COMPARATIVE GROWTH PROPERTIES OF ANALYTIC FUNCTION OF SEVERAL COMPLEX VARIABLES IN THE UNIT POLYDISC

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Abstract. In this paper, we introduced the idea of relative Nevanlinna $L^*$-order and relative Nevanlinna $L^*$-lower order of an analytic function of several complex variables with respect to an entire function of several complex variables in the unit polydisc. Also we study some growth properties of composition of two analytic functions of several complex variables in the unit polydisc on the basis of their relative Nevanlinna $L^*$-order and relative Nevanlinna $L^*$-lower order.

1. Introduction

A function $f$, analytic in the unit disc $U = \{z : |z| < 1\}$, is said to be of finite Nevanlinna order $[2]$, if there exist a number $\mu$ such that the Nevanlinna characteristic function

$$T(r, f) = \frac{1}{2\pi} \int_{0}^{2\pi} \log^+ |f(re^{i\theta})| d\theta$$

satisfies, $T(r, f) < (1 - r)^{-\mu}$ for all $r$ in $0 < r_0 < r < 1$.

The greatest lower bound of all such number $\mu$, is called Nevanlinna order of $f$, Thus the Nevanlinna order $\rho_f$ of $f$ is given by

$$\rho_f = \limsup_{r \to 1} \frac{\log T(r, f)}{-\log(1 - r)}.$$ 

Similarly, Nevanlinna lower order $\lambda_f$ of $f$, is given by

$$\lambda_f = \liminf_{r \to 1} \frac{\log T(r, f)}{-\log(1 - r)}.$$ 

In sec.([3]) introduced the notion of Nevanlinna $L$-order for an analytic function $f$ in the unit disc $U = \{z : |z| < 1\}$, where $L = L(\frac{1}{1-r})$ is a positive continuous
function in the unit disc $U$ increasing slowly i.e $L\left(\frac{a}{1-r}\right) \sim L\left(\frac{1}{1-r}\right)$ as $r \to 1$ for every positive constant $'a'$ in the following manner.

**Definition 1** If $f$ be analytic in $U$, then the Nevanlinna $L$ - order $\rho_f^L$ and the Nevanlinna $L$ - lower order $\lambda_f^L$ of $f$ are defined as

$$\rho_f^L = \limsup_{r \to 1} \frac{\log T(r, f)}{\log \left(\frac{L\left(\frac{1}{1-r}\right)}{1-r}\right)},$$

and

$$\lambda_f^L = \liminf_{r \to 1} \frac{\log T(r, f)}{\log \left(\frac{L\left(\frac{1}{1-r}\right)}{1-r}\right)}.$$  

Now we introduce the concepts of relative Nevanlinna $L^*$ - order and relative Nevanlinna $L^*$ - lower order of an analytic function $g$ in the unit disc $U$ which are as follows.

**Definition 2** If $f$ be analytic in $U$ and $g$ be entire then the relative Nevanlinna $L^*$ - order of $f$ with respect to $g$, denoted by $\rho_g^L(f)$ is defined by

$$\rho_g^L(f) = \inf \{\mu > 0 : T_f(r) < T_g\left(\frac{\exp\{L\left(\frac{1}{1-r}\right)\}}{1-r}\right)^\mu \text{ for all } 0 < r_0(\mu) < r < 1\}.$$  

Similarly, relative Nevanlinna $L^*$ - lower order $f$ with respect to $g$ denoted by $\lambda_g^L(f)$ is given by

$$\lambda_g^L(f) = \liminf_{r \to 1} \frac{\log T_g^{-1} T_f(r)}{\log \left(\frac{\exp\{L\left(\frac{1}{1-r}\right)\}}{1-r}\right)}.$$  

When $g(z) = \exp(z)$, the definition coincides with the definition of the Nevanlinna $L^*$ - order and the Nevanlinna $L^*$ - lower order.

Now we are extending the notion of single variable to several complex variables in ([3], [4]), let $f$ be an non - constant analytic function of several variables $z_1, z_2, \ldots, z_n$ in the unit polydisc,

$U = \{(z_1, z_2, \ldots, z_n) : |z| \leq 1, i = 1, 2, \ldots, n\}$.

