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VALUE DISTRIBUTION OF GENERAL q-DIFFERENCE POLYNOMIALS

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Abstract. In this article, we mainly study the value distribution of more general q-difference polynomials for a transcendental entire function of zero and finite order. These are significant generalization of earlier results. As a very special case, we obtain the results of N. X. Xu and C. P. Zhong and others.

1. Introduction, Definitons and Results

For a meromorphic function f in the complex plane we assume familiarity with the standard notations of Nevanlinna theory such as, $T(r, f)$, $N(r, f)$ and $m(r, f)$ etc., as explained in [7, 19]. We need the following definitions.

Definition 1.1. Let $f(z)$ and $a(z)$ be meromorphic functions in the complex plane. If $T(r, a) = S(r, f)$, then $a(z)$ is called a small function of $f(z)$, where $S(r, f) = o(T(r, f))$ as $r \to \infty$, except possibly on a set of finite linear measure.

Definition 1.2. Let

$$M_j(f(qz)) = f^{l_{0j}} f^{l_{1j}}(q_1 z) f^{l_{2j}}(q_2 z) \cdots f^{l_{kj}}(q_k z) = \prod_{i=0}^{k} f^{l_{ij}}(q_i z), \tag{1}$$

where $q_0 = 1$ and $q_1, q_2, ..., q_k \in \mathbb{C}\backslash\{0\}$, $l_{0j}, l_{1j}, ..., l_{kj}$ are non-negative integers. Let the degree and weight of the monomial be $\gamma_{M_j} = l_{0j} + l_{1j} + \cdots + l_{kj}$ and $\Gamma_{M_j} = l_{0j} + 2l_{1j} + \cdots + (k+1)l_{kj} = \sum_{i=0}^{k}(i+1)l_{ij}$, respectively. If

$$P_q(f(qz)) = \sum_{j=1}^{s} a_j M_j(f(qz)), \tag{2}$$

where $a_j (j = 1, 2, 3, \ldots, s)$ are constants, then $P_q(f(qz))$ is called a difference polynomial in f of degree γ_{P_q} and the weight Γ_{P_q}. We define upper and lower degree of $P_q(f(qz))$ as follows $\overline{\gamma}_{P_q} = max_{1\leq j\leq s}\gamma_{P_q}$, $\underline{\gamma}_{P_q} = min_{1\leq j\leq s}\gamma_{P_q}$ and $\Gamma_{P_q} = max_{1\leq j\leq s}\gamma_{P_q}$. If $\overline{\gamma}_{P_q} = \underline{\gamma}_{P_q} = \gamma_{P_q}$, then $P_q(f(qz))$ is called homogeneous q-difference polynomial in $f(qz)$, otherwise non-homogeneous.

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Definition 1.3.[18] For a meromorphic function $f(z)$, the order and exponent of convergence of zeros is defined respectively as follows

$$\sigma(f) = limsup_{r\to\infty} \frac{logT(r,f)}{log\, r}, \ \ \lambda(f) = limsup_{r\to\infty} \frac{logN(r,\frac{1}{f})}{log\, r}.$$

A Borel exceptional value of $f(z)$ is any value a satisfying $\lambda(f-a) < \sigma(f)$.

In 1959, W. K. Hayman [8], discussed about Picard values of an entire and meromorphic functions and their derivatives. He obtained the following result.

Theorem A. Let $f(z)$ be a transcendental entire function. Then

(1) for $n \geq 3$ and $a \neq 0$, $\psi(z) = f'(z) - a(f(z))^n$ assumes all finite values infinitely often.
(2) For $n \geq 2$, $\phi(z) = f'(z)(f(z))^n$ assumes all finite values except possibly zero infinitely often.

As we have seen in recent years many researchers [4, 3, 5, 6, 9, 11, 13, 14, 16, 17, 15] are showing interest in the study of difference analogue of the Nevanlinna theory. Many articles [10, 14, 12, 18] have focused on the study of difference version of Hayman conjecture.

In 2007, I. Laine and C. C. Yang [10], considered the difference version of Theorem A and obtained the following result.

Theorem B. Let $f(z)$ be a transcendental entire function of finite order, c is a nonzero complex constant and $n \geq 2$, then $f^n(z)f(z+c)$ takes every nonzero value infinitely often.

Again in 2011, K. Liu and X. G. Qi [14] proved the following result by considering q-difference polynomials.

Theorem C. If $f(z)$ is a transcendental meromorphic function of zero order, a, q are nonzero complex constants. If $n \geq 6$, then $f^n(z)f(qz+c)$ assumes every nonzero value $b \in C$ infinitely often. If $n \geq 8$, then $f^n(z) + a[f(qz+c) - f(z)]$ assumes every nonzero value $b \in C$ infinitely often.

In the same year, K. Liu, X. L. Liu and T. B. Cao [12] obtained extension of above results by considering zero distribution of q-difference polynomials.

