PARTIAL SUMS OF $\tau$-CONFLUENT HYPERGEOMETRIC FUNCTION

AMIT SONI AND DEEPAK BANSAL

ABSTRACT. In the present investigation, $\tau$-confluent hypergeometric function with their normalization are considered. In this paper, we will study the ratio of a function of the form $(1, \Phi_1^{1}(b; c; z))_n = z + \frac{\Gamma(c)}{\Gamma(d)} \sum_{k=1}^{n} \frac{\Gamma(b+kr)}{\Gamma(c+kr)} z^{k+1}$ to its sequence of partial sums $(1, \Phi_1^{1}(b; c; z))_n$. We will determine the lower bounds for $\Re \left\{ \frac{1}{(1, \Phi_1^{1}(b; c; z))_n} \right\}$, $\Re \left\{ \frac{1}{(1, \Phi_1^{1}(b; c; z))'_n} \right\}$, $\Re \left\{ \frac{1}{(1, \Phi_1^{1}(b; c; z))''_n} \right\}$ and $\Re \left\{ \frac{1}{(1, \Phi_1^{1}(b; c; z))'''_n} \right\}$.

1. Introduction

Let $\mathcal{H}$ denote the class of analytic functions $f$ defined in the open unit disk $D = \{ z \in \mathbb{C} : |z| < 1 \}$ and $\mathcal{A}$ denote the subclass of $\mathcal{H}$, which are normalized by the condition $f(0) = 0 = f'(0) - 1$ and have representation of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in D.$$  \hfill (1)

It is well known that the series

$$1\Phi_1^{1}(b; c; z) = \sum_{n=0}^{\infty} \frac{(b)_n z^n}{(c)_n n!}$$  \hfill (2)

in which $c$ is neither zero nor a negative integer is convergent for all finite $z$. Here $(b)_n$ denotes the Pochhammer symbol which is defined by

$$(b)_n := \begin{cases} 1, & (n = 0) \\ b(b + 1)...(b + n - 1), & (n \in \mathbb{N}). \end{cases}$$

The Pochhammer symbol is related to the gamma functions by the relation

$$(b)_n = \frac{\Gamma(b+n)}{\Gamma(b)},$$

where $b$ is neither zero nor a negative integer. The function $1\Phi_1^{1}(b; c; z)$ is known as a confluent hypergeometric function for more details one can refer [7]. In 2009 Mathematics Subject Classification. 33E12, 30C45.

Key words and phrases. $\tau$-Confluent Hypergeometric Function, Analytic Function, Univalent Function.

Virchenko [11] introduced $\tau$-confuent hypergeometric function which is defined by (see also [12]):

$$1\phi_1^\tau(b; c; z) = \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(b + \tau n) z^n}{(c + \tau n) \, n!}.$$  \hfill (3)

For $\tau = 1$,

$$1\phi_1^\tau(b; c; z) = 1\phi_1(b; c; z).$$

As the function $1\phi_1^\tau(b; c; z)$ does not belong to the family $\mathcal{A}$, thus it is natural to consider the following normalization of function $1\phi_1^\tau(b; c; z)$ in $\mathbb{D}$:

$$1\Phi_1(b; c; z) = z \frac{1}{1\phi_1(b; c; z)} = z + \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=2}^{\infty} \frac{\Gamma(b + \tau(n - 1)) z^n}{(c + \tau(n - 1)) (n - 1)!}.$$ \hfill (4)

For the present investigation we will study $1\Phi_1(b; c; z)$ for real values of $b$ and $c$ satisfying $c \geq b > 0$ only.

If $f, g$ are analytic functions in $\mathbb{D}$, then $f$ is said to be subordinate to $g$, written as $f(z) \prec g(z)$ ($z \in \mathbb{D}$), if there exists an analytic function $w$ with $w(0) = 0$ and $|w(z)| \leq 1$ ($z \in \mathbb{D}$) such that $f(z) = g(w(z))$. In particular, if $g$ is univalent in $\mathbb{D}$, then we have the following equivalence:

$$f(z) \prec g(z) \iff f(0) = g(0) \quad \text{and} \quad f(\mathbb{D}) \subset g(\mathbb{D}).$$