**Definition 3** Let $T_f(r_1, r_2, \ldots, r_n)$ denote the Nevanlinna's characteristic function of several complex variables and $g$ be an other entire function of several complex variables. Then the relative Nevanlinna $L^*$ - order of $f$ with respect to $g$ , denote by $\rho_g^L(f)$ is defined by

$$\rho_g^L(f) = \limsup_{r_1, r_2, \ldots, r_n \to 1} \frac{\log T_g^{-1} T_f(r_1, r_2, \ldots, r_n)}{\log \left(\frac{\exp\{L\left(\frac{1}{1-r_1(1-r_2)\ldots(1-r_n)}\right)\}}{1-r_1(1-r_2)\ldots(1-r_n)}\right)}.$$  

Similarly , the relative Nevanlinna $L^*$ - lower order of $f$ with respect to $g$, denoted by $\lambda_g^L(f)$ is given by

$$\lambda_g^L(f) = \liminf_{r_1, r_2, \ldots, r_n \to 1} \frac{\log T_g^{-1} T_f(r_1, r_2, \ldots, r_n)}{\log \left(\frac{\exp\{L\left(\frac{1}{1-r_1(1-r_2)\ldots(1-r_n)}\right)\}}{1-r_1(1-r_2)\ldots(1-r_n)}\right)}$$.
where \( L \equiv L(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n}) \) is a positive continuous function in the unit polydisc \( U \) increasing slowly i.e., \( L(\frac{a}{1-r_1}, \frac{a}{1-r_2}, \ldots, \frac{a}{1-r_n}) \sim L(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n}) \) as \( r_1, r_2, \ldots, r_n \to 1 \), for every positive constant \( 'a' \). When \( g(z_1, z_2, \ldots, z_n) = \exp(z_1, z_2, \ldots, z_n) \), then the definition coincides with the definition of the Nevanlinna \( L^* \)-order and the Nevanlinna \( L^* \)-lower order.

In this paper, we study some growth properties of composition of two analytic functions of several complex variables in the unit polydisc \( U \equiv \{(z_1, z_2, \ldots, z_n) : |z_i| < 1, i = 1, 2, \ldots, n\} \), on the basis of their relative Nevanlinna \( L^* \)-order (relative Nevanlinna \( L^* \)-lower order). We do not explain the standard definitions and notations in the theory of entire function of several complex variables as those are available in [1], [6] and [7].

In this section we present the main results of this paper.

**Theorem 1** Let \( f, g \) be any two analytic functions in \( U \) and \( h \) be any entire function of several complex variables such that \( 0 < \lambda_h^{L^*}(f \circ g) \leq \rho_h^{L^*}(f \circ g) < \infty \) and \( \lambda_h^{\ast}(f) \leq \rho_h^{\ast}(f) < \infty \). If \( \lambda_h^{L^*}(f) \leq \rho_h^{L^*}(f) < \infty \) and \( L(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n}) = 0(\log T_h^{-1}T_f(r_1, r_2, \ldots, r_n)) \) as \( r_1, r_2, \ldots, r_n \to 1 \), then

\[
\frac{\lambda_h^{L^*}(f \circ g)}{\rho_h^{L^*}(f)} \leq \liminf_{r_1, r_2, \ldots, r_n \to 1} \frac{\log T_h^{-1}T_{f \circ g}(r_1, r_2, \ldots, r_n)}{\log T_h^{-1}T_f(r_1, r_2, \ldots, r_n) + L(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n})} \leq \frac{\rho_h^{L^*}(f \circ g)}{\lambda_h^{L^*}(f)}.
\]

**Proof.** From the definition of relative Nevanlinna \( L^* \)-order and relative Nevanlinna \( L^* \)-lower order of an analytic function of several complex variables in the unit polydisc \( U \), we have for arbitrary positive \( \epsilon \) and for all sufficiently large values of \( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \) that

\[
\log T_h^{-1}T_{f \circ g}(r_1, r_2, \ldots, r_n) \geq (\lambda_h^{L^*}(f \circ g) - \epsilon)\log \left( \frac{\exp[L(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n})]}{1 - r_1}(1 - r_2)\ldots(1 - r_n) \right),
\]

i.e.,

\[
\frac{\log T_h^{-1}T_{f \circ g}(r_1, r_2, \ldots, r_n)}{(\lambda_h^{L^*}(f \circ g) - \epsilon)} \geq \log \left( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \right)
+ L\left( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \right), (1)
\]