Theorem D. If $f(z)$ is a transcendental meromorphic function of zero order, a, q are nonzero complex constants, $\alpha(z)$ is a nonzero small function with respect to f. If $n \geq 6$, then $f^n(z)(f^m - a)f(qz+c) - \alpha(z)$ has infinitely many zeros. If $n \geq 7$, then $f^n(z)(f^m - a)[f(qz+c) - f(z)] - \alpha(z)$ has infinitely many zeros.

In 2016, N. Xu and C. P. Zhong [18] generalized above results to more general case and proved the following results.

Theorem E. Let f(z) be a transcendental entire function of zero order, a be a nonzero complex constant, $q \in \mathbb{C}\backslash\{0,1\}$, n be any positive integer. Considering q-difference polynomial $H(z) = f(qz) - a(f(z))^n$,

(1) if n = 3, then $H(z) - a(z)$ has infinitely many zeros, where $a(z)$ is a nonzero small function with respect to $f(z)$.
(2) In particular, if $a(z)$ is a nonzero rational function, then the condition $n = 3$ can be reduced to $n > 1$.

Theorem F. Let $f(z)$ be a transcendental entire function of zero order, $q_1, q_2, \ldots, q_m$ be non-zero complex constants such that at least one of them is not equal to 1, $a \in \mathbb{C}-\{0\}, m, n \in N^+$. Considering q-difference polynomial $F(z) = f(q_1z)f(q_2z)\cdots f(q_mz) - a(f(z))^n$,

(1) if $m < \frac{n-1}{2-\frac{1}{n}}$. Then $F(z) - \alpha(z)$ has infinitely many zeros, where $\alpha(z)$ is a nonzero small function with respect to $f(z)$.
(2) In particular, if $\alpha(z)$ is a nonzero rational function, then the condition $m < \frac{n-1}{2-\frac{1}{n}}$ can be reduced to $n > m$.
(3) If $m \neq n$, then also $F(z) - \alpha(z)$ has infinitely many zeros.

Theorem G. Let $f(z)$ be a transcendental entire function of finite and positive order $\sigma(f)$, $q_1, q_2, \ldots, q_m$ be non-zero complex constants such that at least one of them is not equal to 1 and $q_1^{\sigma(f)} + q_2^{\sigma(f)} + \cdots + q_m^{\sigma(f)} \neq n$, $a \in \mathbb{C} - \{0\}, m, n \in N^+$. If $f(z)$ has finitely many zeros, then $F(z) - \alpha(z)$ has infinitely many zeros, where $\alpha(z)$ is a nonzero small function with respect to $f(z)$.
If $G(z)$ be an entire function with order less than one and if $F(z) - a(f(z))^n = G(z)$, then $f(z)$ has infinitely many zeros.

In this article we generalize all the above results to more general q-difference polynomials.

Theorem 1.1. *Let $f(z)$ be a zero order transcendental entire function, $q_1, q_2, \ldots, q_m$ be non-zero complex constants and at least one of them is not equal to 1, $a \in \mathbb{C} - \{0\}, \overline{\gamma}_{P_q}, n \in \mathbb{N}$. Let the q-difference polynomial be $H(z) = P_q(f(qz)) - aP(f)$, where $P_q(f(qz))$ be as defined in (1.2) and $P(f) = a_n f^n + a_{n-1} f^{n-1} + \cdots + a_0$.*

(1) *If $\overline{\gamma}_{P_q} < \frac{n-1}{2-\frac{1}{n}}$, then $H(z) - \alpha(z)$ has infinitely many zeros, where $\alpha(z) \neq 0$ is a small function of f.*
(2) *If $\alpha(z) \neq 0$ is a rational function, then $\overline{\gamma}_{P_q} < \frac{n-1}{2-\frac{1}{n}}$ reduces to $n > \overline{\gamma}_{P_q}$.*

Corollary 1.1. The q-difference polynomial $P_q(f(qz)) - aP(f) - R(z) = 0$ has no zero order transcendental entire solution when $n > \overline{\gamma}_{P_q}$, where $R(z)$ is a nonzero rational function.

Remark 1.1. Substituting $l_{01} = 0$, $l_{11} = 1$ in (1.2) we get $\overline{\gamma}_{P_q} = 1$. Hence we get Theorem E.

Theorem 1.2. *Let $f(z)$ be a zero order transcendental entire function, $q_0 = 1$ and $q_1, q_2, \ldots, q_m$ be non-zero complex constants and at least one of them is not equal to 1, $a \in \mathbb{C} - \{0\}, \overline{\gamma}_{P_q}, n \in \mathbb{N}$. If $(2\overline{\gamma}_{P_q} - \underline{\gamma}_{P_q}) \neq n$, then $H(z) - \alpha(z)$ has infinitely many zeros, where $\alpha(z) \neq 0$ is a small function of f.*

Remark 1.2. Substituting $j = 1$,$l_{01} = 0$, $l_{i1} = l_{i1} = \cdots = l_{ik} = 1$ in (1.2) and considering $P(f) = f^n$ then Theorem 1.1 and 1.2 reduces to Theorem F.

All the previous results are obtained for the case when $f(z)$ is a transcendental entire function of zero order. In Theorem 1.3 and 1.4 by considering $f(z)$ as a finite and positive order transcendental entire function we discuss the value distribution of q-difference polynomial $H(z)$.

