CONVOLUTION PROPERTIES FOR SUBCLASSES OF UNIVALENT FUNCTIONS USING SALAGEAN INTEGRAL OPERATOR

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Abstract. Making use of the Salagean integral operator $I^n$, we defined subclasses of univalent functions and investigated some convolution properties for these subclasses.

1. Introduction

Let $A$ denote the class of analytic functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$  \hspace{1cm} (1.1)

which are analytic in the open unit disc $\mathbb{U} = \{ z \in \mathbb{C} : |z| < 1 \}$, and $S$ is the subclass of $A$ which are univalent.

Let $\Omega$ be the class of functions $w$ analytic in $\mathbb{U}$, satisfying $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \mathbb{U}$.

If $f(z)$ and $g(z)$ are analytic in $\mathbb{U}$, we say that $f(z)$ is subordinate to $g(z)$, written $f(z) \prec g(z)$ if there exists a Schwarz function $w \in \Omega$, such that $f(z) = g(w(z))$, $z \in \mathbb{U}$. Furthermore, if the function $g(z)$ is univalent in $\mathbb{U}$, then we have the following equivalence, (cf., e.g., [8]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

For functions $f(z)$ given by (1.1) and $g(z)$ given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k,$$  \hspace{1cm} (1.2)

the Hadamard product or convolution of $f(z)$ and $g(z)$ is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z).$$  \hspace{1cm} (1.3)

2000 Mathematics Subject Classification. 30C45.

Key words and phrases. Analytic functions, Salagean integral operator, convolution.

For $f(z) \in A$, Salagean [11] introduced the following differential operator:

$$D^0f(z) = f(z), \quad D^1f(z) = zf'(z), \ldots, \quad D^nf(z) = D(D^{n-1}f(z)) \quad (n \in \mathbb{N} = \{1, 2, \ldots\}).$$

We note that

$$D^nf(z) = z + \sum_{k=2}^{\infty} k^na_kz^k = (h_n * f)(z) \quad (f \in A; n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}), \quad (1.4)$$

where

$$h_n(z) = z + \sum_{k=2}^{\infty} k^na_kz^k \quad (n \in \mathbb{N}_0, z \in U). \quad (1.5)$$

Also, Salagean [11] introduced the following integral operator:

$$I^0f(z) = f(z), \quad I^1f(z) = \int_0^zf(t)\frac{dt}{t}, \ldots, \quad I^nf(z) = I(I^{n-1}f(z)) \quad (n \in \mathbb{N}).$$

We note that

$$I^nf(z) = z + \sum_{k=2}^{\infty} k^{-a}n_kz^k = (\lambda_n * f)(z) \quad (n \in \mathbb{N}_0), \quad (1.6)$$

where

$$\lambda_n(z) = z + \sum_{k=2}^{\infty} k^{-a}n_kz^k \quad (n \in \mathbb{N}_0, z \in U). \quad (1.7)$$

We note that

(i) $I^{-n}f(z) = D^nf(z) \quad (n \in \mathbb{N}_0)$ (see [11]) and $I^{-1}f(z) = Df(z)$;

(ii) $(h_n * \lambda_n)(z) * f(z) = f(z) \quad (n \in \mathbb{N}_0)$;

(iii) $z(I^{n+1}f(z)) = I^n f(z) \quad (n \in \mathbb{N}_0)$.

With the help of the Salagean integral operator $I^n$, we say that a function $f \in A$ is in the class $S^n(A, B)$ ($-1 \leq B < A \leq 1$) if it satisfying the subordination condition:

$$\frac{I^n f(z)}{I^{n+1} f(z)} \prec \frac{1 + Az}{1 + Bz} \quad (n \in \mathbb{N}_0). \quad (1.8)$$

Let $C^n(A, B)$ denote the class of the functions $f \in A$ satisfying $zf'(z) \in S^n(A, B)$. We note that $S^{-1}(A, B) = S^+(A, B)$ and $C^{-1}(A, B) = C(A, B)$ (see [4], [6], [7] and [12]).

Denote by $S^n_\alpha(A, B)$ the class of functions $f \in A$ satisfying the subordination condition:

$$\frac{1}{\cos \lambda} \left\{ e^{i\lambda} \frac{I^n f(z)}{I^{n+1} f(z)} - i \sin \lambda \right\} \prec \frac{1 + Az}{1 + Bz} \quad (|\lambda| \leq \frac{\pi}{2}, n \in \mathbb{N}_0), \quad (1.9)$$

and let $C^n_\alpha(A, B)$ be the class of functions $f \in A$ satisfying $zf'(z) \in S^n_\alpha(A, B)$. We note that $S^{-1}_\alpha(A, B) = S^+(A, B)$ (see Nikitin [9] and Aouf [1] with $\alpha = 0$) and $C^{-1}_\alpha(A, B) = C(A, B)$ (see Bhoosurmath and Devadas [2]).