and

\[
\log T_h^{-1}T_f(r_1, r_2, \ldots, r_n) \leq (\rho_h^{L^*}(f) + \epsilon)\log \left( \frac{\exp[L(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n})]}{1 - r_1}(1 - r_2)\ldots(1 - r_n) \right),
\]
As for a sequence of values of \( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \), i.e.,

\[
\log T_h^{-1} T_f(r_1, r_2, \ldots, r_n) \leq (\rho^*_h(f) + \epsilon) \log \left( \frac{1}{1-r_1} \frac{1}{1-r_2} \ldots \frac{1}{1-r_n} \right) + L \left( \frac{1}{1-r_1} \frac{1}{1-r_2} \ldots \frac{1}{1-r_n} \right).\]

Now from equation (1) and (2), it follows for all sufficiently large values of \( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \) that,

\[
\log T_h^{-1} T_{fog}(r_1, r_2, \ldots, r_n) \geq \frac{(\lambda_h^*(f \circ g) - \epsilon)}{(\rho^*_h(f) + \epsilon)} \log T_h^{-1} T_f(r_1, r_2, \ldots, r_n),
\]

i.e.

\[
\frac{\log T_h^{-1} T_{fog}(r_1, r_2, \ldots, r_n)}{\log T_h^{-1} T_f(r_1, r_2, \ldots, r_n) + L \left( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \right)} \geq \frac{(\lambda_h^*(f \circ g) - \epsilon)}{(\rho^*_h(f) + \epsilon)} \times \frac{1}{1 + \frac{L \left( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \right)}{\log T_h^{-1} T_f(r_1, r_2, \ldots, r_n)}},
\]

i.e.,

\[
\frac{\log T_h^{-1} T_{fog}(r_1, r_2, \ldots, r_n)}{\log T_h^{-1} T_f(r_1, r_2, \ldots, r_n) + L \left( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \right)} \geq \frac{(\lambda_h^*(f \circ g) - \epsilon)}{(\rho^*_h(f) + \epsilon)} \times \frac{1}{1 + \frac{L \left( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \right)}{\log T_h^{-1} T_f(r_1, r_2, \ldots, r_n)}},
\]

Since \( L \left( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \right) = 0 \{ \log T_h^{-1} T_f(r_1, r_2, \ldots, r_n) \} \) as \( r_1, r_2, \ldots, r_n \to 1 \), it follows from equation (3) that

\[
\liminf_{r_1, r_2, \ldots, r_n \to 1} \frac{\log T_h^{-1} T_{fog}(r_1, r_2, \ldots, r_n)}{\log T_h^{-1} T_f(r_1, r_2, \ldots, r_n) + L \left( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \right)} \geq \frac{\lambda_h^*(f \circ g) - \epsilon}{\rho^*_h(f) + \epsilon}.
\]

As \( \epsilon > 0 \) is arbitrary we get from equation (4) that from which we have

\[
\liminf_{r_1, r_2, \ldots, r_n \to 1} \frac{\log T_h^{-1} T_{fog}(r_1, r_2, \ldots, r_n)}{\log T_h^{-1} T_f(r_1, r_2, \ldots, r_n) + L \left( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \right)} \geq \frac{\lambda_h^*(f \circ g)}{\rho^*_h(f)}.
\]

Again for a sequence of values of \( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \) tending to infinity

\[
\log T_h^{-1} T_{fog}(r_1, r_2, \ldots, r_n) \leq (\lambda_h^*(f \circ g) + \epsilon) \log \left( \exp \left( \frac{L \left( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \right)}{(1-r_1)(1-r_2)\ldots(1-r_n)} \right) \right).
\]
\[
\begin{align*}
\log T_h^{-1}T_{f \circ g}(r_1, r_2, \ldots, r_n) &\leq (\lambda_h^{L^*}(f \circ g) + \epsilon)\log \left( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \right) \\
&+ L \left( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \right). \tag{6}
\end{align*}
\]

And
\[
\begin{align*}
\log T_h^{-1}T_f(r_1, r_2, \ldots, r_n) &\geq (\lambda_h^{L^*}(f) - \epsilon)\log \left( \frac{\exp\{L \left( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \right)\}}{(1-r_1)(1-r_2)\ldots(1-r_n)} \right),
\end{align*}
\]
i.e.,
\[
\begin{align*}
\log T_h^{-1}T_f(r_1, r_2, \ldots, r_n) &\geq (\lambda_h^{L^*}(f) - \epsilon)\log \left( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \right) \\
&+ L \left( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \right). \tag{7}
\end{align*}
\]

