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ON THE STABILITY OF A FUNCTIONAL EQUATION IN 2-BANACH SPACES BY USING FIXED POINT THEOREM

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Abstract. In this paper, we investigate some stability and hyperstability results for the following functional equation

$$f(ax+by) = \frac{(a+b)}{2} f(x+y) + \frac{(a-b)}{2} f(x-y),$$

where a, b are different integers greater than 1, in 2-Banach spaces by using Brzdęk's fixed point approach.

1. Introduction

In this paper, we will denote the set of natural numbers by $\mathbb{N}$ and the set of real numbers by $\mathbb{R}$. we put $E_0 := E\backslash\{0\}$, $\mathbb{R}_+ := [0, \infty)$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Also, Y^X denotes the set of all functions from a nonempty set X to a nonempty set Y.By $\mathbb{N}_m$, $m \in \mathbb{N}$, we will denote the set of all natural numbers greater than or equal to m.

The problem of the stability of functional equations was first raised by Ulam [18] concerning the stability of group homomorphisms. In the following year, Hyers [10] first partially answered Ulams question, and proved the Ulam stability of Cauchy function in Banach spaces. Hyers' Theorem was generalized by Aoki [3] and by Rassias [15] for additive mappings and linear mappings by considering an unbounded Cauchy difference. Afterward Găvruta [9] obtained generalization of the Th. M. Rassias theorem by replacing the unbounded Cauchy difference by a general control function in the spirit of Th. M. Rassias' approach. During the last decades, many mathematicians have investigated various generalizations, extensions and applications of the Hyers-Ulam stability of functional equations (see [1, 4, 5, 6, 16, 17]). In particular, the stability problem of the functional equations in various spaces was proved in [11, 14]. The first hyperstability result was published in [15] and concerned the ring homomorphisms. We say a functional equation is hyperstable

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if any function f satisfying this equation approximately is a true solution of it. Recently, Brzdęk [7] proved a fixed point theorem in 2-Banach spaces and showed its applications to the Ulam stability of functional equations.

In 2011, Kenary, Jang and Park [12] proved the Hyers-Ulam stability of the following functional equation

$$f(ax+by) = \frac{(a+b)}{2}f(x+y) + \frac{(a-b)}{2}f(x-y), \tag{1}$$

in non-Archimedean normed spaces and in random normed spaces, where a, b are different integers greater than 1. Recently, interesting results concerning the functional equation (1) have been obtained in [2] and [13]. The purpose of this paper is to prove some stability and hyperstability results for the functional equation 1 in a 2-Banach space using Brzdęk and Ciepliński's fixed point results in [7].
We need to recall some basic facts concerning 2-normed spaces and some preliminary results (see, for instance, [8]).

Definition 1 let X be a real linear space with $dimX > 1$ and $\|\cdot,\cdot\| : X \times X \longrightarrow \mathbb{R}_+$ be a function satisfying the following properties:

(1) $\|x, y\| = 0$ if and only if x and y are linearly dependent,
(2) $\|x, y\| = \|y, x\|$,
(3) $\|\lambda x, y\| = |\lambda| \|x, y\|$,
(4) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$,

for all $x, y, z \in X$ and $\lambda \in \mathbb{R}$. Then the function $\|\cdot,\cdot\|$ is called *a 2-norm* on X and the pair $(X, \|\cdot,\cdot\|)$ is called a *linear 2-normed space.* Sometimes the condition (4) called the *triangle inequality.*

Example 1 For $x = (x_1, x_2)$, $y = (y_1, y_2) \in E = \mathbb{R}^2$, the Euclidean 2-norm $\|x, y\|_E$ is defined by

$$\|x, y\|_E = |x_1 y_2 - x_2 y_1| \, .$$

Definition 2 A sequence $\{x_k\}$ in a 2-normed space X is called a *convergent sequence* if there is an $x \in X$ such that

$$\lim_{k\to\infty} \|x_k - x, y\| = 0,$$

for all $y \in X$. If $\{x_k\}$ converges to x, write $x_k \longrightarrow x$ with $k \longrightarrow \infty$ and call x the limit of $\{x_k\}$. In this case, we also write $\lim_{k\to\infty} x_k = x$.