Further, let $M^n(A, B)$ be the class of functions $f \in A$ satisfying the subordination condition:

$$\frac{I^n f(z)}{z} \prec \frac{1 + Az}{1 + Bz} \quad (n \in \mathbb{N}_0), \quad (1.10)$$
and $M^\sigma_r(A, B)$ ($\sigma \geq 0$) be the class of functions $f \in A$ satisfying the subordination condition:

$$
(1 - \sigma) \frac{I^n f(z)}{z} + \sigma \frac{I^{n-1} f(z)}{z} < \frac{1 + Az}{1 + Bz} \quad (n \in \mathbb{N}_0). 
$$

(1.11)

Evidently, $M^\sigma_r(A, B) = M(A, B)$ (see Goel and Mehrok [5]).

Also, we note that

(i) $M^0_r(1 - 2\beta, -1) = M^\sigma_r(\beta)(0 \leq \beta < 1)$ the class of functions $f \in A$ satisfying the condition:

$$
Re \left\{ (1 - \sigma) \frac{I^n f(z)}{z} + \sigma \frac{I^{n-1} f(z)}{z} \right\} > \beta; 
$$

(ii) $M^0_r(1 - 2\beta, -1) = M_s(\beta)(0 \leq \beta < 1)$ the class of functions $f \in A$ satisfying the condition:

$$
Re \left\{ (1 - \sigma) \frac{f(z)}{z} + \sigma f'(z) \right\} > \beta.
$$

Convolution properties for various subclasses of analytic functions have been obtained by several researchers (see [2], [3], [10], [12], [13]). In this paper, we investigate convolution properties of the classes $S^n(A, B)$, $C^n(A, B)$, $S^n_\lambda(A, B)$, $C^n_\lambda(A, B)$, $M^n(A, B)$ and $M^\sigma_r(A, B)$, respectively, associated with the Salagean integral operator.

2. Main Results

Unless otherwise mentioned, we assume throughout this section that $0 \leq \theta < 2\pi$, $n \in \mathbb{N}_0$, $\sigma \geq 0$, $-1 \leq B < A \leq 1$ and $\lambda_n(z)$ given by (1.7).

**Theorem 1.** The function $f(z)$ defined by (1.1) is in the class $S^n(A, B)$ if and only if

$$
\frac{1}{z} \left[ (f * \lambda_{n+1})(z) \ast \frac{z + Cz^2}{(1 - z)^2} \right] \neq 0 \quad (z \in \mathbb{U}) 
$$

(2.1)

for all $C = C_0 = \frac{e^{-i\theta} + A}{(B - A)}$, $\theta \in [0, 2\pi)$, and also for $C = -1$.

**Proof.** First suppose $f(z)$ defined by (1.1) is in the class $S^n(A, B)$, we have

$$
\frac{I^n f(z)}{I^{n+1} f(z)} < \frac{1 + Az}{1 + Bz}.
$$

(2.2)

since the function from the left-hand side of the subordination is analytic in $\mathbb{U}$, it follows $I^{n+1} f(z) \neq 0, z \in \mathbb{U}^* = U \setminus \{0\}$, i.e., $\frac{1}{z} I^{n+1} f(z) \neq 0, z \in \mathbb{U}$, this is equivalent to the fact that (2.1) holds for $C = -1$.

From (2.2) according to the subordination of two functions we say that there exists a function $w(z) \in \Omega$, such that

$$
\frac{I^n f(z)}{I^{n+1} f(z)} = \frac{1 + Aw(z)}{1 + Bw(z)} \quad (z \in \mathbb{U}),
$$

which is equivalent to

$$
\frac{I^n f(z)}{I^{n+1} f(z)} \neq \frac{1 + Ae^{i\theta}}{1 + Be^{i\theta}} \quad (z \in \mathbb{U}; 0 \leq \theta < 2\pi),
$$
or
\[
\frac{1}{z} \{ f_1^1 f(z)(1 + Be^{i\theta}) - f_1^{n+1} f(z)(1 + Ae^{i\theta}) \} \neq 0. \tag{2.3}
\]
Since
\[
I^{n+1} f(z) * \frac{z}{(1 - z)} = I^{n+1} f(z) \tag{2.4}
\]
and
\[
I^{n+1} f(z) * \left[ \frac{z}{(1 - z)^2} \right] = I^n f(z) \tag{2.5}
\]
Now from (2.3), (2.4) and (2.5), we obtain
\[
\frac{1}{z} \left[ (f * \lambda_{n+1})(z) * \frac{z + Cz^2}{(1 - z)^2} \right] \neq 0 \quad (z \in \mathbb{U}; 0 \leq \theta < 2\pi),
\]
which leads to (2.1), which proves the necessary part of Theorem 1.