From equation (6) and (7), we get for a sequence of values of \( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \) tending to infinity that
\[
\begin{align*}
\log T_h^{-1}T_{f \circ g}(r_1, r_2, \ldots, r_n) &\leq \frac{(\lambda_h^{L^*}(f \circ g) + \epsilon)}{(\lambda_h^{L^*}(f) - \epsilon)} \log T_h^{-1}T_f(r_1, r_2, \ldots, r_n) \\
\end{align*}
\]
i.e.,
\[
\begin{align*}
\frac{\log T_h^{-1}T_{f \circ g}(r_1, r_2, \ldots, r_n)}{\log T_h^{-1}T_f(r_1, r_2, \ldots, r_n) + L \left( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \right)} &\leq \frac{(\lambda_h^{L^*}(f \circ g) + \epsilon)}{(\lambda_h^{L^*}(f) - \epsilon)} \\
&\times \frac{\log T_h^{-1}T_f(r_1, r_2, \ldots, r_n)}{\log T_h^{-1}T_f(r_1, r_2, \ldots, r_n) + L \left( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \right)}, \tag{8}
\end{align*}
\]
i.e.,
\[
\begin{align*}
\frac{\log T_h^{-1}T_{f \circ g}(r_1, r_2, \ldots, r_n)}{\log T_h^{-1}T_f(r_1, r_2, \ldots, r_n) + L \left( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \right)} &\leq \frac{(\lambda_h^{L^*}(f \circ g) + \epsilon)}{(\lambda_h^{L^*}(f) - \epsilon)} \\
&\times \frac{1}{1 + L \left( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \right)}. \tag{8}
\end{align*}
\]

As \( L \left( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \right) = 0 \{\log T_h^{-1}T_f(r_1, r_2, \ldots, r_n)\} \) as \( r_1, r_2, \ldots, r_n \to 1 \), we get from equation (8) that
\[
\liminf_{r_1, r_2, \ldots, r_n \to 1} \frac{\log T_h^{-1}T_{f \circ g}(r_1, r_2, \ldots, r_n)}{\log T_h^{-1}T_f(r_1, r_2, \ldots, r_n) + L \left( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \right)} \leq \frac{(\lambda_h^{L^*}(f \circ g) + \epsilon)}{(\lambda_h^{L^*}(f) - \epsilon)}. \tag{9}
\]
Since $\epsilon > 0$, is arbitrary it follows from equation (9) that

$$\liminf_{r_1, r_2, \ldots, r_n \to 1} \frac{\log T^{-1}_h T_{f \circ g}(r_1, r_2, \ldots, r_n)}{\log T^{-1}_h T_f(r_1, r_2, \ldots, r_n) + L \left( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \right)} \leq \frac{\lambda^*_h(f \circ g)}{\lambda^*_h(f)}.$$ (10)

Again, for a sequence of values of $\frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n}$ tending to infinity

$$\log T^{-1}_h T_f(r_1, r_2, \ldots, r_n) \leq (\lambda^*_h(f) + \epsilon) \log \left( \exp \left\{ L \left( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \right) \right\} \right),$$

i.e.

$$\log T^{-1}_h T_f(r_1, r_2, \ldots, r_n) \leq (\lambda^*_h(f) + \epsilon) \log \left( \frac{1}{1-r_1} \frac{1}{1-r_2} \ldots \frac{1}{1-r_n} \right) + L \left( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \right).$$ (11)

Now, from equation (1) and (11), we obtain for a sequence of values of $\frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n}$ tending to infinity that

$$\log T^{-1}_h T_{f \circ g}(r_1, r_2, \ldots, r_n) \geq \frac{(\lambda^*_h(f \circ g) - \epsilon)}{(\lambda^*_h(f) + \epsilon)} \log T^{-1}_h T_f(r_1, r_2, \ldots, r_n),$$

i.e.

