Electronic Journal of Mathematical Analysis and Applications
Vol. 7(2) July 2019, pp. 267-283.
ISSN: 2090-729X(online)
http://math-frac.org/Journals/EJMAA/

GLOBAL ATTRACTING SET AND STABILITY FOR STOCHASTIC INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. This article is concerned with existence, global attracting set and stability of mild solutions for a class of neutral stochastic integro-differential equations. The partial differential equations (PDEs) are driven by a fractional Brownian motion with Hurst index $H \in (\frac{1}{2}, 1)$ in Hilbert spaces. Existence theorems are proved via Banach fixed point theorem and resolvent operator theory for integro-differential equations. The global attracting set is obtained by integral inequalities. In addition, sufficient conditions for exponentially stability in mean square of the mild solution are presented.

1. Introduction

Fractional Brownian motion (fBm) was originally defined and studied by Kolmogorov [25] within a Hilbert space framework and has become a very important tool in modern probability and statistical modeling. FBm is a family of centered Gaussian processes with continuous sample paths indexed by the Hurst parameter $H \in (0, 1)$. Often fBm is equivalently presented as an integral of a deterministic kernel with respect to an ordinary Brownian motion. In fact, there exist at least two such kernels: Mandelbrot-Van Ness kernel with infinite support and Molchan-Golosov kernel with compact support. FBm is a self-similar process with stationary increments and has a long-memory when $H > \frac{1}{2}$. For $H < \frac{1}{2}$ the increments are negatively correlated and for $H = \frac{1}{2}$ the increments are independent i.e., the ordinary Brownian motion case. These significant properties make fBm a natural candidate as a model for noise in a wide variety of physical phenomena, such as mathematical finance, communication networks (see [9, 34] for example). Despite of all these properties, when $H \neq \frac{1}{2}$ fBm is neither semi-martingale nor a Markov process. Hence the traditional tools of Itô stochastic calculus can not be applied effectively in studying solution of equations driven by fBm. Therefore, it is crucial to study stochastic analysis with respect to fBm and related problems. The following is a good set of relevant references: Biagini

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2010 *Mathematics Subject Classification.* 60G18, 60H15, 34K40, 60J65.
Key words and phrases. Resolvent operator, Neutral stochastic integro-differential equations, Global attracting set, Fractional Brownian motion.
Submitted Nov. 13, 2018.

et al. [1] as well as Mishura [32] and the references [3, 10]. On the other hand, there has been intense interest in neutral stochastic partial differential equations and integro-differential equations with resolvent operators. Such interest has created an active research area with applications in physics, chemistry, biology, medicine, economics, etc. ([17, 19, 20, 21, 22, 28] for example). Moreover, the existence, global attracting set and stability of solutions for neutral stochastic partial functional differential equation in infinite dimensional spaces with delays have been extensively studied ([6, 23, 24, 29, 30, 31]). However, results on stochastic PDEs and integro-differential equations with delay driven by fBm are scarce. Ferrante and Rovira [11] established existence and uniqueness of solutions to delayed stochastic differential equations with fBm for $H > \frac{1}{2}$. Caraballo el al. [5] studied existence and exponential behavior of solutions to stochastic delay evolution equations with an fBm. Boufoussi and Hajji [2] presented existence, uniqueness and stability analysis of solutions for neutral stochastic functional differential equations driven by a fBm in a Hilbert space. More recently, Caraballo and Diop [4] investigated the existence and uniqueness of solutions to neutral stochastic delay partial functional integro-differential equations driven by an fBm with Hurst index $H > \frac{1}{2}$. Their results extended similar ones in [2] to neutral integro-differential type equation. Our objective is to prove similar theorems regarding existence, global attracting set and stability of mild solutions for a class of neutral stochastic integro-differential equations under fBm in Hilbert space:

$$\begin{cases} d[x(t) + f(t, x(t - r(t)))] \\ \qquad = A[x(t) + \int_0^t K(t-s)x(s)ds]dt + g(t, x(t - \rho(t)))dt + h(t)dB^H(t) \\ \qquad t \in J := [0, T] \\ x_0(t) = \varphi(t) \in \mathbb{C}_0^{\mathcal{F}_0}([-\tau, 0], \mathcal{L}^2(\Omega, \mathbb{H})), \quad -\tau \le t \le 0, \quad \tau > 0, a.s., \end{cases} \tag{1.1}$$

where the state $x(\cdot)$ takes values in a separable real Hilbert space $\mathbb{H}$ and $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ is the infinitesimal generator of an resolvent family $\{R(t)\}_{t\geq 0}$. We note that $K(t) : D(K(t)) \subset \mathbb{H} \to \mathbb{H}$ is a bounded linear operator; B^H is a fractional Brownian motion in a separable real Hilbert space $\mathbb{K}$; and $r, \rho : J \to [0, \tau]$ are continuous functions. The mappings $f, g : J \times \mathbb{H} \to \mathbb{H}$, $h : J \to \mathcal{L}_2^0$ are appropriate functions to be specified later. We study existence, global attracting set and stability of mild solution of (1.1) via fixed point theorem and theory of analytic resolvent operators for integro-differential equations. We primarily rely on techniques using strongly continuous family of operators $\{R(t), t \geq 0\}$ defined on the Hilbert space $\mathbb{H}$. The resolvent operator is similar to the semi-group operator for abstract differential equations in Banach spaces. Since the resolvent operator does not satisfy all semi-group properties (see [7, 27]), we attempt to apply the theory of analytic resolvent operators proposed by Grimmer [14], Grimmer and Pritchard [15].

In Section 2, we recall briefly the notations, concepts and basic results about the fBm and deterministic integro-differential equations. In Section 3, we present the main results on existence, global attracting set and stability of mild solutions for (1.1).

2. Preliminaries

This section presents some basic concepts, notations, relevant definitions and lemmas. For more details, see [1, 14, 15, 32, 33].

Let $(\mathbb{H}, \|\cdot\|_{\mathbb{H}}, \langle\cdot,\cdot\rangle)$ and $(\mathbb{K}, \|\cdot\|_{\mathbb{K}}, \langle\cdot,\cdot\rangle)$ be two real separable Hilbert spaces, with their vector norms and inner products respectively. We denote by $\mathcal{L}(\mathbb{K};\mathbb{H})$ the set of all linear bounded operators from $\mathbb{K}$ into $\mathbb{H}$ under the usual operator norm $\|\cdot\|$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in J}, \mathbf{P})$ be a complete filtered probability space satisfying the usual condition (i.e., it is continuous from the right and $\mathcal{F}_0$ contains all $\mathbf{P}$-null sets). Denote $\{B^H(t)\}_{t\in J}$ an fBm to the filtration $\{\mathcal{F}_t\}_{t\in J}$.