(ii) Reversely, because the assumption (2.1) holds for 
\[
C = \text{e}^{i\theta} = \{ 0 \leq \theta < 2\pi \}
\]
in Theorem 1, it follows that
\[
\frac{1}{z} I^{n+1} f(z) \neq 0 \quad \text{for all} \quad z \in \mathbb{U},
\]
and hence the function \( \varphi(z) = \frac{I^n f(z)}{I^{n+1} f(z)} \) is analytic in \( \mathbb{U} \) (i.e. it is regular at \( z_0 = 0 \), with \( \varphi(0) = 1 \)).

Since it was shown in the first part of the proof that the assumption (2.1) is equivalent to (2.3), we obtain that
\[
\frac{I^n f(z)}{I^{n+1} f(z)} \neq \frac{1 + A e^{i\theta}}{1 + B e^{i\theta}} \quad (z \in \mathbb{U}; \theta \in [0, 2\pi)), \tag{2.6}
\]
if we denote
\[
\psi(z) = \frac{1 + A z}{1 + B z},
\]
the relation (2.6) shows that \( \varphi(\mathbb{U}) \cap \psi(\partial \mathbb{U}) = \emptyset \). Thus, the simply-connected domain \( \varphi(\mathbb{U}) \) is included in a connected component of \( C \setminus \psi(\partial \mathbb{U}) \). From here, using the fact that \( \varphi(0) = \psi(0) \) together with the univalence of the function \( \psi \), it follows that \( \varphi(z) < \psi(z) \), which represents in fact the subordination (2.2), i.e. \( f \in S^n (A, B) \).

**Theorem 2.** The function \( f(z) \) defined by (1.1) is in the class \( C^n (A, B) \) if and only if
\[
\frac{1}{z} \left[ (f * \lambda_{n+1})(z) * \frac{z + (1 + 2C)z^2}{(1 - z)^3} \right] \neq 0 \quad (z \in \mathbb{U}) \tag{2.7}
\]
for all \( C = C_\theta = \frac{\text{e}^{-i\theta} + A}{B - A}, \theta \in [0, 2\pi) \), and also for \( C = -1 \).

**Proof.** Set
\[
g(z) = \frac{z + Cz^2}{(1 - z)^2}
\]
and we note that
\[
z g'(z) = \frac{z + (1 + 2C)z^2}{(1 - z)^3}. \tag{2.8}
\]
From the identity \( z f'(z) * g(z) = f(z) * z g'(z) \quad (f, g \in A) \) and the fact that
\[
f(z) \in C^n (A, B) \iff z f'(z) \in S^n (A, B).
\]
The result follows from Theorem 1.

**Remark 1.** (i) Putting \( n = -1 \) and \( e^{i\theta} = z \quad (0 \leq \theta < 2\pi) \) in Theorem 1, we obtain the result obtained by Padmanabhan and Ganesan [10, Theorem 2].
(ii) Putting \( n = -1, A = 1 - 2\alpha \) \((0 \leq \alpha < 1)\), \( B = -1 \) and \( e^{-i\theta} = -\pi (0 \leq \theta < 2\pi) \) in Theorem 1, we obtain the result obtained by Silverman et al. [13, Theorem 2];

(iii) Putting \( n = -1 \) and \( e^{i\theta} = \pi (0 \leq \theta < 2\pi) \) in Theorem 2, we obtain the result obtained by Padmanabhan and Ganesan [10, Theorem 1];

(iv) Putting \( n = -1, A = 1 - 2\alpha \) \((0 \leq \alpha < 1)\), \( B = -1 \) and \( e^{-i\theta} = -\pi (0 \leq \theta < 2\pi) \) in Theorem 2, we obtain the result obtained by Silverman et al. [13, Theorem 1].

