CERTAIN PROPERTIES OF A NEW SUBCLASS OF $p$-VALENTLY CLOSE TO CONVEX FUNCTIONS

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Abstract. In the present paper we introduce and investigate an interesting subclass $K_p^{(k)}(\alpha, \beta)$ analytic and $p$-valently close to convex functions in the open unit disk $U$. For functions belonging to $K_p^{(k)}(\alpha, \beta)$, we derive several properties coefficient estimates, sufficient condition, distortion theorem and inclusion relationships.

1. Introduction and definitions

Let $A_p$ denote the class of all functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n}z^{p+n} \quad (p \in N)$$

which are analytic in the open unit disk, $U = \{ z \in \mathbb{C} : |z| < 1 \}$. In particular, we write $A_1 = A$.

For any two analytic functions $f$ and $g$ in $U$, we say that $f$ is subordinate to $g$ in $U$, written as $f(z) \prec g(z)$ if there exist a schwarz function $w(z)$ such that $f(z) = g(w(z))$, for $z \in U$. In particular, if $g$ is univalent in $U$, then $f$ is subordinate to $g$ iff $f(0) = g(0)$ and $f(U) \subset g(U)$.

A function $f \in A_p$, is said to be $p$-valently starlike of order $\gamma \ (0 \leq \gamma < p)$ in $U$ if it satisfies the inequality [5]

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \gamma \quad (z \in U)$$

or equivalently

$$\frac{zf'(z)}{f(z)} < \frac{p + (p - 2\gamma)z}{1 - z} \quad (z \in U).$$

The class of all $p$-valent starlike functions of order $\gamma$ in $U$ is denoted by $S_p^*(\gamma)$. Also, we denote that $S_p^*(0) = S_p^*$, $S_p^*(\gamma) = S^*(\gamma)$ and $S_1^*(0) = S^*$. 

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A function \( f \in A_p \), is said to be \( p \)-valently close-to-convex of order \( \gamma \) \((0 \leq \gamma < p)\) in \( U \) if \( g \in S_p^* (\gamma) \) and satisfies the inequality [9]

\[
Re \left( \frac{zf'(z)}{g(z)} \right) > \gamma \quad (z \in U)
\]
or equivalently

\[
\frac{zf'(z)}{g(z)} < \frac{p + (p - 2\gamma)z}{1 - z} \quad (z \in U).
\]

The class of all \( p \)-valent close-to-convex functions of order \( \gamma \) in \( U \) is denoted by \( K_p(\gamma) \). Also, we denote that

\[
K_p(0) = K_p, \quad K_1(\gamma) = K(\gamma) \quad \text{and} \quad K_1(0) = K.
\]

Recently, Bulut [3] discussed a class \( K^{(k)}_p(\gamma, p) \) for analytic and \( p \)-valently close-to-convex functions. A function \( f \in A_p \) is said to be in the class \( K^{(k)}_s(\gamma, p) \) if there exist a function \( g \in S_p^* (\frac{(k-1)p}{k}) \) \((k \in N \) is a fixed integer), such that

\[
Re \left( \frac{z^{(k-1)p+1}f'(z)}{g_k(z)} \right) > \gamma \quad (z \in U; 0 \leq \gamma < p),
\]

where \( g_k \) is defined by the equality

\[
g_k(z) = \prod_{v=0}^{k-1} \varepsilon^{-vp}g(\varepsilon^vz); \quad \varepsilon = e^{\frac{2\pi}{p}}. \tag{2}
\]

Here assuming \( g \in S_p^* (\frac{(k-1)p}{k}) \) makes \( \frac{g_k(z)}{z^{(k-1)p+1}f'(z)}\) a \( p \)-valent starlike function which in turn implies the close-to-convexity of \( f \). By simple calculation, we see that \( f(z) \in K^{(k)}_s(\gamma, p) \) if and only if

\[
\left| \frac{z^{(k-1)p+1}f'(z)}{g_k(z)} - p \right| < \left| \frac{z^{(k-1)p+1}f'(z)}{g_k(z)} + p - 2\gamma \right| \tag{3}
\]

Recently several similar classes of \( K^{(k)}_s(\gamma, p) \) for analytic and univalent function have been defined and investigated, some of them we refer to [4, 7, 11, 12, 13, 14, 15, 17]. Motivated essentially by the above mentioned class \( K^{(k)}_s(\gamma, p) \) and the above referred works for analytic and univalent functions, we now introduce a new class for \( p \)-valent analytic function in the following manner:

**Definition 1.** For \( 0 \leq \alpha \leq 1 \) and \( 0 < \beta \leq 1 \), a function \( f \in A_p \) is said to be in the class \( K^{(k)}_p(\alpha, \beta) \), if there exist a function \( g \in S_p^* (\frac{(k-1)p}{k}) \) \((k \in N \) is a fixed integer), such that

\[
\left| \frac{z^{(k-1)p+1}f'(z)}{g_k(z)} - p \right| < \left| \frac{\alpha z^{(k-1)p+1}f'(z)}{g_k(z)} + p \right| \tag{4}
\]

where \( g_k \) is defined by the equality (2).