Definition 2.1. *An one-dimensional fBm with Hurst parameter $H \in (0,1)$ is a centered Gaussian process $\beta^H = \{\beta^H(t)\}_{t\in J}$ with covariance function*

$$R(t,s) = \mathbf{E}[\beta^H(t)\beta^H(s)] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}).$$

We note that $\beta^{\frac{1}{2}}$ is a standard Brownian motion. It is known that $\beta^H(t)$ with $H \in (\frac{1}{2}, 1)$ has the following Volterra representation:

$$\beta^H(t) = \int_0^t K_H(t,s)d\beta(s), \tag{2.1}$$

where $\beta = \{\beta(t)\}_{t\in J}$ is a Wiener process and the Volterra kernel $K_H(t,s)$ is given by

$$K_H(t,s) = c_H s^{\frac{1}{2}-H} \int_0^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du$$

where

$$c_H = \sqrt{\frac{H(2H-1)}{B(2-2H, H-\frac{1}{2})}}$$

with $B(\cdot,\cdot)$ being the Beta function for $t > s$. We put $K_H(t,s) = 0$ if $t \leq s$.

For the deterministic function $\varphi \in \mathcal{L}^2(J)$, the fractional Wiener integral of φ with respect to β^H is defined by

$$\int_J \varphi(s)d\beta^H(s) = \int_J K_H^*\varphi(s)d\beta(s),$$

where

$$K_H^*\varphi(s) = \int_s^T \varphi(r)\frac{\partial K}{\partial r}(r,s)dr.$$

We assume that there exists a complete orthonormal system $\{e_k\}_{k\geq 1}$ in $\mathbb{K}$, a sequence of nonnegative real numbers λ_k such that $Qe_k = \lambda_k e_k, k = 1, 2, ...$, where $Q \in \mathcal{L}(\mathbb{K};\mathbb{H})$ with finite trace $tr(Q) = \sum_{k=1}^{\infty} \lambda_k < \infty$. We define the infinite dimensional fractional Brownian motion on $\mathbb{K}$ with covariance Q as

$$B^H(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} e_k \beta_k^H(t),$$

where β_k^H are real, independent fBm's. This process is a $\mathbb{K}$-valued Gaussian starting from 0 with zero mean and covariance:

$$\mathbf{E}\langle B^H(t), x\rangle\langle B^H(s), y\rangle = R(t,s)\langle Q(x), y\rangle \quad \forall x, y \in \mathbb{K}, \quad \forall t, s \in J.$$

Let $\mathcal{L}_2^0 = \mathcal{L}_2(Q^{\frac{1}{2}}\mathbb{K};\mathbb{H})$ be the space of all Hilbert-Schmidt operators from $Q^{\frac{1}{2}}\mathbb{K}$ into $\mathbb{H}$ with the inner product $\langle a, b\rangle_{\mathcal{L}_2^0} = Tr[aQb^*]$, where b^* is the adjoint of the operator b.

Definition 2.2. *The fractional Wiener integral of $\Psi : J \to \mathcal{L}_2^0$ with respect to Q-fBm is defined by*

$$\begin{aligned} \int_0^t \Psi(s)dB^H(s) &= \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n}\Psi(s)e_n d\beta_n^H(s) \\ &= \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n} K_H^*(\Psi e_n) d\beta_n(s) \end{aligned} \tag{2.2}$$

where β_n is the standard Brownian motion used to present β_n^H as in (2.1).

We have the following inequality which is instrumental to prove our results.

Lemma 2.1. *([2], Lemma 2) If $\Psi : J \to \mathcal{L}_2^0$ satisfies $\int_J \|\Psi(s)\|^2_{\mathcal{L}_2^0} ds < \infty$ then the above sum in (2.2) is well defined as a $\mathbb{H}$-valued random variable, and we have*

$$\mathbf{E}\Big\|\Psi(s)dB^H(s)\Big\|^2 \le 2Ht^{2H-1}\int_0^t \|\Psi(s)\|^2_{\mathcal{L}_2^0} ds.$$

To access existence, global attracting set and stability of mild solutions for (1.1), we need to introduce partial integro-differential equations and resolvent operators.

Let X, Z be two Banach spaces such that $\|z\|_Z := \|Az\|_X + \|z\|_X$ for all $z \in Z$; A and $K(t)$ be closed linear operators on X and satisfy the following assumptions:
(**H**1) The operator $A : D(A) \subseteq X \to X$ is the infinitesimal generator of a strongly continuous semigroup on X.
(**H**2) For all $t \ge 0$, $K(t) : D(K(t)) \subseteq X \to X$ is a closed linear operator, $D(A) \subseteq D(K(t))$, and $K(t) \in \mathcal{L}(Z, X)$. For any $z \in Z$, the map $t \to K(t)z$ is bounded, differentiable and the derivative $t \to \frac{dK(t)z}{dt}$ is uniformly continuous and bounded on J.

Remark 2.1. *We observe that in many applications $K(\cdot)$ is a scalar or an appropriate matrix, so (**H**2) is satisfied there.*

By Theorem 2.3 in [14], we see that (**H**1) and (**H**2) imply the following integro-differential abstract Cauchy problem

$$\frac{dx(t)}{dt} = A\Big[x(t) + \int_0^t K(t-s)x(s)ds\Big], \qquad x(0) = x_0 \in X, \tag{2.3}$$

has an associated resolvent operator of bounded linear operators $(R(t))_{t\in J}$ on X.

Definition 2.3. *A one-parameter family of bounded linear operator* $(R(t))_{t\in J}$ *on* X *is called a resolvent operator of* (2.3) *if the following conditions are satisfied.*
(a) $R(0) = I$ *(the identity operator on* X*);*
(b) For all $u \in X$, $R(t)u$ *is continuous for* $t \in J$*;*
(c) $R(t) \in \mathcal{L}(Z)$, $t \in J$. *For* $x \in D(A)$, $R(\cdot)x \in \mathsf{C}(J; D(A)) \cap \mathsf{C}^1(J; X)$, *and*

$$\begin{aligned}\frac{dR(t)x}{dt} &= A\Big[R(t)x + \int_0^t K(t-s)R(s)xds\Big],\\ \frac{dR(t)x}{dt} &= R(t)Ax + \int_0^t R(t-s)AK(s)xds, \quad for \quad t \in J.\end{aligned}$$

Motivated by Grimmer [14], we can give the mild solution for the integro-differential equation

$$\frac{dx(t)}{dt} = A\Big[x(t) + \int_0^t K(t-s)x(s)ds\Big] + \kappa(t), \qquad x(0) = x_0 \in X$$

if x satisfies the following variation of constants formula:

$$x(t) = R(t)x_0 + \int_0^t R(t-s)\kappa(s)ds, \quad for \quad t \in J,$$

where $\kappa : J \to X$ is a continuous function.