Definition 3 A sequence $\{x_k\}$ in a 2-normed space X is said to be a *Cauchy sequence* with respect to the 2-norm if

$$\lim_{k,l\to\infty} \|x_k - x_l, y\| = 0,$$

for all $y \in X$. If every Cauchy sequence in X converges to some $x \in X$, then X is said to be *complete* with respect to the 2-norm. Any complete 2-normed space is said to be a *2-Banach space.*
Now, we present the fixed point theorem concerning 2-Banach spaces given in [7]. First, we need the following hypotheses:

(H1) E is a nonempty set, $\big(Y, \|\cdot,\cdot\|\big)$ is a 2-Banach space, Y_0 is a subset of Y containing two linearly independent vectors, $j \in \mathbb{N}$, $f_i : E \to E$, $g_i : Y_0 \to Y_0$, and $L_i : E \times Y_0 \to \mathbb{R}_+$ for $i = 1, ..., j$;

(H2) $\mathcal{T} : Y^E \to Y^E$ is an operator satisfying the inequality

$$\left\|\mathcal{T}\xi(x)-\mathcal{T}\mu(x), y\right\| \le \sum_{i=1}^{j} L_i(x,y)\left\|\xi\big(f_i(x)\big)-\mu\big(f_i(x)\big), g_i(y)\right\|,\ \xi,\mu \in Y^E, x \in E, y \in Y_0; \tag{2}$$

(H3) $\Lambda : \mathbb{R}_+^{E\times Y_0} \to \mathbb{R}_+^{E\times Y_0}$ is an operator defined by

$$\Lambda\delta(x,y) := \sum_{i=1}^{j} L_i(x,y)\delta\big(f_i(x), g_i(y)\big), \quad \delta \in \mathbb{R}_+^{E\times Y_0},\ x \in E, y \in Y_0. \tag{3}$$

Theorem 1[7] Let hypotheses (H1)-(H3) hold and functions $\varepsilon : E \times Y_0 \to R_+$ and $\varphi : E \to Y$ fulfill the following two conditions:

$$\left\|\mathcal{T}\varphi(x) - \varphi(x), y\right\| \le \varepsilon(x,y) \quad x \in E, y \in Y_0, \tag{4}$$

$$\varepsilon^*(x,y) := \sum_{n=0}^{\infty} \big(\Lambda^n\varepsilon\big)(x,y) < \infty \quad x \in E, y \in Y_0. \tag{5}$$

Then, there exists a unique fixed point ψ of $\mathcal{T}$ for which

$$\left\|\varphi(x) - \psi(x), y\right\| \le \varepsilon^*(x,y) \quad x \in E, y \in Y_0. \tag{6}$$

Moreover,

$$\psi(x) := \lim_{n\to\infty} (\mathcal{T}^n\varphi)(x) \quad x \in E. \tag{7}$$

2. Main results

In this section, we prove some stability and hyperstability results of the functional equation (1) in 2-Banach spaces by using Theorem 1. In what follows $(Y, \|\cdot,\cdot\|)$ is a real 2-Banach space.

Theorem 2 Let $h_1, h_2 : E_0 \times Y_0 \to \mathbb{R}_+$ be two functions such that

$$\mathcal{U} := \{n \in \mathbb{N}\colon \alpha_n < 1\} \ne \phi \tag{8}$$

where

$$\alpha_n := \frac{2}{a+b}\lambda_1(a+an-bn)\lambda_2(a+an-bn) + \frac{|a-b|}{a+b}\lambda_1(1+2n)\lambda_2(1+2n),$$

$$\lambda_i(n) := \inf\{t \in \mathbb{R}_+\colon h_i(nx,z) \le t\, h_i(x,z), \ \ x \in E_0, z \in Y_0\} \tag{9}$$

for all $n \in \mathbb{N}$, where $i = 1, 2$. Assume that $f : E \to Y$ satisfies the inequality

$$\left\|f(ax+by) - \frac{(a+b)}{2}f(x+y) - \frac{(a-b)}{2}f(x-y), z\right\| \le h_1(x,z)h_2(y,z), \tag{10}$$