Theorem 1.3. *Let $f(z)$ be a finite and positive order transcendental entire function $\sigma(f)$, $q_0 = 1$ and $q_1, q_2, \ldots, q_m$ be non-zero complex constants and at least one of them is not equal to 1 and $l_{0j} + l_{1j}q_1^{\sigma(f)} + l_{2j}q_2^{\sigma(f)} + \cdots + l_{mj}q_m^{\sigma(f)} \neq n$, $a \in \mathbb{C} - \{0\}, \overline{\gamma}_{P_q}, n \in \mathbb{N}$. If $f(z)$ has finitely many zeros. Then $H(Z) - \alpha(z)$ has infinitely many zeros, where $\alpha(z) \neq 0$ is a small function of f.*

Theorem 1.4. *Let $f(z)$ be a finite and positive order transcendental entire function and $G(z)$ is an entire function with order less than 1, $q_0 = 1$ and $q_1, q_2, \ldots, q_m$ be* *non-zero complex constants and at least one of them is not equal to 1 and* $l_{0j} + l_{1j}q_1^{\sigma(f)} + l_{2j}q_2^{\sigma(f)} + \cdots + l_{kj}q_m^{\sigma(f)} \neq n$, $a \in \mathbb{C} - \{0\}, \overline{\gamma}_{P_q}, n \in \mathbb{N}$. *If*

$$P_q(f(qz)) - aP(f) = G(z), \tag{3}$$

then $f(z)$ *has infinitely many zeros.*

Remark 1.3. Substituting $j = 1$, $l_{01} = 0, l_{i1} = l_{i1} = \cdots = l_{im} = 1$ in (1.2) and considering $P(f) = f^n$ then Theorem 1.3 and 1.4 reduce to Theorem G.

2. Some Lemmas.

Lemma 2.1. [20] *Let* $f(z)$ *be a transcendental meromorphic function of zero order and* q *be a non-zero complex constant. Then*

$$T(r, f(qz)) = (1 + o(1))T(r, f(z)) \text{ or } T(r, f(qz)) = T(r, f(z)) + S_1(r, f),$$

on a set of lower logarithmic density 1.

Lemma 2.2. [2] *Let* $f(z)$ *be a nonconstant zero order meromorphic function and* $q \in C\backslash\{0\}$. *Then*

$$m\left(r, \frac{f(qz)}{f(z)}\right) = S(r, f),$$

on a set of logarithmic density 1.

Lemma 2.3. [1] *If an entire function* f *has a finite exponent of convergence* $\lambda(f)$ *for its zero-sequence, then* f *has a representation in the form* $f(z) = Q(z)e^{g(z)}$, *satisfying* $\lambda(Q) = \sigma(Q) = \lambda(f)$. *Further, if* f *is of finite order, then* g *in the above form is a polynomial of degree less or equal to the order of* f.

Lemma 2.4. [19] *Suppose that* $f_1(z), f_2(z), \ldots, f_n(z), (n \geq 2)$ *are meromorphic functions and* $g_1(z), g_2(z), \ldots, g_n(z)$ *are entire functions satisfying the following conditions,*

(1) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0$;
(2) $g_j(z) - g_k(z)$ *are not constants for* $1 \leq j < k \leq n$;
(3) *for* $1 \leq j \leq n$, $1 \leq h < k \leq n$, $T(r, f_j) = 0(T(r, e^{g_h - g_k}))(r \to \infty, r \notin E)$. *Then* $f_j(z) \equiv 0(j = 1, 2, \ldots, n)$.

Lemma 2.5. *Let* $f(qz)$ *be a zero-order meromorphic function and* $P_q(f(qz))$ *be a* q*-difference polynomial in* f *of degree* $n \geq 1$ *with coefficients* $a_j(z)$, *upper degree* $\overline{\gamma}_P$ *and lower degree* $\underline{\gamma}_P$, *then*

$$m\left(r, \frac{P_q(f(qz))}{f^{\overline{\gamma}_{P_q}}}\right) \leq (\overline{\gamma}_{P_q} - \underline{\gamma}_{P_q})m\left(r, \frac{1}{f}\right) + S_1(r, f),$$

on a set of logarithmic density 1.

Proof. Let $M_j(f(qz))$ and $P_q(f(qz))$ are defined as in (1.1) and (1.2) respectively, then

$$\left|\frac{P_q(f(qz))}{f^{\overline{\gamma}_P}}\right| = \sum_{j=1}^{s} |a_j| \left|\frac{M_j(f(qz))}{f^{\gamma_{M_j}}}\right| \left|\frac{1}{f}\right|^{\overline{\gamma}_P - \gamma_{M_j}}, \tag{4}$$

where γ_{M_j} is the degree of the monomial $M_j(f)$.