If $0 \in \rho(A)$ (the resolvent set of operator A), then it is possible to define the fractional power A^α, for $0 < \alpha \le 1$, as a closed linear operator on its domain $D(A^\alpha)$. Furthermore, the subspace $D(A^\alpha)$ is dense in X and the expression $\|x\|_\alpha = \|A^\alpha x\|_\alpha, \quad x \in D(A^\alpha)$ defines a norm in $D(A^\alpha)$. Denote the space $(D(A^\alpha), \|\cdot\|_\alpha)$ by $\mathbb{H}_\alpha$, then the following properties are well known (Pazy [33]).

Lemma 2.2. *Under the above conditions, the following properties hold:*
(a) If $A^\alpha : \mathbb{H}_\alpha \to \mathbb{H}_\alpha$, *then* $\mathbb{H}_\alpha$ *is a Banach space for* $0 < \alpha \le 1$*;*
(b) If the resolvent operator of A *is compact, then the injection* $\mathbb{H}_\beta \hookrightarrow \mathbb{H}_\alpha$ *is continuous and compact for* $0 < \alpha \le \beta$*;*
(c) For every $0 < \alpha \le 1$ *there exists* $M_\alpha > 0$ *such that*

$$\|A^\alpha R(t)\| \le M_\alpha t^{-\alpha} e^{-\lambda t}, \quad t > 0, \lambda > 0.$$

In other words, for the resolvent operator $R(t)$ we have the following property.

Lemma 2.3. *(Lemma 2.2* [12]*)* $AR(t)$ *is continuous for* $t > 0$ *in the uniform operator topology of* $\mathcal{L}(X)$.

We need the following lemma by Govindan [13].

Lemma 2.4. *(Lemma 4.1*[13]*) Let* A *be the infinitesimal generator of an analytic semigroup of bounded linear operators* $\{R(t), t \ge 0\}$ *in* X. *For any stochastic process* $F : \mathbb{R}^+ \to X$ *which is strongly measurable with* $\int_0^T \mathbf{E}\|A^\alpha F(t)\|^2 dt < \infty$, $T \in (0, \infty)$, *the following inequality holds:*

$$\mathbf{E}\Big\|\int_0^t AR(t-s)F(s)ds\Big\|^2 \le \frac{M_{1-\alpha}^2 \Gamma(2\alpha-1)}{\lambda^{2\alpha-1}} \int_0^t e^{-\lambda(t-s)}\mathbf{E}\|A^\alpha F(t)\|^2 ds,$$

provided that $\alpha \in (\frac{1}{2}, 1)$, *where* $\Gamma(\cdot)$ *is the Gamma function.*

Definition 2.4. *Denote the space* $\mathbb{C}_T := \mathcal{C}([-\tau, T], \mathcal{L}^2(\Omega, \mathbb{H}))$*-formed by all continuous functions from* $[-\tau, T]$ *into* $\mathcal{L}^2(\Omega, \mathbb{H}))$ *such that for all* $x \in \mathbb{C}_T$,

$$\|x\|^2_{\mathbb{C}_T} := \sup_{t\in[-\tau,T]} \mathbf{E}\|x(t)\|^2.$$

Then $\mathbb{C}_T$ *with the above norm is a Banach space.*

Let $\mathbb{C}_0^{\mathcal{F}_0}([-\tau, 0], \mathcal{L}^2(\Omega, \mathbb{H}))$, $(\mathbb{C}_0^{\mathcal{F}_t}([-\tau, 0], \mathcal{L}^2(\Omega, \mathbb{H})))$ denote the family of all bounded $\mathcal{F}_0$ $(\mathcal{F}_t)$-measurable, $\mathbb{C}_0$-valued random variables φ, satisfying $\|\varphi\|^2_{\mathbb{C}_0} < \infty$. Motivated by Long [29], we offer the following definition.

Definition 2.5. *The set* $G \subset \mathcal{L}^2(\Omega, \mathbb{H})$ *is called a global attracting set of (1.1), if for any initial value* $\varphi \in \mathbb{C}_0^{\mathcal{F}_0}([-\tau, 0], \mathcal{L}^2(\Omega, \mathbb{H}))$, *the solution* $x_t(0, \varphi)$ *converges to* G *as* $t \to \infty$. *That is,*

$$dist\big(x_t(0, \varphi), G\big) \xrightarrow{t \to +\infty} 0,$$

where

$$dist(\phi, G) = \inf_{\psi\in G} \sup_{s\in[-\tau,0]} \mathbf{E}(\|\phi(s) - \psi(s)\|)$$

for $\phi \in \mathbb{C}_0^{\mathcal{F}_0}([-\tau, 0], \mathcal{L}^2(\Omega, \mathbb{H}))$.

Definition 2.6. *The mild solution of (1.1) is said to be exponentially stable in mean square if there exists a pair of positive constants* $\lambda > 0$ *and* $M \geq 1$ *such that for any solution* $x_t(0, \varphi)$ *with the initial value* $\varphi \in \mathbb{C}_0^{\mathcal{F}_0}([-\tau, 0], \mathcal{L}^2(\Omega, \mathbb{H}))$,

$$\mathbf{E}\|x_t(0, \varphi)\|^2 \leq M\|\varphi\|^2_{\mathbb{C}_0} e^{-\lambda t}, \quad t \geq 0.$$

The following is the definition of a mild solution for (1.1).

Definition 2.7. *An* $\mathcal{F}_t$*-adapted stochastic process* $x : [-\tau, T] \to \mathbb{H}$, $0 < T < +\infty$ *is called a mild solution of (1.1) on* $[-\tau, T]$ *if* $x_0(\cdot) = \varphi \in \mathbb{C}_0^{\mathcal{F}_0}([-\tau, 0], \mathcal{L}^2(\Omega, \mathbb{H}))$ *on* $[-\tau, 0]$ *a.s., and for each* $s \in [0, T)$ *the function* $AR(t - s)f(s, x(s - r(s)))$ *is integrable such that:*
(i) $x(\cdot) \in \mathcal{C}([-\tau, T], \mathcal{L}^2(\Omega, \mathbb{H}))$;
(ii) For arbitrary $t \in J$, $x(t)$ *satisfies the following integral equation:*

$$\begin{aligned} x(t) = R(t)[\varphi(0) + f(0, \varphi(-r(0)))] - f(t, x(t - r(t))) \\ - \int_0^t AR(t - s)f(s, x(s - r(s)))ds \\ - \int_0^t AR(t - s) \int_0^s K(s - \xi)f(\xi, x(\xi - r(\xi)))d\xi ds \\ + \int_0^t R(t - s)g(s, x(s - \rho(s)))ds + \int_0^t R(t - s)h(s)dB^H(s), \quad \mathbf{P} - a.s.. \end{aligned} \tag{2.4}$$

For existence, global attracting set and stability of the mild solution to (1.1), we impose the following assumptions throughout.