for all $x, y \in E_0, z \in Y_0$ such that $ax+by \ne 0, x+y \ne 0$ and $x-y \ne 0$. Then there exists a unique function $F : E \to Y$ such that

$$\left\|f(x) - F(x), z\right\| \le \lambda_0 h_1(x,z)h_2(x,z) \tag{11}$$

for all $x \in E_0, z \in Y_0$, where

$$\lambda_0 := \inf_{n\in\mathcal{U}}\left\{\frac{2\lambda_1(1+n)\lambda_2(-n)}{(a+b)(1-\frac{2}{a+b}\lambda_1(a+an-bn)\lambda_2(a+an-bn) - \frac{|a-b|}{a+b}\lambda_1(1+2n)\lambda_2(1+2n))}\right\}.$$

Proof. Replacing x by $(1+m)x$ and y by $-mx$, where $x \in E_0, z \in Y_0$ and $m \in \mathbb{N}$, in inequality (10) we get

$$\left\| \frac{2}{a+b} f((a+am-bm)x) - \frac{(a-b)}{a+b} f((1+2m)x) - f(x), z \right\| \leq \frac{2}{a+b} h_1((1+m)x, z) h_2(-mx, z), \tag{12}$$

for all $x \in E_0, z \in Y_0$. For each $m \in \mathbb{N}$, we define the operator $\mathcal{T}_m : Y^{E_0} \to Y^{E_0}$ by

$$\mathcal{T}_m \xi(x) := \frac{2}{a+b} \xi((a+am-bm)x) - \frac{(a-b)}{a+b} \xi((1+2m)x), \quad \xi \in Y^{E_0}, x \in E_0. \tag{13}$$

Further put

$$\varepsilon_m(x,z) := \frac{2}{a+b} h_1((1+m)x, z) h_2(-mx, z), \quad x \in E_0, z \in Y_0, \tag{14}$$

and observe that

$$\begin{aligned} \varepsilon_m(x,z) &= \frac{2}{a+b} h_1((1+m)x, z) h_2(-mx, z) \\ &\leq \frac{2}{a+b} \lambda_1(1+m) \lambda_2(-m) h_1(x,z) h_2(x,z), \end{aligned} \tag{15}$$

for all $x \in E_0, z \in Y_0, m \in \mathbb{N}$. Then, inequality (12) can be rewritten as

$$\left\| f(x) - \mathcal{T}_m f(x), z \right\| \leq \varepsilon_m(x,z), \quad x \in E_0, z \in Y_0. \tag{16}$$

Furthermore, for every $x \in E_0, z \in Y_0$, $\xi, \mu \in Y^{E_0}$, we obtain

$$\begin{aligned} &\left\| \mathcal{T}_m \xi(x) - \mathcal{T}_m \mu(x), z \right\| = \\ &\quad \Big\| \frac{2}{a+b} \xi((a+am-bm)x) - \frac{(a-b)}{a+b} \xi((1+2m)x) - \\ &\qquad \frac{2}{a+b} \mu((a+am-bm)x) + \frac{(a-b)}{a+b} \mu((1+2m)x), z \Big\| \\ &\leq \frac{2}{a+b} \left\| (\xi-\mu)\big((a+am-bm)x\big), z \right\| + \frac{|a-b|}{a+b} \left\| (\xi-\mu)((1+2m)x), z \right\|. \end{aligned}$$

So, (H2) is valid for $\mathcal{T}_m$. This brings us to define the operator $\Lambda_m : \mathbb{R}_+^{E_0 \times Y_0} \to \mathbb{R}_+^{E_0 \times Y_0}$ by

$$\Lambda_m \delta(x,z) := \frac{2}{a+b} \delta((a+am-bm)x, z) + \frac{|a-b|}{a+b} \delta((1+2m)x, z), \tag{17}$$

for all $\delta \in \mathbb{R}_+^{E_0 \times Y_O}, x \in E_0, z \in Y_0$. For each $m \in \mathbb{N}$, the above operator has the form described in (H3) with $f_1(x) = (a+am-bm)x$, $f_2(x) = (1+2m)x$, $g_1(z) = g_2(z) = z$, $L_1(x) = \frac{2}{a+b}$ and $L_2(x) = \frac{|a-b|}{a+b}$ for all $x \in E_0, z \in Y_0$. By using the mathematical induction, we will show that for each $x \in E_0, z \in Y_0$, $n \in \mathbb{N}_0$, and $m \in \mathcal{U}$ we have