Case 1: When $|f(qz)| \leq 1$, $|\frac{1}{f(qz)}| \geq 1$ and $\left|\frac{1}{f(qz)}\right|^{\overline{\gamma}_P - \gamma_{M_j}} \geq 1$, and we have

$$\left|\frac{1}{f(qz)}\right|^{\overline{\gamma}_P - \gamma_{M_j}} \leq \left|\frac{1}{f(qz)}\right|^{\overline{\gamma}_P - min_{1\leq j \leq s}\gamma_{M_j}} = \left|\frac{1}{f(qz)}\right|^{\overline{\gamma}_P - \underline{\gamma}_P}.$$

Hence we get from (2.1),

$$\left|\frac{P_q(f(qz))}{f^{\overline{\gamma}_P}}\right| \leq \left|\frac{1}{f}\right|^{\overline{\gamma}_{P_q} - \underline{\gamma}_{P_q}} \left[\sum_{j=1}^{s} |a_j| \left|\frac{f(q_1 z)}{f}\right|^{l_{1j}} \left|\frac{f(q_2 z)}{f}\right|^{l_{2j}} \cdots \left|\frac{f(q_k z)}{f}\right|^{l_{kj}}\right].$$

Using the logarithmic derivative lemma, we get

$$m\left(r, \frac{P_q(f(qz))}{f^{\overline{\gamma}_{P_q}}}\right) \leq (\overline{\gamma}_{P_q} - \underline{\gamma}_{PP_q}) m\left(r, \frac{1}{f(qz)}\right) + S_1(r, f(qz)).$$

Since f is a meromorphic function of zero order, we have

$$S_1(r, f(qz)) = S_1(r, f). \tag{5}$$

Hence

$$m\left(r, \frac{P_q(f(qz))}{f^{\overline{\gamma}_{P_q}}}\right) \leq (\overline{\gamma}_{P_q} - \underline{\gamma}_{P_q}) m\left(r, \frac{1}{f}\right) + S_1(r, f).$$

Outside of a possible exceptional set with the finite logarithmic measure.

Case 2: When $|f(qz)| > 1$ we have $\left|\frac{1}{f(qz)}\right| \leq 1$, $\left|\frac{1}{f(qz)}\right|^{\overline{\gamma}_{P_q} - \gamma_{M_j}} \leq 1$ and $log^+ \left|\frac{1}{f(qz)}\right|^{\overline{\gamma}_{P_q} - \gamma_{M_j}} = 0$.

Hence from (2.1) and logarithmic derivative lemma we get,

$$m\left(r, \frac{P_q(f(qz))}{f^{\overline{\gamma}_P}}\right) \leq S_1(r, f(qz)).$$

Proceeding as in Case 1, we get,

$$\begin{aligned} m\left(r, \frac{P_q(f(qz))}{f^{\overline{\gamma}_P}}\right) &\leq S_1(r, f), \\ &\leq (\overline{\gamma}_{P_q} - \underline{\gamma}_{P_q}) m\left(r, \frac{1}{f}\right) + S_1(r, f). \end{aligned}$$

3. Proof of the Theorems.

Proof of Theorem 1.1.

(1) Let $\Phi(z) = \frac{P_q(f(qz)) - \alpha(z)}{a\, P(f)}$. From the condition $\overline{\gamma}_{P_q} < \frac{n-1}{2-\frac{1}{n}}$ we get $n > \overline{\gamma}_{P_q}$. Since $f(z)$ is a zero order transcendental entire function, by Lemma 2.1, we get,

$$\begin{aligned} T(r, P(z)) &= \frac{P_q(f(qz)) - \alpha(z)}{a\, \Phi(z)}, \\ nT(r, f) &\leq T(r, P_q(f(qz))) + T(r, \alpha(z)) + T(r, \Phi(z)) + O(1), \\ &\leq \overline{\gamma}_{P_q} T(r, f) + T(r, \Phi(z)) + S(r, f). \end{aligned}$$

From the above equation, we obtain

$$(n - \overline{\gamma}_{P_q}) T(r, f) \leq T(r, \Phi(z)) + S(r, f), \tag{6}$$

on a set of logarithmic density 1. Since $n > \overline{\gamma}_{P_q}$ we can note that $\Phi(z)$ is transcendental. On the other hand,

$$\begin{aligned} T(r,\Phi(z)) = T\left(r, \frac{P_q(f(qz)) - \alpha(z)}{a\,P(f)}\right) &\leq T(r, P_q(f(qz))) + T(r,\alpha(z)) + T(r,P(f)) + O(1) \\ &\leq \overline{\gamma}_{P_q}T(r,f) + nT(r,f) + S(r,f). \end{aligned}$$

Therefore

$$T(r,\Phi(z)) \leq (n + \overline{\gamma}_{P_q})T(r,f) + S(r,f). \tag{7}$$

From (3.1), (3.2) and the condition $n > \overline{\gamma}_{P_q}$, we get $T(r,\Phi(z)) = O(T(r,f))$.
Suppose $H(z) - \alpha(z)$ has finitely many zeros, then $\Phi(z)$ has only finite 1-points. Hence

$$N\left(r, \frac{1}{\Phi(z) - 1}\right) = S(r,\Phi(z)) = S(r,f).$$

We can note from the second fundamental theorem

$$\begin{aligned} T(r,\Phi(z)) &\leq \overline{N}(r,\Phi) + \overline{N}\left(r,\frac{1}{\Phi}\right) + \overline{N}\left(r,\frac{1}{\Phi-1}\right) + S(r,\Phi) \\ &\leq \frac{1}{n}N(r,\Phi) + \overline{\gamma}_{P_q}T(r,f) + S(r,f), \end{aligned}$$

$$\left(1 - \frac{1}{n}\right)T(r,\Phi) \leq \overline{\gamma}_{P_q}T(r,f) + S(r,f). \tag{8}$$