(**H**3) There exist positive constants C, M_k, $M_{1-\alpha}$ such that for all $t \in J$,

(i) $\|R(t)\| \leq C$, $\|K(t)\| \leq M_k$;

(ii) $\|A^{1-\alpha}R(t)\| \leq \frac{M_{1-\alpha}}{t^{1-\alpha}}$.

(**H**3*) There exist positive constants M_k, λ, M such that for all $t \in J$,

(i) $\|K(t)\| \leq M_k$;

(ii) $\|R(t)\| \leq Me^{-\lambda t}$.

(**H**4) For $f : J \times \mathbb{H} \to \mathbb{H}$, there exist constants $\alpha \in (\frac{1}{2}, 1)$, M_f, $L_f > 0$ such that the function $f(\cdot)$ is $\mathbb{H}_\alpha$-valued and satisfies for all $t \in J$, $\nu_1, \nu_2 \in \mathbb{H}$,

(i) $\|A^\alpha f(t, \nu_1) - A^\alpha f(t, \nu_2)\| \leq M_f\|\nu_1 - \nu_2\|$;

(ii) $\|A^\alpha f(t, \nu_1)\|^2 \leq L_f(\|\nu_1\|^2 + 1)$.

(**H**5) The function $g : J \times \mathbb{H} \to \mathbb{H}$ satisfies the following conditions: there exist positive constants M_g, L_g such that for all $t \in J$, $\nu_1, \nu_2 \in \mathbb{H}$,

(i) $\|g(t, \nu_1) - g(t, \nu_2)\| \leq M_g\|\nu_1 - \nu_2\|$;

(ii) $\|g(t, \nu_1)\|^2 \leq L_g(\|\nu_1\|^2 + 1)$.

(**H**6) The function $A^\alpha f$ is continuous in the quadratic mean sense: for all $\nu \in \mathcal{C}(J, \mathcal{L}^2(\Omega, \mathbb{H}))$,

$$\lim_{t \to s} \mathbf{E}\|A^\alpha f(t, \nu(t)) - A^\alpha f(s, \nu(s))\|^2 = 0.$$

(**H**7) The function $h : J \to \mathcal{L}_2^0$ satisfies the following conditions:

(i) $\int_0^T \|h(s)\|^2_{\mathcal{L}_2^0} < \infty$;

(ii) $\int_0^T e^{\lambda s}\|h(s)\|^2_{\mathcal{L}_2^0} < \infty$, for some $\lambda > 0$.

3. Main Theorems

In this section, we prove theorems for existence, global attracting set and stability of mild solutions for a class of neutral stochastic integro-differential equations under fractional Brownian motion.

We first obtain the following existence theorem.

Theorem 3.1. *Assume that* (**H1**) – (**H7**)(i) *hold. If*

$$4\Big[M_f^2\Big(\|A^{-\alpha}\|^2 + \frac{T^{2\alpha+1}M_{1-\alpha}^2}{2\alpha - 1}[1 + TM_k^2]\Big) + T^2M_g^2C^2\Big] < 1, \tag{3.1}$$

then there exists a unique mild solution to (1.1) on $[-\tau, T]$.

Proof. We consider the space $\Upsilon_T = \{x \in \mathbb{C}_T : x(s) = \varphi(s), s \in [-\tau, 0]\}$. Υ_T is a closed subset of $\mathbb{C}_T$ endowed with the norm $\|\cdot\|_{\mathbb{C}_T}$ and define the operator $\Pi : \Upsilon_T \to \Upsilon_T$ by $(\Pi x)(t) = \varphi(t)$ for $t \in [-\tau, 0]$ and

$$\begin{aligned}(\Pi x)(t) =& R(t)[\varphi(0) + f(0, \varphi(-r(0)))] - f(t, x(t - r(t))) \\ &- \int_0^t AR(t - s)f(s, x(s - r(s)))ds\end{aligned}$$

$$
\begin{aligned}
&-\int_0^t AR(t-s)\int_0^s K(s-\xi)f(\xi,x(\xi-r(\xi)))d\xi ds \\
&+\int_0^t R(t-s)g(s,x(s-\rho(s)))ds+\int_0^t R(t-s)h(s)dB^H(s), \quad t\in J,
\end{aligned}
$$

It suffices to show that the operator Π has a unique fixed point in Υ_T. For this purpose, we use the Banach fixed point theorem and prove the existence by two steps.

Step 1. Π_2 is continuous on J in the $\mathcal{L}^2(\Omega,\mathbb{H})$-sense. Let $t\in(0,T)$ and $|\epsilon|$ be sufficiently small. Then for any fixed $x\in\mathbb{C}_T$, we have

$$
\begin{aligned}
&\big\|(\Pi x)(t+\epsilon)-(\Pi x)(t)\big\| \\
&\leq \big\|R(t+\epsilon)-R(t)[\varphi(0)+f(0,\varphi(-r(0)))]\big\| \\
&+\big\|f(t+\epsilon,x(t+\epsilon-r(t+\epsilon)))-f(t,x(t-r(t)))\big\| \\
&+\Big\|\int_0^{t+\epsilon}AR(t+\epsilon-s)f(s,x(s-r(s)))ds-\int_0^t AR(t-s)f(s,x(s-r(s)))ds\Big\| \\
&+\Big\|\int_0^{t+\epsilon}AR(t+\epsilon-s)\int_0^s K(s-\xi)f(\xi,x(\xi-r(\xi)))d\xi ds \\
&-\int_0^t AR(t-s)\int_0^s K(s-\xi)f(\xi,x(\xi-r(\xi)))d\xi ds\Big\| \\
&+\Big\|\int_0^{t+\epsilon}R(t+\epsilon-s)g(s,x(s-\rho(s)))ds-\int_0^t R(t-s)g(s,x(s-\rho(s)))ds\Big\| \\
&+\Big\|\int_0^{t+\epsilon}R(t+\epsilon-s)h(s)dB^H(s)-\int_0^t R(t-s)h(s)dB^H(s)\Big\| \\
&:=\sum_{k=1}^{6}Q_k(\epsilon).
\end{aligned}
$$