$$(\Lambda_m^n \varepsilon_m)(x,z) \leq \frac{2}{a+b} \lambda_1(1+m) \lambda_2(-m) \alpha_m^n h_1(x,z) h_2(x,z) \tag{18}$$

where

$$\alpha_m = \frac{2}{a+b} \lambda_1(a+am-bm) \lambda_2(a+am-bm) + \frac{|a-b|}{a+b} \lambda_1(1+2m) \lambda_2(1+2m).$$

From (14) and (15), we obtain that the inequality (18) holds for $n = 0$. Next, we will assume that (18) holds for $n = k$, where $k \in \mathbb{N}$. Then we have

$$\begin{aligned}
(\Lambda_m^{k+1}\varepsilon_m)(x,z) &= \Lambda_m\left((\Lambda_m^k\varepsilon_m)(x,z)\right)\\
&= \frac{2}{a+b}(\Lambda_m^k\varepsilon_m)\big((a+am-bm)x,z\big) + \frac{|a-b|}{a+b}(\Lambda_m^k\varepsilon_m)((1+2m)x,z)\\
&\le \frac{2}{a+b}\Big(\frac{2}{a+b}\lambda_1(1+m)\lambda_2(-m)\alpha_m^k h_1((a+am-bm)x,z)h_2((a+am-bm)x,z)\\
&\quad + \frac{|a-b|}{a+b}\lambda_1(1+m)\lambda_2(-m)\alpha_m^k h_1((1+2m)x,z)h_2((1+2m)x,z)\Big)\\
&\le \frac{2}{a+b}\lambda_1(1+m)\lambda_2(-m)\alpha_m^{k+1}h_1(x,z)h_2(x,z),
\end{aligned}$$

for all $x \in E_0, z \in Y_0$, $m \in \mathcal{U}$. This shows that (18) holds for $n = k+1$. Now we can conclude that the inequality (18) holds for all $n \in \mathbb{N}_0$. Therefore, by (18), we obtain that

$$\begin{aligned}
\varepsilon_m^*(x,z) &= \sum_{n=0}^{\infty}(\Lambda_m^n\varepsilon_m)(x,z)\\
&\le \sum_{n=0}^{\infty}\frac{2}{a+b}\lambda_1(1+m)\lambda_2(-m)\alpha_m^n h_1(x,z)h_2(x,z)\\
&= \frac{2\lambda_1(1+m)\lambda_2(-m)h_1(x,z)h_2(x,z)}{(a+b)(1-\alpha_m)} < \infty
\end{aligned}$$

for all $x \in E_0, z \in Y_0$, $m \in \mathcal{U}$. Therefore, according to Theorem 1 with $\varphi = f$, we get that the limit

$$F_m(x) := \lim_{n\to\infty}\left(\mathcal{T}_m^n f\right)(x)$$

exists for each $x \in E_0$ and $m \in \mathcal{U}$, and

$$\left\|f(x) - F_m(x), z\right\| \le \frac{2\lambda_1(1+m)\lambda_2(-m)h_1(x,z)h_2(x,z)}{(a+b)(1-\alpha_m)}, \tag{19}$$

for all $x \in E_0, z \in Y_0$, $m \in \mathcal{U}$. To prove that F_m satisfies the functional equation (1), just prove the following inequality

$$\begin{aligned}
\Big\|\left(\mathcal{T}_m^n f\right)(ax+by) - \frac{(a+b)}{2}\left(\mathcal{T}_m^n f\right)(x+y) - \frac{(a-b)}{2}(\mathcal{T}_m^n f)(x-y), z\Big\|&\\
\le \alpha_m^n h_1(x,z)h_2(y,z)&
\end{aligned} \tag{20}$$

