I_p^{\alpha} z f' (z)}{I_p^{\alpha} g (z)} = \frac{z (I_p^{\alpha+1} z f' (z))'}{z (I_p^{\alpha+1} g (z))'} + \frac{I_p^{\alpha+1} z f' (z)}{I_p^{\alpha+1} g (z)}
\] \[= \frac{z (I_p^{\alpha+1} g(z))'}{I_p^{\alpha+1} g(z)} + 1
\] \[= \frac{t(z) p(z) + z p' (z) + p(z)}{t(z) + 1}
\] \[= p(z) + \frac{z p' (z)}{t(z) + 1}.
\] (38)

Since \( k \geq 0, 0 \leq \eta < p \) and \( \Re \{ t(z) \} > \frac{k p + \eta}{k + 1} \), we see that
\[\Re \{ t(z) + 1 \} > 0 \quad (z \in U).\]
Hence, applying Lemma 2, we can show that \( p(z) \prec q_{p,k,\gamma}(z) \) so that \( f \in UK_p(\alpha; k; \gamma, \eta) \). Therefore, we complete the proof of Theorem 3. \( \square \)

**Theorem 4** Let \( k \geq 0 \) and \( 0 \leq \gamma, \eta < p \). Then,

\[
UK^*_p(\alpha; k; \gamma, \eta) \subset UK^*_p(\alpha + 1; k; \gamma, \eta).
\] (2.18)

**Proof.** Just as we derived Theorem 2 as consequence of Theorem 1 by using the equivalence (21), we can also prove Theorem 4 by using Theorem 3 and the equivalence (??). \( \square \)

### 3. Inclusion Properties Involving the Integral Operator \( F_{c,p} \)

In this section, we present several integral-preserving properties of the \( p \)-valent function classes introduced here. We consider the generalized Libera integral operator \( F_{c,p}(f) \) (see [5] and [4]) defined by

\[
F_{c,p}(f)(z) = \frac{c + p}{z^c} \int_{c}^{t} f(z) \, dt \quad (c > -p).
\] (39)

**Theorem 5** Let \( kp + \gamma + c(k + 1) \geq 0 \). If \( f \in US^*_p(\alpha; k; \gamma) \), then \( F_{c,p}(f) \in US^*_p(\alpha; k; \gamma) \).

**Proof.** Let \( f \in US^*_p(\alpha; k; \gamma) \) and set

\[
p(z) = \frac{z \left(I^\alpha_{p} F_{c,p}(f)(z)\right)'}{I^\alpha_{p} F_{c,p}(f)(z)} \quad (z \in \mathbb{U}),
\] (40)

where \( p(z) \) is analytic in \( \mathbb{U} \) with \( p(0) = p \). From (39), we have

\[
z \left(I^\alpha_{p} F_{c,p}(f)(z)\right)' = (c + p) I^\alpha_{p} f(z) - cI^\alpha_{p} F_{c,p}(f)(z).
\] (41)

Then, by using (40) and (41), we obtain

\[
(c + p) \frac{I^\alpha_{p} f(z)}{I^\alpha_{p} F_{c,p}(f)(z)} = p(z) + c.
\] (42)

Taking the logarithmic differentiation on both sides of (42) and multiplying by \( z \), we have

\[
z \left(I^\alpha_{p} f(z)\right)' = p(z) + \frac{zp'(z)}{p(z) + c} < q_{k,\gamma}(z).
\] (43)

Hence, by virtue of Lemma 1, we conclude that \( p(z) \prec q_{k,\gamma}(z) \) in \( \mathbb{U} \), which implies that \( F_{c,p}(f) \in US^*_p(\alpha; k; \gamma) \). \( \square \)

Next, we derive an inclusion property involving \( F_{c,p}(f) \), which is given by the following.