For more details one can refer [4]. In the present paper, we will study the ratio of a function of the form (4) to its sequence of partial sums

$$(1\Phi_1(b; c; z))_n = z + \frac{\Gamma(c)}{\Gamma(b)} \sum_{k=1}^{n} \frac{\Gamma(b + k\tau) z^{k+1}}{(c + k\tau) \, k!} = z + \sum_{k=1}^{n} b_k z^{k+1},$$ \hfill (5)

$$(1\Phi_1(b; c; z))'_n = 1 + \frac{\Gamma(c)}{\Gamma(b)} \sum_{k=1}^{n} \frac{(k + 1)\Gamma(b + k\tau) z^{k}}{(c + k\tau) \, k!} = 1 + \sum_{k=1}^{n} (k + 1) b_k z^{k},$$ \hfill (6)

$$(1\Phi_1(b; c; z))_0 = z \quad \text{and} \quad (1\Phi_1(b; c; z))'_0 = 1.$$ \hfill (7)

We will determine lower bounds for $\Re \left\{ \frac{1}{1\Phi_1(b; c; z)} \right\}$, $\Re \left\{ \frac{1}{1\Phi_1(b; c; z)_n} \right\}$, $\Re \left\{ \frac{1}{1\Phi_1(b; c; z)}' \right\}$, $\Re \left\{ \frac{1}{1\Phi_1(b; c; z)}'_n \right\}$ and $\Re \left\{ \frac{1}{1\Phi_1(b; c; z)}'' \right\}$. For various known results concerning with partial sums of analytic univalent functions one can refer the works of Bansal and Orhan [1], Çağlar and Deniz [2], Choi [3], Orhan and Yaşmur [5], Owa et. al [6], Sheil-Small [8], Silverman [9] and Silvia [10].

To prove main results we need following Lemma:

**Lemma 1.** If $\tau > 0$ and $c \geq b > \max\{2 - \tau, 0\}$ then,

$$|1\Phi_1^\tau(b; c; z)| \leq 1 + \frac{2\Gamma(c)}{\Gamma(b)} \quad (z \in \mathbb{D})$$ \hfill (8)

and

$$|1\Phi_1^\tau(b; c; z)'| \leq 1 + \frac{11 \Gamma(c)}{2 \Gamma(b)} \quad (z \in \mathbb{D}).$$ \hfill (9)
Proof. To prove this lemma, we use the following inequalities

\[
\frac{n}{(n-1)!} \leq \left(\frac{2}{3}\right)^{n-3} \quad \forall n \geq 4
\]

\[
\Gamma(c+\tau(n-1)) \geq \Gamma(b+\tau(n-1)) \quad \text{for } \tau > 0, \ c \geq b > \max\{2-\tau, 0\} \text{ and } n \in \{2, 3, 4\ldots\}
\]

and \(n! \geq 2^{n-1}\) for all \(n \in \mathbb{N}\). Using (4), we have

\[
|1_{\Phi}^{(1)}(b; c; z)| \leq |z| + \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=2}^{\infty} \frac{1}{(n-1)!} \frac{\Gamma(b+\tau(n-1))}{\Gamma(c+\tau(n-1))} |z|^n
\]

\[
\leq 1 + \frac{\Gamma(c)}{\Gamma(b)} \left[ 1 + \sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^{n-2} \right]
\]

\[
= 1 + \frac{2\Gamma(c)}{\Gamma(b)}.
\]

Similarly

\[
|1_{\Phi}^{(1)}(b; c; z)| \leq 1 + \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=2}^{\infty} \frac{n}{(n-1)!} \frac{\Gamma(b+\tau(n-1))}{\Gamma(c+\tau(n-1))} |z|^{n-1}
\]

\[
< 1 + \frac{\Gamma(c)}{\Gamma(b)} \left[ 1 + \frac{7}{2} + \sum_{n=4}^{\infty} \left(\frac{2}{3}\right)^{n-3} \right]
\]

\[
= 1 + \frac{11\Gamma(c)}{2\Gamma(b)}.
\]

\[
2. \text{ Main results}
\]

**Theorem 1.** If \(\tau > 0, \ c \geq b > \max\{2-\tau, 0\}\) and \(\Gamma(b) \geq 2\Gamma(c)\), then

\[
\Re\left\{ \frac{1_{\Phi}^{(1)}(b; c; z)}{(1_{\Phi}^{(1)}(b; c; z))} \right\} \geq \left(1 - \frac{2\Gamma(c)}{\Gamma(b)}\right) (z \in \mathbb{D}) \quad (10)
\]

and

\[
\Re\left\{ \frac{(1_{\Phi}^{(1)}(b; c; z))}{1_{\Phi}^{(1)}(b; c; z)} \right\} \geq \frac{\Gamma(b)}{\Gamma(b) + 2\Gamma(c)} (z \in \mathbb{D}). \quad (11)
\]