for every $x, y \in E_0$ such that $ax+by \ne 0, x+y \ne 0$ and $x-y \ne 0$, $z \in Y_0$, $n \in \mathbb{N}_0$, and $m \in \mathcal{U}$. Since the case $n = 0$ follows immediately from (10), take $k \in \mathbb{N}$ and assume that (20) holds for $n = k$ and every $x, y \in E_0$ such that $ax+by \ne 0, x+y \ne 0$ and $x-y \ne 0$, $z \in Y_0$, $m \in \mathcal{U}$. Then, for each $x, y \in E_0, z \in Y_0$ and $m \in \mathcal{U}$, we get

$$
\begin{aligned}
&\Big\|(\mathcal{T}_m^{k+1}f)(ax+by)-\frac{(a+b)}{2}(\mathcal{T}_m^{k+1}f)(x+y)-\frac{(a-b)}{2}(\mathcal{T}_m^{k+1}f)(x-y),z\Big\|\\
&=\Big\|\frac{2}{a+b}(\mathcal{T}_m^{k}f)f\left((a+am-bm)(ax+by)\right)-\frac{(a-b)}{a+b}(\mathcal{T}_m^{k}f)f\left((1+2m)(ax+by)\right)\\
&\quad-(\frac{2}{a+b})\frac{(a+b)}{2}(\mathcal{T}_m^{k}f)f\left((a+am-bm)(x+y)\right)\\
&\quad+(\frac{a-b}{a+b})\frac{(a+b)}{2}(\mathcal{T}_m^{k}f)f\left((1+2m)(x+y)\right)\\
&\quad-(\frac{2}{a+b})\frac{(a-b)}{2}(\mathcal{T}_m^{k}f)f\left((a+am-bm)(x-y)\right)\\
&\quad+(\frac{a-b}{a+b})\frac{(a-b)}{2}(\mathcal{T}_m^{k}f)f\left((1+2m)(x-y)\right),z\Big\|\\
&\leq\frac{2}{a+b}\Big\|(\mathcal{T}_m^{k}f)f\left((a+am-bm)(ax+by)\right)-\frac{(a+b)}{2}(\mathcal{T}_m^{k}f)f\left((a+am-bm)(x+y)\right)\\
&\quad-\frac{(a-b)}{2}(\mathcal{T}_m^{k}f)f\left((a+am-bm)(x-y)\right),z\Big\|+\frac{|a-b|}{a+b}\Big\|(\mathcal{T}_m^{k}f)f\left((1+2m)(ax+by)\right)\\
&\quad-\frac{(a+b)}{2}(\mathcal{T}_m^{k}f)f\left((1+2m)(x+y)\right)-\frac{(a-b)}{2}(\mathcal{T}_m^{k}f)f\left((1+2m)(x-y)\right),z\Big\|\\
&\leq\frac{2}{a+b}\alpha_m^k h_1\big((a+am-bm)x,z\big)h_2\big((a+am-bm)y,z\big)\\
&\quad+\frac{|a-b|}{a+b}\alpha_m^k h_1\big((1+2m)x,z\big)h_2\big((1+2m)y,z\big)\\
&\leq\alpha_m^{k+1}h_1(x,z)h_2(y,z).
\end{aligned}
$$

Thus, By using the mathematical induction, we have shown that (20) holds for every $x,y\in E_0$ such that $ax+by\neq 0, x+y\neq 0$ and $x-y\neq 0$, $z\in Y_0$, $n\in\mathbb{N}_0$, and $m\in\mathcal{U}$. Letting $n\to\infty$ in (20), we obtain the equality

$$F_m(ax+by)=\frac{a+b}{2}F_m(x+y)+\frac{a-b}{2}F_m(x-y), \tag{21}$$

for all $x,y\in E_0$ such that $ax+by\neq 0, x+y\neq 0$ and $x-y\neq 0, m\in\mathcal{U}$. This implies that $F_m:E_0\to Y$, defined in this way, is a solution of the equation

$$F(x)=\frac{2}{a+b}f((a+am-bm)x)-\frac{|a-b|}{a+b}f((1+2m)x),\quad x\in E_0, m\in\mathcal{U}. \tag{22}$$