From (3.1), (3.3), we get

$$(1 - \frac{1}{n} - \frac{\overline{\gamma}_{P_q}}{n - \overline{\gamma}_{P_q}})T(r,\Phi) \leq S(r,\Phi). \tag{9}$$

which is a contradiction, since $\overline{\gamma}_{P_q} < \frac{n-1}{2-\frac{1}{n}}$. Hence $H(Z) - \alpha(z)$ has infinitely many zeros.

(2.) By Lemma 2.1, we have

$$\begin{aligned} T(r,H(z)) &\leq T(r, P_q(f(qz)) - aP(f)) \leq T(r, P_q(f(qz))) + T(r,P(f)) + S(r,f) \\ &\leq (n + \overline{\gamma}_{P_q})T(r,f) + S(r,f). \end{aligned} \tag{10}$$

On the other side,

$$T(r, aP(f)) \leq T(r, P_q(f(qz)) - H(z)) \leq T(r, P_q(f(qz))) + T(r,H(z)),$$

$$nT(r,f) \leq \overline{\gamma}_{P_q}T(r,f) + T(r,H(z)) + S(r,f). \tag{11}$$

From (3.5) and (3.6) we obtain,

$$(n - \overline{\gamma}_{P_q})T(r,f) + S(r,f) \leq T(r,H) \leq (n + \overline{\gamma}_{P_q})T(r,f) + S(r,f). \tag{12}$$

From the above equation we obtain, $T(r,H) = O(T(r,f))$. Since $n > \overline{\gamma}_{P_q}$ and $\sigma(f) = 0$, clearly $H(z)$ is of zero order.
Let us assume that $R(z) = H(z) - \alpha(z)$ has finitely many zeros. Then $R(z)$ becomes a rational function, since $H(z)$ is a function of zero order and $\alpha(z)$ is a non-zero rational function. Then we get $T(r,H) = S(r,f)$, which is a contradiction to our assumption. Hence, $H(z) - \alpha(z)$ has infinitely many zeros.

Proof of Theorem 1.2.

Let us assume that $H(Z)-\alpha(z)$ has finitely many zeros, by Lemma 2.1, we obtain

$$\begin{aligned} T(r,H(z)-\alpha(z)) &= T(r,P_q(f(qz))-aP(f)-\alpha(z)) \\ &\leq T(r,P_q(f(qz)))+T(r,P(f))+T(r,\alpha(z))+S(r,f), \\ &\leq \overline{\gamma}_{P_q}T(r,f)+nT(r,f)+S(r,f), \\ &\leq (n+\overline{\gamma}_{P_q})T(r,f)+S(r,f). \end{aligned}$$

From the above inequality we get $\sigma(H(z)-\alpha(z))=0$.
From the Hadamard factorization theorem, we obtain

$$H(z)-\alpha(z)=P_q(f(qz))-aP(f)-\alpha(z)=P_1(z), \tag{13}$$

where $P_1(z)$ is a polynomial. Rewriting (3.8), we get

$$aP(f)=P_q(f(qz))-P_1(z)-\alpha(z). \tag{14}$$

When $n>(2\overline{\gamma}_{P_q}-\underline{\gamma}_{P_q})$, from (3.9) and Lemma 2.1, we have

$$\begin{aligned} T(r,aP(f)) &= T(r,P_q(f(qz))-P_1(z)-\alpha(z)) \\ nT(r,f) &\leq \overline{\gamma}_{P_q}T(r,f)+S(r,f), \\ &\leq (2\overline{\gamma}_{P_q}-\underline{\gamma}_{P_q})T(r,f)+S(r,f). \end{aligned}$$

Which is a contradiction to the assumption.
When $n<2\overline{\gamma}_{P_q}-\underline{\gamma}_{P_q}$, from (3.9), Lemma 2.2 and Lemma 2.5, we have

$$\begin{aligned} T(r,P_q(f(qz))) &= m(r,P_q(f(qz)))=m\left(r,f^{\overline{\gamma}_{P_q}}\frac{P_q(f(qz))}{f^{\overline{\gamma}_{P_q}}}\right) \\ &\geq m(r,f^{\overline{\gamma}_{P_q}})-m\left(r,\frac{f^{\overline{\gamma}_{P_q}}}{P_q(f(qz))}\right) \\ &\geq \overline{\gamma}_{P_q}m(r,f)-(\underline{\gamma}_{P_q}-\overline{\gamma}_{P_q})m(r,f)+S(r,f) \\ &\geq (2\overline{\gamma}_{P_q}-\underline{\gamma}_{P_q})m(r,f)+S(r,f). \end{aligned}$$

On the other hand by (3.9), we get

$$\begin{aligned} T(r,P_q(f(qz))) &= T(r,aP(f)+P_1(z)+\alpha(z)) \\ (2\overline{\gamma}_{P_q}-\underline{\gamma}_{P_q})T(r,f) &\leq nT(r,f)+S(r,f). \end{aligned}$$

Which is a contradiction to our assumption. Hence $H(z)-\alpha(z)$ has infinitely many zeros.