From strong continuity of $R(t)$, we conclude

$$\lim_{\epsilon\to 0}[R(t+\epsilon)-R(t)][\varphi(0)+f(0,\varphi(-r(0)))]=0.$$

By (**H3**) we obtain

$$\|[R(t+\epsilon)-R(t)][\varphi(0)+f(0,\varphi(-r(0)))]\|\leq 2C\|\varphi(0)+f(0,\varphi(-r(0)))\|\in\mathcal{L}^2(\Omega).$$

Subsequently the Lebesgue dominated theorem implies that

$$\lim_{\epsilon\to 0}\mathbf{E}\|Q_1(\epsilon)\|^2=0.$$

Since the operator $A^{-\alpha}$ is bounded, we have

$$\mathbf{E}\|Q_2(\epsilon)\|^2\leq\|A^{-\alpha}\|^2\mathbf{E}\|A^{\alpha}f(t+\epsilon,x(t+\epsilon-r(t+\epsilon)))-A^{\alpha}f(t,x(t-r(t)))\|^2.$$

Based on (**H6**), we conclude that

$$\lim_{\epsilon\to 0}\mathbf{E}\|Q_2(\epsilon)\|^2 = 0.$$

Now, we consider $Q_3(\epsilon)$. Without loss of generality, we assume that $\epsilon > 0$ (the case $\epsilon < 0$ is quite similar). We have

$$\begin{aligned}Q_3(\epsilon) &\le \Big\|\int_0^t [A^{1-\alpha}R(t+\epsilon-s) - A^{1-\alpha}R(t-s)]A^\alpha f(s,x(s-r(s)))ds\Big\| \\ &\quad + \Big\|\int_t^{t+\epsilon}[A^{1-\alpha}R(t+\epsilon-s)A^\alpha f(s,x(s-r(s)))ds\Big\| \\ &:= Q_{31}(\epsilon) + Q_{32}(\epsilon).\end{aligned}$$

Apply Hölder inequality on $Q_{31}(\epsilon)$ we get

$$\mathbf{E}\|Q_{31}(\epsilon)\|^2 \le t\mathbf{E}\int_0^t \big\|[A^{1-\alpha}R(t+\epsilon-s) - A^{1-\alpha}R(t-s)]A^\alpha f(s,x(s-r(s)))\big\|^2 ds.$$

By Lemma 2.3, for each $s \in [0,t]$, we have

$$\lim_{\epsilon\to 0}[A^{1-\alpha}R(t+\epsilon-s) - A^{1-\alpha}R(t-s)]A^\alpha f(s,x(s-r(s))) = 0.$$

By (**H3**), (**H4**), we obtain

$$\begin{aligned}&\|[A^{1-\alpha}R(t+\epsilon-s) - A^{1-\alpha}R(t-s)]A^\alpha f(s,x(s-r(s)))\| \\ &\le M_{1-\alpha}\big((t+\epsilon-s)^{\alpha-1} + (t-s)^{\alpha-1}\big)\|A^\alpha f(s,x(s-r(s)))\| \in \mathcal{L}^2([0,t]\times\Omega).\end{aligned}$$

Again, Lebesgue dominated theorem implies that

$$\lim_{\epsilon\to 0}\mathbf{E}\|Q_{31}(\epsilon)\|^2 = 0.$$

Moreover, by Hölder's inequality and (**H3**), (**H4**), we obtain

$$\mathbf{E}\|Q_{32}(\epsilon)\|^2 \le \frac{L_f M_{1-\alpha}^2}{2\alpha-1}\epsilon^{2\alpha}\int_0^T (\mathbf{E}\|x(s-r(s))\|^2 + 1)ds \xrightarrow{\epsilon\to 0} 0.$$

Therefore,

$$\lim_{\epsilon\to 0}\mathbf{E}\|Q_3(\epsilon)\|^2 = 0.$$

Similarly, to estimate $Q_4(\epsilon)$ we assume that $\epsilon > 0$.

$$\begin{aligned}Q_4(\epsilon) &\le \Big\|\int_0^t [A^{1-\alpha}R(t+\epsilon-s) - A^{1-\alpha}R(t-s)]\int_0^s K(s-\xi)A^\alpha f(s,x(s-r(s)))d\xi ds\Big\| \\ &\quad + \Big\|\int_t^{t+\epsilon}[A^{1-\alpha}R(t+\epsilon-s)\int_0^s K(s-\xi)A^\alpha f(s,x(s-r(s)))d\xi ds\Big\| \\ &:= Q_{41}(\epsilon) + Q_{42}(\epsilon).\end{aligned}$$

By Hölder inequality, we get

$$\begin{aligned}&\mathbf{E}\|Q_{41}(\epsilon)\|^2\\ &\leq t\mathbf{E}\int_0^t \Big\|[A^{1-\alpha}R(t+\epsilon-s)-A^{1-\alpha}R(t-s)]\int_0^s K(s-\xi)A^\alpha f(s,x(s-r(s)))d\xi\Big\|^2 ds.\end{aligned}$$

By Lemma 2.3, for each $s\in[0,t]$, we have

$$\lim_{\epsilon\to 0}[A^{1-\alpha}R(t+\epsilon-s)-A^{1-\alpha}R(t-s)]\int_0^s K(s-\xi)A^\alpha f(s,x(s-r(s)))d\xi=0.$$

From (**H3**), (**H4**), we obtain

$$\begin{aligned}&\|[A^{1-\alpha}R(t+\epsilon-s)-A^{1-\alpha}R(t-s)]\int_0^s K(s-\xi)A^\alpha f(s,x(s-r(s)))d\xi\|\\ &\leq tM_kM_{1-\alpha}\big((t+\epsilon-s)^{\alpha-1}+(t-s)^{\alpha-1}\big)\|A^\alpha f(s,x(s-r(s)))\|\in\mathcal{L}^2([0,t]\times\Omega).\end{aligned}$$

By Lebesgue dominated theorem, we see that

$$\lim_{\epsilon\to 0}\mathbf{E}\|Q_{41}(\epsilon)\|^2=0.$$

In the same fashion by Hölder's inequality and (**H3**), (**H4**), we obtain

$$\mathbf{E}\|Q_{42}(\epsilon)\|^2\leq\frac{tL_fM_k^2M_{1-\alpha}^2}{2\alpha-1}\epsilon^{2\alpha}\int_0^T(\mathbf{E}\|x(s-r(s))\|^2+1)ds\xrightarrow{\epsilon\to 0}0.$$