**Theorem 3.** The function \( f(z) \) defined by (1.1) is in the class \( S^n_\lambda(A, B) \) if and only if

\[
\frac{1}{z} \left[ (f * \lambda_{n+1})(z) * \frac{z + Ez^2}{(1 - z)^2} \right] \neq 0 \quad (z \in U),
\]

(2.9)

for all \( E = E_\theta = \frac{e^{-i\theta} + e^{-i\lambda}(A \cos \lambda + iB \sin \lambda)}{(B - e^{-i\lambda}(A \cos \lambda + iB \sin \lambda))}, \theta \in [0, 2\pi) \), and also for \( E = -1 \).

**Proof.** First suppose \( f(z) \) defined by (1.1) is in the class \( S^n_\lambda(A, B) \), we have

\[
\frac{1}{\cos \lambda} \left\{ e^{i\lambda} \frac{I^n f(z)}{I^{n+1} f(z)} - i \sin \lambda \right\} = \frac{1 + Az}{1 + Bz} \quad (|\lambda| < \frac{\pi}{2}; n \in \mathbb{N}_0),
\]

(2.10)

since the function from the left-hand side of the subordination is analytic in \( \mathbb{U} \), it follows \( I^{n+1} f(z) \neq 0, z \in \mathbb{U} \setminus \{0\} \), i.e., \( \frac{1}{z} I^{n+1} f(z) \neq 0, z \in \mathbb{U} \), this is equivalent to the fact that (2.9) holds for \( E = -1 \).

From (2.10) according to the subordination of two functions we say that there exists a function \( w(z) \in \Omega \), such that

\[
\frac{1}{\cos \lambda} \left\{ e^{i\lambda} \frac{I^n f(z)}{I^{n+1} f(z)} - i \sin \lambda \right\} = \frac{1 + Aw(z)}{1 + Bw(z)} \quad (z \in \mathbb{U}),
\]

which is equivalent to

\[
\frac{1}{\cos \lambda} \left\{ e^{i\lambda} \frac{I^n f(z)}{I^{n+1} f(z)} - i \sin \lambda \right\} \neq \frac{1 + Ae^{i\theta}}{1 + Be^{i\theta}} \quad (z \in \mathbb{U}; 0 \leq \theta < 2\pi),
\]

or

\[
\frac{1}{z} (e^{i\lambda} I^n f(z)(1 + Be^{i\theta}) - I^{n+1} f(z)(1 + Ae^{i\theta})) \neq 0. \quad (2.11)
\]

By simplifying (2.11), we obtain (2.9). This completes the proof of Theorem 3.

**Theorem 4.** The function \( f(z) \) defined by (1.1) is in the class \( C^n_\lambda(A, B) \) if and only if

\[
\frac{1}{z} \left[ (f * \lambda_{n+1})(z) * \frac{z + (1 + 2E)z^2}{(1 - z)^3} \right] \neq 0 \quad (z \in U)
\]

(2.12)

for all \( E = E_\theta = \frac{e^{-i\theta} + e^{-i\lambda}(A \cos \lambda + iB \sin \lambda)}{(B - e^{-i\lambda}(A \cos \lambda + iB \sin \lambda))}, \theta \in [0, 2\pi) \), and also for \( E = -1 \).

**Proof.** Set

\[
g(z) = \frac{z + Ez^2}{(1 - z)^2},
\]

and we note that

\[
zg'(z) = \frac{z + (1 + 2E)z^2}{(1 - z)^3}.
\]

From the identity \( zf(z) * g(z) = f(z) * zg'(z) \) \((f, g \in \mathcal{A})\) and the fact that

\[
f(z) \in Q^n_\lambda(A, B) \iff zf'(z) \in S^n_\lambda(A, B).
\]
The result follows from Theorem 3.

**Remark 2.** (i) Putting \(n = -1\) and \(e^{i\theta} = \kappa(0 \leq \theta < 2\pi)\) in Theorem 3, we obtain the result obtained by Padmanabhan and Ganesan [10, Theorem 4];

(ii) Putting \(n = -1, A = 1 - 2\alpha(0 \leq \alpha < 1), B = -1\) and \(e^{-i\theta} = -\kappa(0 \leq \theta < 2\pi)\) in Theorem 3, we obtain the result obtained by Silverman et al. [13, Theorem 4];

(iii) Putting \(n = -1\) and \(e^{i\theta} = \kappa(0 \leq \theta < 2\pi)\) in Theorem 4, we obtain the result obtained by Padmanabhan and Ganesan [10, Theorem 3];

(iv) Putting \(n = -1, A = 1 - 2\alpha(0 \leq \alpha < 1), B = -1\) and \(e^{-i\theta} = -\kappa(0 \leq \theta < 2\pi)\) in Theorem 4, we obtain the result obtained by Silverman et al. [13, Theorem 3].