$$\frac{\log T^{-1}_h T_{f \circ g}(r_1, r_2, \ldots, r_n)}{\log T^{-1}_h T_f(r_1, r_2, \ldots, r_n) + L \left( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \right)} \geq \frac{(\lambda^*_h(f \circ g) - \epsilon)}{(\lambda^*_h(f) + \epsilon)}$$

$$\times \frac{\log T^{-1}_h T_f(r_1, r_2, \ldots, r_n)}{\log T^{-1}_h T_f(r_1, r_2, \ldots, r_n) + L \left( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \right)}.$$ (12)

In view of the condition $L \left( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \right) \neq 0 \{ \log T^{-1}_h T_f(r_1, r_2, \ldots, r_n) \}$, as $r_1, r_2, \ldots, r_n \to 1$, we obtain from (12) that

$$\limsup_{r_1, r_2, \ldots, r_n \to 1} \frac{\log T^{-1}_h T_{f \circ g}(r_1, r_2, \ldots, r_n)}{\log T^{-1}_h T_f(r_1, r_2, \ldots, r_n) + L \left( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \right)} \geq \frac{(\lambda^*_h(f \circ g) - \epsilon)}{(\lambda^*_h(f) + \epsilon)}.$$ (13)
Since $\epsilon > 0$, is arbitrary it follows from (13) that

$$
\limsup_{r_1, r_2, \ldots, r_n \to 1} \frac{\log T_h^{-1}T_f^g(r_1, r_2, \ldots, r_n)}{\log T_h^{-1}T_f(r_1, r_2, \ldots, r_n) + L\left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n}\right)} \geq \frac{\lambda_{h}^{L^*}(f \circ g)}{\lambda_{h}^{L^*}(f)}.
$$

(14)

Also, for all sufficiently large values of $\frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n},$

$$
\log T_h^{-1}T_f^g(r_1, r_2, \ldots, r_n) \leq (\rho_h^{L^*}(f \circ g) + \epsilon)\log \left(\frac{\exp\{L\left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n}\right)\}}{(1-r_1)(1-r_2)\ldots(1-r_n)}\right),
$$
i.e.

$$
\log T_h^{-1}T_f^g(r_1, r_2, \ldots, r_n) \leq (\rho_h^{L^*}(f \circ g) + \epsilon)\{\log T_h^{-1}T_f(r_1, r_2, \ldots, r_n) + L\left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n}\right)\}.
$$

(15)

So from equation (7) and (15), it follows for all sufficiently large values of $\frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n}$ that

$$
\log T_h^{-1}T_f^g(r_1, r_2, \ldots, r_n) \leq \frac{(\rho_h^{L^*}(f \circ g) + \epsilon)}{(\lambda_{h}L^*(f) - \epsilon)}\log T_h^{-1}T_f(r_1, r_2, \ldots, r_n),
$$
i.e.

$$
\frac{\log T_h^{-1}T_f^g(r_1, r_2, \ldots, r_n)}{\log T_h^{-1}T_f(r_1, r_2, \ldots, r_n) + L\left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n}\right)} \leq \frac{\rho_h^{L^*}(f \circ g) + \epsilon}{\lambda_{h}L^*(f) - \epsilon}
$$

\begin{align*}
&\times \frac{\log T_h^{-1}T_f(r_1, r_2, \ldots, r_n) + L\left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n}\right)}{\log T_h^{-1}T_f(r_1, r_2, \ldots, r_n) + L\left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n}\right)}, \\
&\frac{\log T_h^{-1}T_f^g(r_1, r_2, \ldots, r_n)}{\log T_h^{-1}T_f(r_1, r_2, \ldots, r_n) + L\left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n}\right)} \leq \frac{\rho_h^{L^*}(f \circ g) + \epsilon}{\lambda_{h}L^*(f) - \epsilon}
\end{align*}

(16)

Using $L\left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n}\right) = 0\{\log T_h^{-1}T_f(r_1, r_2, \ldots, r_n)\}$ as $r_1, r_2, \ldots, r_n \to 1$, we obtain from (16) that

$$
\limsup_{r_1, r_2, \ldots, r_n \to 1} \frac{\log T_h^{-1}T_f^g(r_1, r_2, \ldots, r_n)}{\log T_h^{-1}T_f(r_1, r_2, \ldots, r_n) + L\left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n}\right)} \leq \frac{(\rho_h^{L^*}(f \circ g) + \epsilon)}{(\lambda_{h}L^*(f) - \epsilon)}
$$