Hence

$$\lim_{\epsilon\to 0}\mathbf{E}\|Q_4(\epsilon)\|^2=0.$$

The computation of $Q_3(\epsilon)$ also leads to

$$\lim_{\epsilon\to 0}\mathbf{E}\|Q_5(\epsilon)\|^2=0.$$

Finally,

$$\begin{aligned}Q_6(\epsilon)&\leq\Big\|\int_0^t[R(t+\epsilon-s)-R(t-s)]h(s)dB^H(s)\Big\|\\ &\quad+\Big\|\int_t^{t+\epsilon}R(t+\epsilon-s)h(s)dB^H(s)\Big\|\\ &:=Q_{61}(\epsilon)+Q_{62}(\epsilon).\end{aligned}$$

Lemma 2.1 implies that

$$\mathbf{E}\|Q_{61}(\epsilon)\|^2\leq 2Ht^{2H-1}\int_0^t\|[R(t+\epsilon-s)-R(t-s)]h(s)\|^2_{\mathcal{L}_2^0}d(s).$$

Based on strong continuity of $R(t)$, for each $s\in[0,t]$ the following limit holds.

$$\lim_{\epsilon\to 0}\|[R(t+\epsilon-s)-R(t-s)]h(s)\|^2_{\mathcal{L}_2^0}=0$$

By (**H3**) and Lebesgue dominated theorem, we have

$$\|[R(t+\epsilon-s)-R(t-s)]h(s)\|^2_{\mathcal{L}_2^0} \leq 2C^2\|h(s)\|^2_{\mathcal{L}_2^0} \in \mathcal{L}^1(J,ds).$$

and

$$\lim_{\epsilon\to 0}\mathbf{E}\|Q_{61}(\epsilon)\|^2 = 0.$$

Applying Lemma 2.1 to $Q_{62}(\epsilon)$ we obtain

$$\mathbf{E}\|Q_{62}(\epsilon)\|^2 \leq 2HC^2\epsilon^{2H-1}\int_t^{t+\epsilon}\|h(s)\|^2_{\mathcal{L}_2^0}d(s) \xrightarrow{\epsilon\to 0} 0.$$

From the above estimates we see that

$$\lim_{\epsilon\to 0}\mathbf{E}\Big\|(\Pi x)(t+\epsilon)-(\Pi x)(t)\Big\|^2 = 0.$$

This concludes Step 1. We notice that the function $t \to (\Pi x)(t)$ is continuous on J in the $\mathcal{L}^2$-sense.

Step 2. Since Π_2 is a contraction mapping in $\Upsilon_{\overline{T}}$ with some small enough $\overline{T} < T$, let $x, y \in \Upsilon_T$. By (**H**3), (**H**4), (**H**5) and Hölder's inequality, for all $t \in J$ we have

$$\begin{aligned}
&\mathbf{E}\Big\|(\Pi x)(t)-(\Pi y)(t)\Big\|^2\\
&\leq 4\mathbf{E}\Big\|f(t,x(t-r(t)))-f(t,y(t-r(t)))\Big\|^2\\
&+4\mathbf{E}\Big\|\int_0^t AR(t-s)[f(s,x(s-r(s)))-f(s,y(s-r(s)))]ds\Big\|^2\\
&+4\mathbf{E}\Big\|\int_0^t AR(t-s)\int_0^s K(s-\xi)[f(\xi,x(\xi-r(\xi)))-f(\xi,y(\xi-r(\xi)))]d\xi ds\Big\|^2\\
&+4\mathbf{E}\Big\|\int_0^t R(t-s)[g(s,x(s-\rho(s)))-g(s,y(s-\rho(s)))]ds\Big\|^2\\
&\leq 4\|A^{-\alpha}\|^2M_f^2\mathbf{E}\|x(s-r(s))-y(s-r(s))\|^2\\
&+\frac{4T^{2\alpha}M_f^2M_{1-\alpha}^2}{2\alpha-1}\int_0^t\mathbf{E}\|x(s-r(s))-y(s-r(s))\|^2ds\\
&+\frac{4T^{2\alpha+1}M_f^2M_k^2M_{1-\alpha}^2}{2\alpha-1}\int_0^t\mathbf{E}\|x(s-r(s))-y(s-r(s))\|^2ds\\
&+4TM_g^2C^2\int_0^t\mathbf{E}\|x(s-\rho(s))-y(s-\rho(s))\|^2ds.
\end{aligned}$$

Taking supremum over t we obtain

$$\|(\Pi x)-(\Pi y)\|^2_{\mathbb{C}_T} \leq 4\Big[M_f^2\Big(\|A^{-\alpha}\|^2+\frac{T^{2\alpha+1}M_{1-\alpha}^2}{2\alpha-1}[1+TM_k^2]\Big)+T^2M_g^2C^2\Big]\|x-y\|^2_{\mathbb{C}_T}.$$

By (3.1), Π is a contractive mapping on $\Upsilon_{\overline{T}}$. So, applying Banach fixed point principle we conclude that there exists a unique fixed point, which is a mild solution of (1.1) on $[-\tau,\overline{T}]$. This procedure can be repeated to extend the solution to the entire interval $[-\tau, T]$ in finitely many steps. Thus we have completed the proof of Theorem 3.1.

Next we establish global attracting set and stability of mild solutions to neutral stochastic integro-differential equations under fractional Brownian motion.

Theorem 3.2. *Assume that $f(t,0)=0$, $\|g(t,0)\| \le C_g$, where $C_g \ge 0$, $t \in J$, the assumptions $(\mathbf{H}1)$, $(\mathbf{H}2)$, $(\mathbf{H}3^{\star})$, $(\mathbf{H}4)(i)$, $(\mathbf{H}5)(i)$, $(\mathbf{H}7)(ii)$ hold and that*

$$\widehat{\Lambda} := 6\|A^{-\alpha}\|^2 M_f^2 + 6\Big[\frac{M_{1-\alpha}^2 M_f^2 \Gamma(2\alpha-1)}{\lambda^{2\alpha}}(M_k^2 T + 1) + 2^{sgn(C_g)}\frac{M^2 M_g^2}{\lambda^2}\Big] < 1. \tag{3.2}$$

Then set $G = \{x(t) \in \mathcal{L}^2(\Omega, \mathbb{H}) : \mathbf{E}\|x(t)\|^2 \le \frac{\widehat{\Delta}}{1-\widehat{\Lambda}}\}$ is a global attracting set of (1.1), where $\widehat{\Delta} := \frac{12M^2C_g^2}{\lambda^2}$.

To prove Theorem 3.2 we state the following integral inequality which is sharper than the one established by Chen [8] and is more effective for studying neutral systems.