Proof of Theorem 1.3.

Let $f(z)$ be a finite and positive order transcendental entire function and has finitely many zeros, then from Lemma 2.3, $f(z)$ can be expressed in the form

$$f(z)=g(z)e^{h(z)}, \tag{15}$$

where $g(z)\,(\not\equiv 0), h(z)$ are polynomials. Set

$$h(z)=a_kz^k+\cdots+a_0, \tag{16}$$

where $a_k(\neq 0),\ldots,a_0$ are constants. Given that $\sigma(f)\neq 0$, hence $\sigma(f)=degh(z)=k\geq 1$.

From (1.2), (3.10) and (3.11) we have

$$\begin{aligned} P_q(f(qz)) &= \sum_{j=1}^{s}\prod_{i=0}^{k} a_j f(q_i z)^{l_{ij}} \\ &= \sum_{j=1}^{s}\prod_{i=0}^{k} a_j g(q_i z)^{l_{ij}} e^{h(q_i z)l_{ij}} \\ &= \sum_{j=1}^{s}\prod_{i=0}^{k} a_j g(q_i z)^{l_{ij}} e^{a_k(l_{0j}+l_{1j}q_1^k+l_{2j}q_2^k+\cdots+l_{mj}q_m^k)z^k} e^{a_{k-1}(l_{0j}+l_{1j}q_1^{k-1}+l_{2j}q_2^{k-1}+\cdots+l_{mj}q_m^{k-1})z^{k-1}} \\ &\cdot \quad \ldots e^{a_0(l_{0j}+l_{1j}+l_{2j}+\cdots+l_{mj})}. \end{aligned}$$

$$P_q(f(qz)) = \sum_{j=1}^{s} P_2(z) e^{a_k(l_{0j}+l_{1j}q_1^k+l_{2j}q_2^k+\cdots+l_{kj}q_m^k)z^k}, \tag{17}$$

where $P_2(z) = \prod_{i=1}^{k} a_j g(q_i z)^{l_{ij}} e^{a_{k-1}(l_{0j}+l_{1j}q_1^{k-1}+l_{2j}q_2^{k-1}+\cdots+l_{mj}q_m^{k-1})z^{k-1}} \ldots e^{a_0(l_{0j}+l_{1j}+l_{2j}+\cdots+l_{mj})}$.
Thus $\sigma(P_2) \le k-1 < k$. On the other side, from (3.10) and (3.11) we have

$$\begin{aligned} P(f) &= a_n f^n + a_{n-1} f^{n-1} + \cdots + a_0 \\ &= a_n g^n e^{nh} + a_{n-1} g^{n-1} e^{(n-1)h} + \cdots + a_0 \\ &= a_n g^n e^{n(a_k z^k+\cdots+a_0)} + a_{n-1} g^{n-1} e^{(n-1)(a_k z^k+\cdots+a_0)} + \cdots + a_0 \\ &= e^{na_k z^k}[a_n g^n e^{na_{k-1}z^{k-1}+\cdots+na_0} + a_{n-1} g^{n-1} e^{-a_k z^k+(n-1)a_{k-1}z^{k-1}+\cdots+(n-1)a_0} + \cdots + a_0 e^{-na_k z^k}]. \end{aligned}$$

$$P(f) = P_3(z) e^{na_k z^k}, \tag{18}$$

where $P_3(z) = a_n g^n e^{na_{k-1}z^{k-1}+\cdots+na_0} + a_{n-1} g^{n-1} e^{-a_k z^k+(n-1)a_{k-1}z^{k-1}+\cdots+(n-1)a_0} + \cdots + a_0 e^{-na_k z^k}$. From (3.12) and (3.13), we get

$$H(z) = \sum_{j=1}^{s} P_2(z) e^{a_k(l_{0j}+l_{1j}q_1^k+l_{2j}q_2^k+\cdots+l_{mj}q_m^k)z^k} - aP_3(z) e^{na_k z^k} \setminus \{0\}. \tag{19}$$

Since $P_2(z)(\not\equiv 0), P_3(z)(\not\equiv 0), \sigma(P_2) < k, \sigma(P_3) < k$, $l_{0j} + l_{1j}q_1^{\sigma(f)} + l_{2j}q_2^{\sigma(f)} + \cdots + l_{kj}q_m^{\sigma(f)} \neq n$, it follows that $H(z)$ is a transcendental entire function and $\sigma(H) = \sigma(f) = k$.
Suppose $H(z) - \alpha(z)$ has finitely many zeros, then $\sigma(H - \alpha(z)) < \sigma(H) = \sigma(f)$.
Hence $H(z) - \alpha(z)$ can be expressed as

$$H(z) - \alpha(z) = S(z) e^{tz^k}, \tag{20}$$

where $S(z)$ is an entire function with $\sigma(S) < k$, $t \neq 0$ is a constant. From (3.14) and (3.15), we get

$$\sum_{j=1}^{s} P_2(z) e^{a_k(l_{1j}q_1^k+l_{2j}q_2^k+\cdots+l_{kj}q_m^k)z^k} - aP_3(z) e^{na_k z^k} - S(z) e^{tz^k} - \alpha(z) = 0. \tag{21}$$