Proof. It is easy to see from (8) of Lemma 1 that

\[
1 + \sum_{k=1}^{\infty} b_k \leq \frac{\Gamma(b) + 2\Gamma(c)}{\Gamma(b)}
\]

which is equivalent to

\[
\frac{\Gamma(b)}{2\Gamma(c)} \sum_{k=1}^{\infty} b_k \leq 1 \quad \text{(where } b_k = \frac{\Gamma(c)\Gamma(b+\tau n)}{n!\Gamma(b)\Gamma(c+\tau n)}) \quad (12)
\]

To prove 10, we have to show that

\[
\frac{\Gamma(b)}{2\Gamma(c)} \left[ 1_{\Phi}^{(1)}(b; c; z) - \frac{(\Gamma(b) - 2\Gamma(c))}{\Gamma(b)} \right] < \frac{1+z}{1-z}. \quad (13)
\]
Using definition of subordination, and putting the values of \(\Phi_1^1(b; c; z)\) and \((1.\Phi_1^1(b; c; z))_n\), we have
\[
1 + \sum_{k=1}^{n} b_k z^k + \frac{\Gamma(b)}{2\Gamma(c)} \sum_{k=n+1}^{\infty} b_k z^k = \frac{1 + w(z)}{1 - w(z)}.
\]

Our assertion (10) is true if we show that \(w(0) = 0\) and \(|w(z)| < 1\) provided \(z \in \mathbb{D}\).

Simplifying for \(w(z)\), we get
\[
w(z) = \frac{\frac{\Gamma(b)}{2\Gamma(c)} \sum_{k=n+1}^{\infty} b_k z^k}{2 + 2 \sum_{k=1}^{n} b_k z^k + \frac{\Gamma(b)}{2\Gamma(c)} \sum_{k=n+1}^{\infty} b_k z^k}.
\]

Obviously \(w(0) = 0\) and
\[
|w(z)| \leq \frac{\frac{\Gamma(b)}{2\Gamma(c)} \sum_{k=n+1}^{\infty} b_k}{2 - 2 \sum_{k=1}^{n} b_k - \frac{\Gamma(b)}{2\Gamma(c)} \sum_{k=n+1}^{\infty} b_k} \leq 1
\]

provided
\[
\sum_{k=1}^{n} b_k + \frac{\Gamma(b)}{2\Gamma(c)} \sum_{k=n+1}^{\infty} b_k \leq 1. \quad (14)
\]

It suffices to show that the left hand side of (14) is bounded above by left hand side of (12), which is equivalent to
\[
\left(\frac{\Gamma(b)}{2\Gamma(c)} - 1\right) \sum_{k=1}^{n} b_k \geq 0.
\]

This is true as \(\Gamma(b) \geq 2\Gamma(c)\).

To prove the result (11), we write
\[
\frac{\Gamma(b) + 2\Gamma(c)}{2\Gamma(c)} \left[ (\Phi_1^1(b; c; z))_n \right] - \frac{\Gamma(b) + 2\Gamma(c)}{\Gamma(b) + 2\Gamma(c)} = \frac{1 + w(z)}{1 - w(z)}.
\]

Substituting the values of \(\Phi_1^1(b; c; z)\) and \((1.\Phi_1^1(b; c; z))_n\) and simplifying for \(w(z)\), we have
\[
w(z) = \frac{-\frac{\Gamma(b) + 2\Gamma(c)}{2\Gamma(c)} \sum_{k=n+1}^{\infty} b_k z^k}{2 + 2 \sum_{k=1}^{n} b_k z^k - \frac{\Gamma(b) + 2\Gamma(c)}{2\Gamma(c)} \sum_{k=n+1}^{\infty} b_k z^k}.
\]

Obviously \(w(0) = 0\) and
\[
|w(z)| \leq \frac{\frac{\Gamma(b) + 2\Gamma(c)}{2\Gamma(c)} \sum_{k=n+1}^{\infty} b_k}{2 - 2 \sum_{k=1}^{n} b_k - \frac{\Gamma(b) + 2\Gamma(c)}{2\Gamma(c)} \sum_{k=n+1}^{\infty} b_k} \leq 1 \quad (15)
\]
as (14) is true for \(\Gamma(b) \geq 2\Gamma(c)\).
Theorem 2. If $\tau > 0$, $c \geq b > \max\{2 - \tau, 0\}$ and $2\Gamma(b) \geq 11\Gamma(c)$, then

$$\Re\left\{ \frac{(1_i\Phi_1'(b; c; z))'_n}{(1_i\Phi_1'(b; c; z))'_n} \right\} \geq \frac{2\Gamma(b) - 11\Gamma(c)}{2\Gamma(b)} \quad (z \in \mathbb{D})$$