Lemma 3.1. *Let $z : \mathbb{R}^+ \to \mathbb{R}^+$ be Borel measurable. Assume that (i) $z(t)$ is a solution of the integral inequality*

$$z(t) \le \begin{cases} \|\phi\|_{\mathcal{C}} e^{-\gamma t} + c_1 \sup_{\vartheta\in[-\tau,0]} z(t+\vartheta) + c_2\int_0^t e^{-\gamma(t-s)} \sup_{\vartheta\in[-\tau,0]} z(s+\vartheta)ds + \Delta, t \in J, \\ \phi(t), \quad t\in[-\tau,0], \end{cases}$$

where $\phi(t) \in \mathsf{C}([-\tau,0],\mathbb{R}^+)$-the family of all continuous $\mathbb{R}^+$-valued functions ϕ defined on $[-\tau,0]$ with the norm $\|\phi\|_{\mathcal{C}} := \sup_{-\tau\le s\le 0}\|\phi(s)\|$; $\gamma > 0$; c_1, c_2, Δ are nonnegative constants; (ii) $\|\phi\|_{\mathcal{C}} \le L$ for some positive L and $\Lambda := c_1 + \frac{c_2}{\gamma} < 1$.

Then there exist constants $\lambda \in (0,\gamma)$ and $L_z \ge L$ such that for all $t \in J$,

$$z(t) \le L_z e^{-\lambda t} + \frac{\Delta}{1-\Lambda},$$

where λ and L_z satisfy

$$\Lambda_\lambda := c_1 e^{\lambda\tau} + \frac{c_2 e^{\lambda\tau}}{\gamma-\lambda} < 1, \quad L_z \ge \frac{L}{1-\Lambda_\lambda}$$

or $c_2 \ne 0$, $\Lambda_\lambda \le 1$ and

$$L_z \ge \frac{(\gamma-\lambda)\big[L - \frac{c_2\Delta}{\gamma(1-\Lambda)}\big]}{c_2 e^{\lambda\tau}}.$$

Proof. Lemma 3.1 is a consequence of Lemma 3.1 in Long el al. [30]. □

Proof of Theorem 3.2. From (2.4) for $t \in J$, we get

$$\begin{aligned} \mathbf{E}\|x(t)\|^2 \le 6\mathbf{E}\|R(t)[\varphi(0) + f(0,\varphi(-r(0)))]\|^2 + 6\mathbf{E}\|f(t,x(t-r(t)))\|^2 \\ +6\mathbf{E}\Big\|\int_0^t AR(t-s)f(s,x(s-r(s)))ds\Big\|^2 \\ +6\mathbf{E}\Big\|\int_0^t AR(t-s)\int_0^s K(s-\xi)f(\xi,x(\xi-r(\xi)))d\xi ds\Big\|^2 \end{aligned} \tag{3.3}$$

$$+6\mathbf{E}\Big\|\int_0^t R(t-s)g(s,x(s-\rho(s)))ds\Big\|^2+6\mathbf{E}\Big\|\int_0^t R(t-s)h(s)dB^H(s)\Big\|^2$$
$$=: 6\sum_{i=1}^{6} S_i.$$

By $(\mathbf{H}3^{\star})(ii)$, $(\mathbf{H}4)(i)$, it follows that

$$S_1 \leq 2M^2\big[\mathbf{E}\|\varphi\|^2+\|A^{-\alpha}\|^2M_f^2\mathbf{E}\|\varphi(-r(0))\|^2\big]e^{-2\lambda t} \tag{3.4}$$
$$\leq 2M^2(1+\|A^{-\alpha}\|^2M_f^2)\|\varphi\|^2_{\mathbb{C}_0}e^{-\lambda t}.$$

From $(\mathbf{H}4)(i)$, ones easily has

$$S_2 \leq \|A^{-\alpha}\|^2M_f^2\mathbf{E}\|x(t-r(t))\|^2 \leq \|A^{-\alpha}\|^2M_f^2\sup_{-\tau\leq\xi\leq 0}\mathbf{E}\|x(t+\xi)\|^2. \tag{3.5}$$

By Lemma 2.4 and $(\mathbf{H}4)(i)$, we have

$$S_3 \leq \frac{M_{1-\alpha}^2M_f^2\Gamma(2\alpha-1)}{\lambda^{2\alpha-1}}\int_0^t e^{-\lambda(t-s)}\sup_{-\tau\leq\xi\leq 0}\mathbf{E}\|x(s+\xi)\|^2ds. \tag{3.6}$$

From $(\mathbf{H}3^{\star})(i)$, $(\mathbf{H}4)(i)$ and Lemma 2.4, we obtain

$$S_4 \leq \frac{M_{1-\alpha}^2M_f^2M_k^2T\Gamma(2\alpha-1)}{\lambda^{2\alpha-1}}\int_0^t e^{-\lambda(t-s)}\sup_{-\tau\leq\xi\leq 0}\mathbf{E}\|x(s+\xi)\|^2ds. \tag{3.7}$$

By $(\mathbf{H}5)(i)$ and Hölder's inequality, we get

$$S_5 \leq \mathbf{E}\Big(\int_0^t\|R(t-s)g(s,x(s-\rho(s)))\|ds\Big)^2$$
$$\leq \mathbf{E}\Big(\int_0^t Me^{-\lambda(t-s)}[M_g\|x(s-\rho(s))\|+g(s,0)]ds\Big)^2$$
$$\leq 2^{sgn(C_g)}\frac{M^2M_g^2}{\lambda}\int_0^t e^{-\lambda(t-s)}\sup_{-\tau\leq\xi\leq 0}\mathbf{E}\|x(s+\xi)\|^2ds+\frac{2M^2C_g^2}{\lambda^2}, \tag{3.8}$$

where $sgn(\cdot)$ is the *sign* function defined on $\mathbb{R}$.
From Lemma 2.1 and $(\mathbf{H}3^{\star})(ii)$, we obtain

$$S_6 \leq 2M^2Ht^{2H-1}\int_0^t e^{-2\lambda(t-s)}\|h(s)\|^2_{\mathcal{L}_2^0}ds$$
$$\leq \Big[2M^2Ht^{2H-1}\int_0^t e^{\lambda s}\|h(s)\|^2_{\mathcal{L}_2^0}ds\Big]e^{-\lambda t}.$$

Therefore, $(\mathbf{H}7)(ii)$ ensures the existence of a positive constant $C=\|\varphi\|^2_{\mathbb{C}_0}$ such that for all $t\in J$,

$$2M^2Ht^{2H-1}\int_0^t e^{\lambda s}\|h(s)\|^2_{\mathcal{L}_2^0}ds \leq \|\varphi\|^2_{\mathbb{C}_0}.$$