Since $l_{1j}q_1^{\sigma(f)} + l_{2j}q_2^{\sigma(f)} + \cdots + l_{kj}q_m^{\sigma(f)} \neq n$.
Case (i): $a_k(l_{0j} + l_{1j}q_1^k + l_{2j}q_2^k + \cdots + l_{mj}q_m^k)z^k \neq t, na_k z^k \neq t$. By Lemma 2.4, we obtain $P_2(z) = 0, P_3(z) = 0, S(z) = 0, \alpha(z) = 0$. This is a contradiction.

Case (ii): $a_k(l_{0j}+l_{1j}q_1^k+l_{2j}q_2^k+\cdots+l_{mj}q_m^k)z^k=t$. Then (3.16) can be written as

$$\left(\sum_{j=1}^{s}P_2(z)-S(z)\right)e^{a_k(l_{0j}+l_{1j}q_1^k+l_{2j}q_2^k+\cdots+l_{mj}q_m^k)z^k}-aP_3(z)e^{na_kz^k}-\alpha(z)=0.$$

By Lemma 2.4, we obtain $P_2(z)-S(z)=0, P_3(z)=0, \alpha(z)=0$. This is a contradiction.

Case (iii): $na_k=t$, following the same procedure as above, we arrive at a contradiction. Hence, $H(z)-\alpha(z)$ has infinitely many zeros.

Proof of Theorem 1.4.

Let us assume that $f(z)$ has finitely many zeros.

Using (3.12) and (3.13) in (1.3), we get

$$\sum_{j=1}^{s}P_2(z)e^{a_k(l_{0j}+l_{1j}q_1^k+l_{2j}q_2^k+\cdots+l_{mj}q_m^k)z^k}-aP_3(z)e^{na_kz^k}=G(z), \tag{22}$$

where $P_2(z)$ and $P_3(z)$ are defined as in Theorem 1.3.

Since $P_2(z)(\not\equiv 0), P_3(z)(\not\equiv 0), \sigma(P_2)<k, \sigma(P_3)<k$, $l_{0j}+l_{1j}q_1^k+l_{2j}q_2^k+\cdots+l_{mj}q_m^k\neq n$ we get $\sigma(G)<1<k$. From (3.17) and Lemma 2.4, we get $P_2(z)=0, P_3(z)=0, G(z)=0$, which is a contradiction. Hence $f(z)$ has infinitely many zeros.

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References

[1] R. B. Ash, Complex Variables, Academic Press, New York-London, 1971.

[2] D. C. Barnett, R. G. Halburd, R. J. Korhonen and W. Morgan, Nevanlinna theory for the q-difference operator and meromorphic solutions of q-difference equations, Proc. Roy. Soc. Edinburgh Sert. A 137, 457-474, 2007.

[3] D. C. Barnett, R. G. Halburd and W. Morgan, Nevanlinna theory for the q-difference operator and meromoprhic solutions of q-difference equations, Proc. Roy. Soc. ED., 137A, 457-474, 2007.

[4] Y. M. Chiang and S. J. Feng, On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane, Ramanujan J., 16, 105-129, 2008.

[5] R. G. Halburd and R. J. Korhonen, Difference analogue of the lemma on the logarithmic derivative with applications to difference equations, J.Math. Anal. Appl., 314, 477-487,2006.

[6] R. G. Halburd and R. J. Korhonen, Nevanlinna theory for the difference operator, Ann. Acad. Sci. Fenn. Math., 31, 463-478, 2006.

[7] W. K. Hayman, Meromorphic Functions, The Clarendon Press, Oxford, 1964.

[8] W. K. Hayman, Picard values of meromorphic Functions and their derivatives, Ann. Math. 70, 9-42, 1959.

[9] J. Heittokangas, R. J. Korhonen, I. Laine, J. Rieppo and J. Zhang, Value sharing results for shifts of meromorphic functions, and sufficient conditions for periodicity, J. Math. Anal. Appl., 355, 352-363, 2009.

[10] I. Laine and C. C. Yang, Value distribution of difference polynomials, Proc. Japan Acad. Ser. A , 83, 148-151, 2007.

[11] K. Liu, X. Liu, and T. Cao, Value distribution and uniqueness of difference polynomials, Adv. Difference Equ., Article ID 234215, 12 pages, 2011.

[12] K. Liu, X. Liu, and T. Cao, Uniqueness and zeros of q-shift difference polynomials, Proc. Indian Acad. Math. Sci., 121, no. 3, 301-310, 2011.

[13] K. Liu and L. Z. Yang, Value distribution of the difference operator, Arch. Math., 92, 270-278, 2009.