Next, we will prove that each function $F:E\to Y$ satisfying the inequality

$$\big\|f(x)-F(x),z\big\|\leq L\ h_1(x,z)h_2(x,z),\quad x\in E_0, z\in Y_0 \tag{23}$$

with some $L>0$, is equal to F_m for each $m\in\mathcal{U}$. To this end, we fix $m_0\in\mathcal{U}$ and $F:E\to Y$ satisfying (23). From (19), for each $x\in E_0$, we get

$$
\begin{aligned}
\big\|F(x)-F_{m_0}(x),z\big\| &\leq \big\|F(x)-f(x),z\big\|+\big\|f(x)-F_{m_0}(x),z\big\|\\
&\leq L\ h_1(x,z)h_2(x,z)+\varepsilon^*_{m_0}(x,z)\\
&\leq L_0\ h_1(x,z)h_2(x,z)\sum_{n=0}^{\infty}\alpha_{m_0}^n,
\end{aligned} \tag{24}
$$

where $L_0 := (1-\alpha_{m_0})L + \frac{2}{a+b}\lambda_1(1+m_0)\lambda_2(-m_0) > 0$ and we exclude the case that $h_1(x,z) \equiv 0$ or $h_2(x,z) \equiv 0$ which is trivial. Observe that F and F_{m_0} are solutions to equation (22) for all $m \in \mathcal{U}$. Now, we will see that, for any $j \in \mathbb{N}_0$,

$$\Big\| F(x) - F_{m_0}(x), z \Big\| \le L_0\ h_1(x,z)h_2(x,z)\sum_{n=j}^{\infty}\alpha_{m_0}^n, \quad x \in E_0, z \in Y_0. \tag{25}$$

The case $j = 0$ is exactly (24). Fix a $k \in \mathbb{N}$ and assume that (25) holds for $j = k$. Then, in view of (24), for each $x \in E_0, z \in Y_0$, we get

$$\begin{aligned}
\big\|F(x) - F_{m_0}(x), z\big\| &= \Big\| \frac{2}{a+b}F((a+am_0-bm_0)x) - \frac{(a-b)}{a+b}F((1+2m_0)x) \\
&\quad - \frac{2}{a+b}F_{m_0}((a+am_0-bm_0)x) + \frac{(a-b)}{a+b}F_{m_0}((1+2m_0)x), z\Big\| \\
&\le \frac{2}{a+b}\big\|F((a+am_0-bm_0)x) - F_{m_0}((a+am_0-bm_0)x), z\big\| \\
&\quad + \frac{|a-b|}{a+b}\big\|F((1+2m_0)x) - F_{m_0}((1+2m_0)x), z\big\| \\
&\le \frac{2}{a+b}L_0\ h_1((a+am_0-bm_0)x,z)h_2((a+am_0-bm_0)x,z)\sum_{n=k}^{\infty}\alpha_{m_0}^n \\
&\quad + \frac{|a-b|}{a+b}L_0\ h_1((1+2m_0)x,z)h_2((1+2m_0)x,z)\sum_{n=k}^{\infty}\alpha_{m_0}^n \\
&= L_0\ \Big(\frac{2}{a+b}h_1((a+am_0-bm_0)x,z)h_2((a+am_0-bm_0)x,z) \\
&\quad + \frac{|a-b|}{a+b}h_1((1+2m_0)x,z)h_2((1+2m_0)x,z)\Big)\sum_{n=k}^{\infty}\alpha_{m_0}^n \\
&\le L_0\ \alpha_{m_0}h_1(x,z)h_2(x,z)\sum_{n=k}^{\infty}\alpha_{m_0}^n \\
&= L_0\ h_1(x,z)h_2(x,z)\sum_{n=k+1}^{\infty}\alpha_{m_0}^n.
\end{aligned}$$

This shows that (25) holds for $j = k+1$. Now we can conclude that the inequality (25) holds for all $j \in \mathbb{N}_0$. Now, letting $j \to \infty$ in (25), we get

$$F = F_{m_0}. \tag{26}$$

Thus, we have also proved that $F_m = F_{m_0}$ for each $m \in \mathcal{U}$, which (in view of (19)) yields

$$\big\|f(x) - F_{m_0}(x), z\big\| \le \frac{2\lambda_1(1+m)\lambda_2(-m)h_1(x,z)h_2(x,z)}{(a+b)(1-\alpha_m)}, \ x \in E_0, z \in Y_0, m \in \mathcal{U}. \tag{27}$$