(17)

As $\epsilon > 0$ is arbitrary, it follows from (17) that

$$
\limsup_{r_1, r_2, \ldots, r_n \to 1} \frac{\log T_h^{-1}T_f^g(r_1, r_2, \ldots, r_n)}{\log T_h^{-1}T_f(r_1, r_2, \ldots, r_n) + L\left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n}\right)} \leq \frac{\rho_h^{L^*}(f \circ g)}{\lambda_{h}L^*(f)}.
$$

(18)
Thus, the theorem follows from (5),(10),(14) and (18). Similarly, in view Theorem 1, we may state the following theorem without proof for the right \( g \) of the composite function \( f \circ g \).

**Theorem 2** Let \( f, g \) be any two analytic functions of several complex variables in \( U \) and \( h \) be any entire function with \( 0 < \lambda^L_h (f \circ g) \leq \rho^L_h (f \circ g) < \infty \), and \( 0 < \lambda^L_h (g) \leq \rho^L_h (g) < \infty \). If \( L \left( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \right) = 0 \{ \log T^{-1}_h (r_1, r_2, \ldots, r_n) \} \), as \( r_1, r_2, \ldots, r_n \rightarrow 1 \), then

\[
\frac{\lambda^L_h (f \circ g)}{\rho^L_h (g)} \leq \liminf_{r_1, r_2, \ldots, r_n \rightarrow 1} \frac{\log T^{-1}_h T_{f_{og}} (r_1, r_2, \ldots, r_n)}{\log T^{-1}_h T_g (r_1, r_2, \ldots, r_n) + L \left( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \right)} \leq \frac{\lambda^L_h (f)}{\lambda^L_h (g)}.
\]

**Theorem 3** Let \( f, g \) be any two analytic functions of several complex variables in \( U \) and \( h \) be an entire function of several complex variables such that \( 0 < \rho^L_h (f \circ g) < \infty \), and \( 0 < \rho^L_h (f) < \infty \). If

\[
\liminf_{r_1, r_2, \ldots, r_n \rightarrow 1} \frac{\log T^{-1}_h T_{f_{og}} (r_1, r_2, \ldots, r_n)}{\log T^{-1}_h T_f (r_1, r_2, \ldots, r_n) + L \left( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \right)} \leq \frac{\rho^L_h (f \circ g)}{\rho^L_h (f)}
\]

\[
\leq \limsup_{r_1, r_2, \ldots, r_n \rightarrow 1} \frac{\log T^{-1}_h T_{f_{og}} (r_1, r_2, \ldots, r_n)}{\log T^{-1}_h T_f (r_1, r_2, \ldots, r_n) + L \left( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \right)}.
\]

**Proof.** From the definition of \( \rho^L_h (f) \), the relative Nevanlinna \( L^* \)-order of an analytic function \( f \) of several complex variables in the unit polydisc \( U \) with respect to an entire function \( h \) of several complex variables we get for a sequence of values of \( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \) tending to infinity that

\[
\log T^{-1}_h T_f (r_1, r_2, \ldots, r_n) \geq (\rho^L_h (f) - \epsilon) \log \left( \frac{\exp \left( L \left( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \right) \right)}{(1-r_1)(1-r_2)\ldots(1-r_n)} \right),
\]

i.e.

\[
\frac{\log T^{-1}_h T_f (r_1, r_2, \ldots, r_n)}{(\rho^L_h (f) - \epsilon)} \geq \log \left( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \right)
\]

\[
+ L \left( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \right).
\]

Now from equation (15) and (19), it follows for a sequence of values of \( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \) tending to infinity that

\[
\log T^{-1}_h T_{f_{og}} (r_1, r_2, \ldots, r_n) \leq \frac{(\rho^L_h (f \circ g) + \epsilon)}{(\rho^L_h (f) - \epsilon)} \log T^{-1}_h T_f (r_1, r_2, \ldots, r_n),
\]
Again for a sequence of values of \(r_1, r_2, \ldots, r_n\), we obtain from (20) that
\[
\liminf_{n \to 1} \frac{\log T_{h^{-1}f_{og}}(r_1, r_2, \ldots, r_n)}{\log T_{h^{-1}f}(r_1, r_2, \ldots, r_n) + L \left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n}\right)} \leq \frac{\log L_h^+(f \circ g) + \epsilon}{\log L_h^+(f) - \epsilon}.
\]