(16)

and

$$\Re\left\{ \frac{(1_i\Phi_1'(b; c; z))'_n}{(1_i\Phi_1'(b; c; z))'_n} \right\} \geq \frac{2\Gamma(b)}{2\Gamma(b) + 11\Gamma(c)} \quad (z \in \mathbb{D}).$$

(17)

Proof. It is easy to see from (9) of Lemma 1 that

$$1 + \sum_{k=1}^{\infty} b_k(k + 1) \leq \frac{2\Gamma(b) + 11\Gamma(c)}{2\Gamma(b)}$$

which is equivalent to

$$\frac{2\Gamma(b)}{11\Gamma(c)} \sum_{k=1}^{\infty} b_k(k + 1) \leq 1 \quad \text{(where } b_k = \frac{\Gamma(c)\Gamma(b + \tau n)}{\Gamma(b)n\Gamma(c + \tau n)} \text{).}$$

(18)

To prove (16), we have to show that

$$\frac{2\Gamma(b)}{11\Gamma(c)} \left[ \frac{(1_i\Phi_1'(b; c; z))'}{(1_i\Phi_1'(b; c; z))'_n} - \left( \frac{2\Gamma(b) - 11\Gamma(c)}{2\Gamma(b)} \right) \right] \leq \frac{1 + z}{1 - z}.$$  

(19)

Using definition of subordination, and putting the values of $1_i\Phi_1'(b; c; z)$ and $(1_i\Phi_1'(b; c; z))'_n$, we have

$$1 + \sum_{k=1}^{n} b_k(k + 1)z^k + \frac{2\Gamma(b)}{11\Gamma(c)} \sum_{k=n+1}^{\infty} (k + 1)b_kz^k$$

$$= \frac{1 + w(z)}{1 - w(z)}.$$  

Our assertion (10) is true if we show that $w(0) = 0$ and $|w(z)| < 1$ provided $z \in \mathbb{D}$. Simplifying for $w(z)$, we get

$$w(z) = \frac{2\Gamma(b)}{11\Gamma(c)} \sum_{k=n+1}^{\infty} (k + 1)b_kz^k$$

$$2 + 2\sum_{k=1}^{n} (k + 1)b_kz^k - 2\sum_{k=n+1}^{\infty} (k + 1)b_kz^k$$

Obviously $w(0) = 0$ and

$$|w(z)| \leq \frac{2\Gamma(b)}{11\Gamma(c)} \sum_{k=n+1}^{\infty} (k + 1)b_k$$

$$2 - 2\sum_{k=1}^{n} (k + 1)b_k - 2\Gamma(b) \sum_{k=n+1}^{\infty} (k + 1)b_k \leq 1$$

provided

$$\sum_{k=1}^{n} (k + 1)b_k + \frac{2\Gamma(b)}{11\Gamma(c)} \sum_{k=n+1}^{\infty} (k + 1)b_k \leq 1.$$  

(20)

It suffices to show that the left hand side of (20) is bounded above by left hand side of (18), which is equivalent to

$$\left( \frac{2\Gamma(b)}{11\Gamma(c)} - 1 \right) \sum_{k=1}^{n} b_k \geq 0.$$
This is true in view of hypothesis.

To prove the result (17), we write

$$\frac{2\Gamma(b) + 11\Gamma(c)}{11\Gamma(c)} \left[ (\Phi_1^1(b; c; z))'_n - \frac{2\Gamma(b)}{2\Gamma(b) + 11\Gamma(c)} \right] = \frac{1 + w(z)}{1 - w(z)}.$$

Substituting the values of $(\Phi_1^1(b; c; z))'$ and $(\Phi_1^1(b; c; z))'_n$ and simplifying for $w(z)$, we have

$$w(z) = \frac{-(1 + \frac{2\Gamma(b)}{11\Gamma(c)}) m \sum_{k=n+1}^{\infty} (k + 1)b_k z^k}{2 + 2 \sum_{k=1}^{n} (k + 1)b_k z^k + \left(1 - \frac{2\Gamma(b)}{11\Gamma(c)}\right) m \sum_{k=n+1}^{\infty} (k + 1)b_k z^k}.$$

Obviously $w(0) = 0$ and

$$|w(z)| \leq \frac{(1 + \frac{2\Gamma(b)}{11\Gamma(c)}) m \sum_{k=n+1}^{\infty} (k + 1)b_k z^k}{2 - 2 \sum_{k=1}^{n} (k + 1)b_k z^k - \left(2\frac{\Gamma(b)}{11\Gamma(c)} - 1\right) m \sum_{k=n+1}^{\infty} (k + 1)b_k z^k} \leq 1 \quad (21)$$

as (20) is true under the hypothesis.

\[\Box\]

References