This implies (11) with $F = F_{m_0}$ and (26) confirms the uniqueness of F.
The following theorem concerns the η-hyperstability of (1) in 2-Banach spaces. Namely, We consider functions $f : E \to Y$ fulfilling (1) approximately, i.e., satisfying the inequality

$$\left\| f(ax+by) - \frac{(a+b)}{2}f(x+y) - \frac{(a-b)}{2}f(x-y), z\right\| \le \eta(x,y,z), \tag{28}$$

for all $x, y \in E_0$ such that $ax + by \neq 0, x + y \neq 0$ and $x - y \neq 0, z \in Y_0$, with $\eta : E_0 \times E_0 \times Y_0 \to \mathbb{R}_+$ is a given mapping. Then we find a unique function $F : E \to Y$ which is close to f. Then, under some additional assumptions on η, we prove that the conditional functional equation (1) is η-hyperstable in the class of functions $f : E \to Y$, i.e., each $f : E \to Y$ satisfying inequality (28), with such η, must fulfil equation (1).

Theorem 3 Let h_1, h_2 and $\mathcal{U}$ be as in Theorem 2. Assume that

$$\begin{cases} \lim_{n\to\infty} \lambda_1(1+n)\lambda_2(-n) = 0, \\ \lim_{n\to\infty} \lambda_1(1+2n)\lambda_2(1+2n) = 0. \end{cases} \tag{29}$$

Then every $f : E \to Y$ satisfying (10) is a solution of (1).

Proof. Suppose that $f : E \to Y$ satisfies (10). Then, by Theorem 2, there exists a mapping $F : E \to Y$ satisfies (1) and

$$\|f(x) - F(x), z\| \leq \lambda_0 h_1(x, z) h_2(x, z) \tag{30}$$

for all $x \in E_0, z \in Y_0$, where

$$\lambda_0 := \inf_{n\in\mathcal{U}} \left\{ \frac{2\lambda_1(1+n)\lambda_2(-n)}{(a+b)(1 - \frac{2}{a+b}\lambda_1(a+an-bn)\lambda_2(a+an-bn) - \frac{|a-b|}{a+b}\lambda_1(1+2n)\lambda_2(1+2n))} \right\}.$$

Since, in view of (29), $\lambda_0 = 0$. This means that $f(x) = F(x)$ for all $x \in E_0$, whence

$$f(ax + by) = \frac{(a+b)}{2} f(x+y) + \frac{(a-b)}{2} f(x-y),$$

for all $x, y \in E_0$ such that $ax + by \neq 0, x + y \neq 0$ and $x - y \neq 0$, which implies that f satisfies the functional equation (1) on E_0.

Corollary 1 Let $\theta \geq 0$, $s \geq 0$, $p, q \in \mathbb{R}$ such that $p + q < 0$. Suppose that $f : E \to Y$ such that $f(0) = 0$ satisfy the inequality

$$\left\| f(ax+by) - \frac{(a+b)}{2} f(x+y) - \frac{(a-b)}{2} f(x-y), z \right\| \leq \theta |x|^p \ |y|^q \ |z|^s \ , \tag{31}$$

for all $x, y \in E_0$ such that $ax + by \neq 0, x + y \neq 0$ and $x - y \neq 0$, $z \in Y_0$. Then f satisfies the functional equation (1) on E.

Proof. The proof follows from Theorem 2 defining

$h_1, h_2 : E_0 \times Y_0 \to \mathbb{R}_+$ by $h_1(x, z) = \theta_1 |x|^p |z|^{s_1}$, $h_2(y, z) = \theta_2 |y|^q |z|^{s_2}$, and $h_1(0, z) = h_2(0, z) = 0$ with $\theta_1, \theta_2 \in \mathbb{R}_+$, $s_1, s_2 \in \mathbb{R}_+$ and $p, q \in \mathbb{R}$ such that $\theta_1\theta_2 = \theta$, $s_1 + s_2 = s$ and $p + q < 0$.