Using (20) that
\[
\liminf_{n \to 1} \frac{\log T_{h^{-1}f_{og}}(r_1, r_2, \ldots, r_n)}{\log T_{h^{-1}f}(r_1, r_2, \ldots, r_n) + L \left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n}\right)} \leq \frac{\log L_h^+(f \circ g) + \epsilon}{\log L_h^+(f) - \epsilon}.
\]

As \(\epsilon > 0\), is arbitrary it follows from equation (21) that
\[
\liminf_{n \to 1} \frac{\log T_{h^{-1}f_{og}}(r_1, r_2, \ldots, r_n)}{\log T_{h^{-1}f}(r_1, r_2, \ldots, r_n) + L \left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n}\right)} \leq \frac{\log L_h^+(f \circ g) + \epsilon}{\log L_h^+(f) - \epsilon}.
\]
Theorem 4 Let \( f, g \) be any two analytic function of several complex variables in \( U \) and \( h \) be an entire function of several complex variables with \( 0 < \rho_h^L (f) < \infty \), and \( 0 < \rho_h^L (f) < \infty \). If \( L \left( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \right) = 0 \{ \log T_h^{-1}T_f(r_1, r_2, \ldots, r_n) \} \) as \( r_1, r_2, \ldots, r_n \to 1 \), then

\[
\lim \sup_{r_1, r_2, \ldots, r_n \to 1} \frac{\log T_h^{-1}T_f(r_1, r_2, \ldots, r_n)}{\log T_h^{-1}T_f(r_1, r_2, \ldots, r_n) + L \left( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \right)} \geq \frac{\rho_h^L (f \circ g) - \epsilon}{\rho_h^L (f) + \epsilon}.
\]

Thus, the theorem follows from equation (22) and (26).

Theorem 5 Let \( f, g \) be any two analytic function of several complex variables in the unit polydisc \( U \) and \( h \) be an entire function of several complex variables such that \( 0 < \lambda_h^L (f \circ g) \leq \rho_h^L (f \circ g) < \infty \), and \( 0 < \lambda_h^L (f) \leq \rho_h^L (f) < \infty \). If \( L \left( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \right) = 0 \{ \log T_h^{-1}T_f(r_1, r_2, \ldots, r_n) \} \) as \( r_1, r_2, \ldots, r_n \to 1 \), then

\[
\lim \inf_{r_1, r_2, \ldots, r_n \to 1} \frac{\log T_h^{-1}T_f(r_1, r_2, \ldots, r_n)}{\log T_h^{-1}T_f(r_1, r_2, \ldots, r_n) + L \left( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \right)} \leq \frac{\rho_h^L (f \circ g)}{\lambda_h^L (f)}.
\]
Then, we can find a constant 
Let us suppose that the conclusion of the theorem do not hold. 
Proof. 
Again from the definition of $\rho_h^L(f)$, it follows that for all sufficiently large values of $\frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n}$ that,

$$\log T^{-1}_h(r_1, r_2, \ldots, r_n) \leq (\rho_h^L(f) + \epsilon) \log \left( \frac{\exp \{L \left( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \right) \}}{(1-r_1)(1-r_2)\ldots(1-r_n)} \right).$$

(28)

Thus, from (27) and (28), we have for a sequence of values of $\frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n}$ tending to infinity that,

$$\log T^{-1}_h(r_1, r_2, \ldots, r_n) \leq \beta(\rho_h^L(f) + \epsilon) \log \left( \frac{\exp \{L \left( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \right) \}}{(1-r_1)(1-r_2)\ldots(1-r_n)} \right),$$

Theorem 6 Let $f, g$ be any two analytic function of several complex variables in the unit polydisc $U$ and $h$ be an entire function of several complex variables with $0 < \lambda_h^L(f \circ g) \leq \rho_h^L(f \circ g) < \infty$, and $0 < \lambda_h^L(g) \leq \rho_h^L(g) < \infty$. If $L \left( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \right) = 0(\log T^{-1}_h(r_1, r_2, \ldots, r_n))$, as $r_1, r_2, \ldots, r_n \to 1$, then

$$\liminf_{r_1, r_2, \ldots, r_n \to 1} \frac{\log T^{-1}_h f \circ g(r_1, r_2, \ldots, r_n)}{\log T^{-1}_h g(r_1, r_2, \ldots, r_n) + L \left( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \right)} \leq \limsup_{r_1, r_2, \ldots, r_n \to 1} \frac{\log T^{-1}_h f \circ g(r_1, r_2, \ldots, r_n)}{\log T^{-1}_h g(r_1, r_2, \ldots, r_n) + L \left( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \right)}.$$