**Theorem 5.** The function \(f(z)\) defined by (1.1) is in the class \(M^n(A, B)\) if and only if

\[
\frac{1}{z} \left[ \left( f \ast \lambda_{n+1} \right)(z) + \frac{z + C(2z^2 - z^3)}{(1 - z)^2} \right] \neq 0 \quad (z \in \mathbb{U}),
\]

(2.13)

for all \(C = C_0 = \frac{e^{-i\theta} + A}{B - A}, \theta \in [0, 2\pi]\), and also for \(C = -1\).

**Proof.** First suppose \(f(z)\) defined by (1.1) is in the class \(M^n(A, B)\), we have

\[
\frac{I^n f(z)}{z} = \frac{1 + Az}{1 + Bz}
\]

(2.14)

From (2.14) according to the subordination of two functions we say that there exists a function \(w(z) \in \Omega\), such that

\[
\frac{I^n f(z)}{z} = \frac{1 + Aw(z)}{1 + Bw(z)} \quad (z \in \mathbb{U}),
\]

which is equivalent to

\[
\frac{I^n f(z)}{z} \neq \frac{1 + Ae^{i\theta}}{1 + Be^{i\theta}} \quad (z \in \mathbb{U}; 0 \leq \theta < 2\pi),
\]

or

\[
\frac{1}{z} \left( I^n f(z)(1 + Be^{i\theta}) - z(1 + Ae^{i\theta}) \right) \neq 0.
\]

Since

\[
\frac{1}{z} I^{n+1} f(z) * \left\{ (1 + Be^{i\theta}) \frac{z}{(1 - z)^2} - z(1 + Ae^{i\theta}) \frac{(1 - z)^2}{(1 - z)^2} \right\} \neq 0
\]

then

\[
\frac{1}{z} \left[ I^{n+1} f(z) * \frac{z + C(2z^2 - z^3)}{(1 - z)^2} \right] \neq 0 \quad (z \in \mathbb{U}; 0 \leq \theta < 2\pi),
\]

which proves Theorem 5.

**Theorem 6.** The function \(f(z)\) defined by (1.1) is in the class \(M^n_{\sigma}(A, B)\) if and only if

\[
\frac{1}{z} \left[ \left( f \ast \lambda_{n+1} \right)(z) + \frac{z(1 - (1 - 2\sigma)z)(1 + Be^{i\theta}) - z(1 - z)^3(1 + Ae^{i\theta})}{(1 - z)^3} \right] \neq 0 \quad (z \in \mathbb{U}).
\]

(2.15)

**Proof.** First suppose \(f(z)\) defined by (1.1) is in the class \(M^n_{\sigma}(A, B)\), we have

\[
(1 - \sigma) \frac{I^n f(z)}{z} + \sigma \frac{I^{n-1} f(z)}{z} \leq \frac{1 + Az}{1 + Bz} \quad (\sigma \geq 0; n \in \mathbb{N}_0).
\]

(2.16)
From (2.16) according to the subordination of two functions we say that there exists a function \( w(z) \in \Omega \), such that

\[
(1 - \sigma) \frac{I^n f(z)}{z} + \sigma \frac{I^{n-1} f(z)}{z} = \frac{1 + Aw(z)}{1 + Bw(z)} \quad (z \in \mathbb{U})
\]

which is equivalent to

\[
(1 - \sigma) \frac{I^n f(z)}{z} + \sigma \frac{I^{n-1} f(z)}{z} \neq \frac{1 + Ae^{i\theta}}{1 + Be^{i\theta}} \quad (z \in \mathbb{U}; 0 \leq \theta < 2\pi),
\]

or

\[
\frac{1}{z} \left\{ [(1 - \sigma)I^n f(z) + \sigma I^{n-1} f(z)f(z)](1 + Be^{i\theta}) - z(1 + Ae^{i\theta}) \right\} \neq 0.
\]

Since

\[
\frac{1}{z} \left( I^{n+1} f(z) \ast \left\{ (1 + Be^{i\theta}) \left[ \frac{(1 - \sigma)z}{(1 - z)^2} + \frac{\sigma z(1 + z)}{(1 - z)^3} \right] - z(1 + Ae^{i\theta}) \frac{(1 - z)^3}{(1 - z)^3} \right\} \right) \neq 0
\]

which proves Theorem 6.

Acknowledgments
The author thanks the referees for their comments and suggestions.

References

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