For each $n \in \mathbb{N}$, we get

$$\begin{aligned} \lambda_1(n) &= \inf\{t \in \mathbb{R}_+ : h_1(nx, z) \leq t\, h_1(x, z), \ \ x \in E_0, z \in Y_0\} \\ &= \inf\{t \in \mathbb{R}_+ : \theta_1 |nx|^p |z|^{s_1} \leq t\, \theta_1 |x|^p |z|^{s_1}, \ \ x \in E_0, z \in Y_0\} \\ &= n^p. \end{aligned}$$

Also, we have $\lambda_2(n) = n^q$ for all $n \in \mathbb{N}$. Clearly, we can find $n_0 \in \mathbb{N}$ such that

$$\lambda_1(a+an-bn)\lambda_2(a+an-bn)+\lambda_1(1+2n)\lambda_2(1+2n) = (a+an-bn)^{p+q}+(1+2n)^{p+q} < 1, \quad n \geq n_0. \tag{32}$$

According to Theorem 2, there exists a unique function $F : E \to Y$ such that

$$\left\| f(x) - F(x), z \right\| \leq \theta \lambda_0 h_1(x,z) h_2(x,z) \tag{33}$$

for all $x \in E_0, z \in Y_0$, where

$$\lambda_0 := \inf_{n \in \mathcal{U}} \left\{ \frac{2\lambda_1(1+n)\lambda_2(-n)}{(a+b)(1 - \frac{2}{a+b}\lambda_1(a+an-bn)\lambda_2(a+an-bn) - \frac{|a-b|}{a+b}\lambda_1(1+2n)\lambda_2(1+2n))} \right\}.$$

On the other hand, Since $p+q<0$, It is sufficient to consider that $p+q<0$. Then

$$\begin{cases} \lim_{n\to\infty} \lambda_1(1+n)\lambda_2(-n) = \lim_{n\to\infty}(1+n)^p(-n)^q = 0, \\ \lim_{n\to\infty} \lambda_1(1+2n)\lambda_2(1+2n) = \lim_{n\to\infty}(1+2n)^{p+q} = 0. \end{cases} \tag{34}$$

Thus by Theorem 2, we get the desired results. The next corollary prove the hyperstability results for the inhomogeneous functional equation.

Corollary 2 Let $\theta, p, q, s \in \mathbb{R}$ such that $\theta \geq 0$ and $p+q<0$. Assume that $G : E^2 \to Y$ and $f : E \to Y$ such that $f(0) = 0$ and satisfy the inequality

$$\left\| f(ax+by) - \frac{(a+b)}{2} f(x+y) - \frac{(a-b)}{2} f(x-y) - G(x,y), z \right\| \leq \theta |x|^p \ |y|^q \ |z|^s \ , \tag{35}$$

for all $x, y \in E_0$ such that $ax+by \neq 0, x+y \neq 0$ and $x-y \neq 0, z \in Y_0$. If the functional equation

$$f(ax+by) = \frac{(a+b)}{2} f(x+y) + \frac{(a-b)}{2} f(x-y) + G(x,y), \tag{36}$$

for all $x, y \in E_0$ such that $ax+by \neq 0, x+y \neq 0$ and $x-y \neq 0$, has a solution $f_0 : E \to Y$, then f is a solution to (36).

Proof. From (35) we get that the function $K : E \to Y$ defined by $K := f - f_0$ satisfies (31). Consequently, Corollary 2 implies that K is a solution to functional equation (1). Therefore,

$$\begin{aligned} f(ax+by) - \frac{(a+b)}{2} f(x+y) - \frac{(a-b)}{2} f(x-y) - G(x,y) = K(ax+by) \\ + f_0(ax+by) - \frac{(a+b)}{2} K(x+y) - \frac{(a+b)}{2} f_0(x+y) \\ - \frac{(a-b)}{2} K(x-y) - \frac{(a-b)}{2} f_0(x-y) - G(x,y) \\ = 0, \end{aligned}$$

for all $x, y, z \in E_0$ such that $ax+by \neq 0, x+y \neq 0$ and $x-y \neq 0$, which means f is a solution to (36).

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