Theorem 7 Let $f, g$ be any two analytic function of several complex variables in the unit polydisc $U$ and $h$ be an entire function of several complex variables such that $\rho_h^L(f) < \infty$, also let $g$ be analytic function of several complex variables in $U$. If $\lambda_h^L(f \circ g) < \infty$, that

$$\liminf_{r_1, r_2, \ldots, r_n \to 1} \frac{\log T^{-1}_h f \circ g(r_1, r_2, \ldots, r_n)}{\log T^{-1}_h g(r_1, r_2, \ldots, r_n)} = \infty.$$
i.e.,

\[
\frac{\log T_h^{-1}T_{fog}(r_1, r_2, \ldots, r_n)}{\log \left( \frac{\exp \left( L \left( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \right) \right)}{(1-r_1)(1-r_2)(1-r_n)} \right)} \leq \beta (\rho_h^{(f)}(f) + \epsilon) \log \left( \frac{\exp \left( L \left( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \right) \right)}{(1-r_1)(1-r_2)(1-r_n)} \right),
\]

\[
\liminf_{r_1, r_2, \ldots, r_n} \frac{\log T_h^{-1}T_{fog}(r_1, r_2, \ldots, r_n)}{\log \left( \frac{\exp \left( L \left( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \right) \right)}{(1-r_1)(1-r_2)(1-r_n)} \right)} = \lambda_h^{L^*}(f \circ g) < \infty,
\]

this is a contradiction.

This proves the theorem.

Remark. Theorem 7 is also valid with "limit superior" instead of "limit" if \( \lambda_h^{L^*}(f \circ g) = \infty \), is replaced by \( \rho_h^{L^*}(f \circ g) = \infty \), and the other conditions are remaining the same.

**Corollary 1** Under the assumptions of Theorem 2.7

\[
\limsup_{r_1, r_2, \ldots, r_n} \frac{T_h^{-1}T_{fog}(r_1, r_2, \ldots, r_n)}{T_h^{-1}T_f(r_1, r_2, \ldots, r_n)} = \infty.
\]

Proof. From Theorem 7, we obtain for all sufficiently large values of \( \frac{1}{1-r_1}, \frac{1}{1-r_2}, \ldots, \frac{1}{1-r_n} \) and \( K > 0 \), that

\[
T_h^{-1}T_{fog}(r_1, r_2, \ldots, r_n) > KT_h^{-1}T_f(r_1, r_2, \ldots, r_n),
\]

i.e.,

\[
T_h^{-1}T_{fog}(r_1, r_2, \ldots, r_n) > \{T_h^{-1}T_f(r_1, r_2, \ldots, r_n)\}^K,
\]

from which the corollary follows.

**Theorem 8** Let \( f, g \) be any two analytic functions of several complex variables in the unit polydisc \( U \) and \( h \) be any entire function of several complex variables such that \( \rho_h^{L^*}(g) < \infty \), and \( \lambda_h^{L^*}(f \circ g) = \infty \), then

\[
\lim_{r_1, r_2, \ldots, r_n} \frac{\log T_h^{-1}T_{fog}(r_1, r_2, \ldots, r_n)}{\log T_h^{-1}T_g(r_1, r_2, \ldots, r_n)} = \infty.
\]

Remark. Theorem 8 is also valid with "limit superior" instead of "limit" if \( \lambda_h^{L^*}(f \circ g) = \infty \), is replaced by \( \rho_h^{L^*}(f \circ g) = \infty \), and the other conditions are remaining the same.

In the line of corollary 1, we may easily verify the following.

**Corollary 2** Under the assumptions of Theorem 8

\[
\limsup_{r_1, r_2, \ldots, r_n} \frac{T_h^{-1}T_{fog}(r_1, r_2, \ldots, r_n)}{T_h^{-1}T_f(r_1, r_2, \ldots, r_n)} = \infty.
\]

**References**


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