**Theorem 6** Let \( kp + \gamma + c(k + 1) \geq 0 \). If \( f \in UC_p(\alpha; k; \gamma) \), then \( F_{c,p}(f) \in UC_p(\alpha; k; \gamma) \).
Proof. By applying Theorem 5, it follows that

\[ f \in UC_p (\alpha; k; \gamma) \iff \frac{zf'}{p} \in US_p^* (\alpha; k; \gamma) \]

\[ \Rightarrow F_{c,p} \left( \frac{zf'}{p} \right) \in US_p^* (\alpha; k; \gamma) \]

\[ \iff \frac{z(F_{c,p}(f))}{p} \in US_p^* (\alpha; k; \gamma) \]

\[ \iff F_{c,p}(f) \in UC_p (\alpha; k; \gamma), \]

which proves Theorem 6. \qed

**Theorem 7** Let \( kp + \eta + c(k + 1) \geq 0 \). If \( f \in UK_p (\alpha; k; \gamma, \eta) \), then \( F_{c,p}(f) \in UK_p (\alpha; k; \gamma, \eta) \).

Proof. Let \( f \in UK_p (\alpha; k; \gamma, \eta) \). Then, in view of the definition of the class \( UK_p (\alpha; k; \gamma, \eta) \), there exists a function \( g \in US_p^* (\alpha; k; \gamma) \) such that

\[ z (I_p f (z))' \frac{1}{I_p g (z)} < q_{k,\gamma} (z). \tag{44} \]

Thus, we set

\[ p (z) = \frac{z (I_p F_{c,p} (f) (z))'}{I_p F_{c,p} (g) (z)} (z \in U), \tag{45} \]

where \( p (z) \) is analytic in \( U \) with \( p(0) = p \). Since \( g \in US_p^* (\alpha; k; \gamma) \), we see from Theorem 5 that \( F_{c,p}(f) \in US_p^* (\alpha; k; \gamma) \). Let

\[ t (z) = \frac{z (I_p F_{c,p} (g) (z))'}{I_p F_{c,p} (g) (z)} (z \in U), \tag{46} \]

where \( t (z) \) is analytic in \( U \) with \( \Re \{ t (z) \} > \frac{k p + \eta}{k + 1} \). Also, from (45), we note that

\[ I_p^\alpha z F_{c,p}' (f) (z) = I_p^\alpha F_{c,p} (g) (z) \cdot p (z). \tag{47} \]

Differentiating both sides of (47) with respect to \( z \), we obtain

\[ \frac{z (I_p^\alpha z F_{c,p}' (f) (z))}{I_p^\alpha F_{c,p} (g) (z)} = \frac{z (I_p^\alpha F_{c,p} (g) (z))'}{I_p^\alpha F_{c,p} (g) (z)} p (z) + z p' (z) \]

\[ = t (z) p (z) + z p' (z). \tag{48} \]
Now using the identity (41) and (48), we obtain
\[
\frac{z \left( I_p^a f(z) \right)'}{I_p^a g(z)} = \frac{z \left( I_p^a z F_{c,p}'(f)(z) \right)'}{z \left( I_p^a F_{c,p}(g)(z) \right)} + c \frac{z \left( I_p^a F_{c,p}(f)(z) \right)'}{z \left( I_p^a F_{c,p}(g)(z) \right)} + c
\]
\[
= \frac{z \left( I_p^a z F_{c,p}'(g)(z) \right)'}{z \left( I_p^a F_{c,p}(g)(z) \right)} + c
\]
\[
= \frac{t(z) p(z) + z p'(z) + cp(z)}{t(z) + c}
\]
\[
= p(z) + \frac{z p'(z)}{t(z) + c}.
\]
Since \( k p + \eta + c (k + 1) \geq 0 \) and \( \Re \{ t(z) \} > \frac{k p + \eta}{k + 1} \), we see that
\[
\Re \{ t(z) + c \} > 0 \quad (z \in \mathbb{U}).
\]
Hence, applying Lemma 2 to (49), we can show that \( p(z) \prec q_{p,k,\gamma}(z) \) so that \( f \in UK_p(\alpha; k; \gamma, \eta) \).

**Theorem 8** Let \( k p + \eta + c (k + 1) \geq 0 \). If \( f \in UK_p^*(\alpha; k; \gamma, \eta) \), then \( F_{c,p}(f) \in UK_p^*(\alpha; k; \gamma, \eta) \).

**Proof.** Just as we derived Theorem 6 as consequence of Theorem 5, we easily deduce the integral-preserving property asserted by Theorem 8 by using Theorem 7.

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**References**


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