[14] K. Liu and X. G. Qi, Meromorphic solutions of q-shift difference equations, Ann. Polon. Math., 101, 215-225, 2011.

[15] Y. Liu, Yinhong Cao, Xiaoguang Qi and H. X. Yi, Value sharing results for q-shifts difference polynomials, Discrete Dyanamics in Nature and Soc., 2013. Article Id:152069.
[16] L. N. Wang, H. X. Xu and T. S. Zhan, Properties of q-shift difference- differential polynomials of meromorphic functions, Adv. Difference Equ., 2014. Art. 249, doi:10.1186/1687-1847-2014-249.
[17] H. Y. Xu, On the value distribution and uniqueness of difference polynomials of meromorphic functions, Adv. Difference Equ., Art 290, doi:10.1186/1687-1847-2013-90, 2013.
[18] N. Xu and C. Zhong, Value distribution of some q-difference polynomials, Bull. Korean Math. Soc., 53, 29-38, 2016.
[19] C. C. Yang and H. X. Yi, Uniqueness theory of meromorphic functions, Kluwer, Dordrect, 2004.
[20] J. L. Zhang and R. J. Korhonen, On the Nevanlinna characteristic of $f(qz)$ and its applications, J. Math. Anal. Appl., 369, 537-544, 2010.
[21] R. P. Agarwal, B. de Andrade, C. Cuevas,"On type of periodicity and ergodicity to a class of fractional order differential equations," Advances in Difference Equations, Hindawi Publ. Corp. NY, USA, Vol. 2010, Article ID 179750, 1-25.
[22] E. Ahmed, A. M. A. El-Sayed, E. M. El-Mesiry and H. A. A. El-Saka, Numerical solution for the fractional replicator equation, IJMPC, 19, 7, 1-9, 2005.
[23] E. Ahmed, A. M. A. El-Sayed, and H. A. A. El-Saka, Equilibrium points, stability and numerical solutions of fractional-order predator-prey and rabies models JMAA. to appear.
[24] E. Ahmed, A. M. A. El-Sayed, and H. A. A. El-Saka, On some Routh-Hurwitz condition for fractional order differential equations and thier application in Lorentz, Rosiler, Chua and Chen systems. Physics Letteres A, 358, 1, 2006.
[25] B. de Andrade,J.P.C. dos Santos, C. Cuevas, "Existence results for a fractional equations with state-dependent delay," Advances in Difference equations, Hindawi Publ. Corp. NY, USA, Volume 2011, Article ID 642013, 15 pages, doi:10.1155/2011/642013
[26] A. M. A. El-Sayed, Fractional Order Evolution Equations. Journal of Fractional Calculus, Vol. 7. 1995.
[27] A. M. A. El-Sayed, Fractional Order Diffusion-Wave Equation. International J. of Theo. Physics, Vol. 35, No. 2. 1996.
[28] A. M. A. El-Sayed, and F. M. Gaafar, Fractional calculus and some intermediate physical processes Appl. Math. and Compute. Vol. 144. 2003.
[29] A. M. A. El-Sayed, fractional differential-difference equations, Journal of Fractional Calculus, 10, 101- 106, 1996.
[30] A. M. A. El-Sayed, Nonlinear functional differential equations of arbitrary orders, Nonlinear Analysis: Theory, Methods and Applications., 33, 2, 181- 186, 1998.
[31] A. M. A. El-Sayed and F. M. Gaafar, Fractional order differential equations with memory and fractional-order relaxation-oscillation model (PU.M.A) Pure Math. and Appl. Vol. 12 2001.
[32] A. M. A. El-Sayed, E. M. El-Mesiry and H. A. A. El-Saka, Numerical solution for multi-term fractional (arbitrary) orders differential equations, Comput., and Appl. Math., 23, 1, 33-54, 2004.
[33] A. M. A. El-Sayed, F. M. Gaafar and H. H. Hashem, On the maximal and minimal solutions of arbitrary orders nonlinear functional integral and differential equations. *Math Sci Res J.* 8(11) pp. 336-348. 2004.
[34] A. M. A. El-Sayed, E. M. El-Mesiry and H. A. A. El-Saka, On the fractional-order logistic equation, J. Appl. Math. Letters Vol. 20 817-823, 2007.
[35] P. Erdi, Complexity Explained, Springer, 2008.
[36] R. Gorenflo and F.Mainardi, Fractional Calculus: Integral and Differential Equations of Fractional Order, in A. Carpinteri and F. Mainardi (Eds), Fractals and Fractional Calculus in Continuum Mechanics, Springer, 223-276, 1997, Wien.
[37] J.Hofbauer and K Sigmund, Evolutionary Games and Population Dynamics, Cambridge Univ. Press 1998.
[38] I. Podlubny and A. M. A. El-Sayed, On two definitions of fractional calculus, Solvak Academy of science-institute of experimental phys. UEF-03-96 ISBN 80-7099-252-2, 1996.
[39] I. Podlubny, Fractional differential equations, Academic Press, 1999.
[40] C.Taylor and A.M.Nowak, Evolutionary game dynamics with non-uniform interaction rate, Theor. Pop.Biol 69,243, 2006.

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