Therefore

$$S_6 \leq \|\varphi\|^2_{\mathbb{C}_0} e^{-\lambda t}. \tag{3.9}$$

Combining (3.3)-(3.9), it follows that

$$\begin{aligned}\mathbf{E}\|x(t)\|^2 \leq\ & 6\big[2M^2(1+\|A^{-\alpha}\|^2M_f^2)+1\big]\|\varphi\|^2_{\mathbb{C}_0}e^{-\lambda t}\\ & + 6\|A^{-\alpha}\|^2M_f^2 \sup_{-\tau\leq\xi\leq 0}\mathbf{E}\|x(t+\xi)\|^2\\ & + 6\Big[\frac{M_{1-\alpha}^2M_f^2\Gamma(2\alpha-1)}{\lambda^{2\alpha-1}}(M_k^2T+1)+2^{sgn(C_g)}\frac{M^2M_g^2}{\lambda}\Big]\\ & \times\int_0^t e^{-\lambda(t-s)}\sup_{-\tau\leq\xi\leq 0}\mathbf{E}\|x(s+\xi)\|^2ds+\frac{12M^2C_g^2}{\lambda^2}. \end{aligned}\tag{3.10}$$

Define

$$\begin{aligned}\widehat{c} &:= 6\big[2M^2(1+\|A^{-\alpha}\|^2M_f^2)+1\big],\\ \widehat{c}_1 &:= 6\|A^{-\alpha}\|^2M_f^2,\\ \widehat{c}_2 &:= 6\Big[\frac{M_{1-\alpha}^2M_f^2\Gamma(2\alpha-1)}{\lambda^{2\alpha-1}}(M_k^2T+1)+2^{sgn(C_g)}\frac{M^2M_g^2}{\lambda}\Big].\end{aligned}$$

By (3.2), we know $\widehat{\Lambda} := \widehat{c}_1 + \frac{\widehat{c}_2}{\lambda} < 1$. Since $\varphi \in \mathbb{C}_0^{\mathcal{F}_0}([-\tau,0],\mathcal{L}^2(\Omega,\mathbb{H}))$, there exist $\widehat{L}\geq 0$, $\widehat{L}_z > 0$, and $\widehat{\lambda}\in(0,\lambda)$ such that

$$\widehat{c}\|\varphi\|^2_{\mathbb{C}_0}\leq \widehat{L},\quad \widehat{\Lambda}_{\widehat{\lambda}} := \widehat{c}_1e^{\widehat{\lambda}\tau}+\frac{\widehat{c}_2e^{\widehat{\lambda}\tau}}{\lambda-\widehat{\lambda}}\leq 1,\quad \widehat{L}_z \geq \frac{(\lambda-\widehat{\lambda})\Big[\widehat{L}-\frac{\widehat{c}_2\widehat{\Delta}}{\lambda(1-\widehat{\Lambda})}\Big]}{\widehat{c}_2e^{\widehat{\lambda}\tau}}.$$

By Lemma 3.1, there exists a set $G = \{x(t)\in\mathcal{L}^2(\Omega,\mathbb{H}) : \mathbf{E}\|x(t)\|^2\leq \frac{\widehat{\Delta}}{1-\widehat{\Lambda}}\}$ as a global attracting set of (1.1). Thus Theorem 3.2 is proved.

Remark 3.1. *It is interesting to compare our results with those by Caraballo and Diop* [4]*:*

$$\begin{cases} d[x(t)+f(t,x(t-r(t)))]\\ \qquad = A[x(t)+f(t,x(t-r(t)))]dt+\int_0^t K(t-s)[x(s)+f(s,x(s-r(s)))]dsdt\\ \qquad\quad +g(t,x(t-\rho(t)))dt+h(t)dB^H(t),\quad t\in J:=[0,T]\\ x_0(\cdot)=\varphi,\quad -\tau\leq t\leq 0,\quad \tau>0.\end{cases}$$

We analyzed the solvability of (1.1) and presented the existence of mild solution of (1.1) under stronger assumptions on the operator A. Thus, our work is a more natural extension of that by Caraballo and Diop [4].

Remark 3.2. *When $K\equiv 0$ the equation (1.1) reduces to the equation which is investigated by Boufoussi and Hajji* [2].

When $K\equiv 0$, $f\equiv 0$ the equation (1.1) reduces to the equation which investigated by Caraballo el al. [5]. *In order to obtain similar results, we assume that $\rho : J\to[0,\tau]$ is differentiable, and there exists a positive ρ^* such that for all $t\in J$: $\left|\frac{1}{1-\rho'(t)}\right|\leq\rho^*$. This improves the results obtained by Boufoussi and Hajji* [2]*, Caraballo el al.* [5].

Remark 3.3. *In this article as well as those by Boufoussi and Hajji* [2]*, Caraballo and Diop* [4]*, Caraballo el al.* [5]*, the equation considered involves additive but not multiplicative noise. The rational is from the following example:*

$$dx(t) = \mu x(t)dt + hx(t)dB^H(t), \quad x(0) = x_0.$$

Then $x(t) = x_0 e^{\mu t - \frac{h^2}{2} t^{2H + hB^H(t)}}$ *is a unique solution of this equation. We easily calculate for all* $t \in J$*,*

$$\mathbf{E}\|x(t)\|^2 = x_0^2 e^{2\mu t + h^2 t^{2H}}.$$

Hence, $x(t)$ does not $\to 0$ in mean square for all μ and h.

Remark 3.4. *Differential equations with impulsive conditions constitute an important field of research due to their numerous applications in ecology, medicine biology, electrical engineering, and other areas of science. There has been a significant development in impulsive theory especially in the area of impulsive differential equation with fixed moments (see the monograph by Lakshmikantham et al.* [26]*). However, in addition to stochastic effects, impulsive effects likewise exist in real systems. Therefore, it is necessary and important to consider the existence, global attracting set and stability of mild solutions for a class of neutral stochastic integro-differential equations under fractional Brownian motion and impulsive effects. The approach outlined in this article can be utilized in that direction.*

Remark 3.5. *Our article studies the properties of solution with finite delay. Besides impulsive effects, the effect of infinite delay on state equations is also very popular. Properties of equations with finite and infinite delays are completely different. For infinite delay, the properties of solutions depend on the choice of the phase space which is proposed by Hale and Kato in Ref.* [16]*. For the fundamental theory related to functional differential equations with infinite delay, see Ref.* [18]*. Our technique in this article can be extended to study the existence, global attracting set and stability of neutral stochastic integro-differential equations under fractional Brownian motion with infinite delay.*

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