**Remark.** (i) For \( p = 1 \), we get the class \( K^{(1)}_p(\alpha, \beta) \) studied by Wang [16].
(ii) For \( p = 1 \) and \( k = 2 \), we get the class \( K^{(2)}_1(\alpha, \beta) \) studied by Wang [15].

In the present paper, we derive several properties including coefficient estimates, sufficient condition, distortion theorem and inclusion relationships for function belonging to the class \( K^{(k)}_p(\alpha, \beta) \).

In order to prove our main result for the function class \( K^{(k)}_p(\alpha, \beta) \), we need the
following lemmas:

Lemma 1. [3] If
\[ g(z) = z^p + \sum_{n=1}^{\infty} b_{p+n} z^{p+n} \in S_p^* \left( \frac{(k-1)p}{k} \right), \]
then
\[ G(z) = \frac{g_k(z)}{z^{(k-1)p}} = z^p + \sum_{n=1}^{\infty} B_{p+n} z^{p+n} \in S_p^*, \] (5)
where \( g_k \) is given by (2).

Lemma 2. [2] Let \( G(z) \in S_p^* \) given by (5) and \( \mu \) be a complex number, then
\[ |B_{p+2} - \mu B_{p+1}^2| \leq p \left( \max \{1, |1+2p(1-2\mu)| \} \right). \]

Let \( \Omega \) be class of analytic functions of the form:
\[ w(z) = w_1 z + w_2 z^2 + \ldots \quad (z \in \mathbb{U}), \] (6)
in the unit disk \( \mathbb{U} \) satisfying the condition \(|w(z)| < 1\).

Lemma 3. ([6], p.10) If \( w(z) \in \Omega \), then for any complex number \( \mu \):
\[
|w_1| \leq 1, |w_2 - \mu w_1^2| \leq 1 + (|\mu| - 1)|w_1^2| \leq \max \{1, |\mu|\}.
\]
The result is sharp for the functions \( w(z) = z \) or \( w(z) = z^2 \).

Lemma 4. Let the function \( K(z) = p + k_1 z + k_2 z^2 + k_3 z^3 + \ldots \) \((z \in \mathbb{U})\) be analytic in the unit disk \( \mathbb{U} \), and satisfies the condition
\[
\left| \frac{K(z) - p}{\alpha K(z) + p} \right| < \beta \quad (z \in \mathbb{U}),
\]
for \( 0 \leq \alpha \leq 1 \) and \( 0 < \beta < 1 \), if and only if there exist an analytic function \( \phi \) in the unit disk \( \mathbb{U} \), such that \(|\phi(z)| \leq \beta \) \((z \in \mathbb{U})\), and
\[
K(z) = \frac{p - pz\phi(z)}{1 + \alpha z\phi(z)}, \quad (z \in \mathbb{U}).
\]

Proof. Assume that the function
\[
\frac{zf'(z)}{G(z)} = p + k_1 z + k_2 z^2 + k_3 z^3 + \ldots = K(z) \quad (z \in \mathbb{U}),
\]
satisfies the condition
\[
\left| \frac{K(z) - p}{\alpha K(z) + p} \right| < \beta \quad (z \in \mathbb{U}).
\]
Setting
\[
k(z) = \frac{p - K(z)}{p + \alpha K(z)},
\]
we see that the function \( k(z) \) is analytic in \( \mathbb{U} \), satisfies the inequality \(|k(z)| < \beta \) for \( z \in \mathbb{U} \) and \( k(0) = 0 \). Now, by using schwartz’s lemma, we get that the function \( k(z) \) has of the form \( k(z) = z\phi(z) \), where \( \phi(z) \) is analytic in \( \mathbb{U} \) and satisfies \(|\phi(z)| \leq \beta \) for \( z \in \mathbb{U} \). Thus, we obtain
\[
K(z) = \frac{p - pk(z)}{1 + \alpha k(z)} = \frac{p - pz\phi(z)}{1 + \alpha z\phi(z)}.
\]
Conversely, if
\[ K(z) = \frac{p - p\phi(z)}{1 + \alpha z\phi(z)} \]
and \(|\phi(z)| \leq \beta\) for \(z \in U\), then \(K\) is analytic in the unit disk \(U\). So we get
\[ \left| \frac{K(z) - p}{\alpha K(z) + p} \right| = |z\phi(z)| \leq \beta|z| < \beta \quad (z \in U), \]
which completes the proof of our lemma.

**Lemma 5.** [8] Let \(-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1\). Then
\[ 1 + A_1 z \leq 1 + A_2 z \]
\[ 1 + B_1 z \preceq 1 + B_2 z. \]
Let \(f(z) = \sum_{n=1}^{\infty} a_n z^n\) and \(g(z) = \sum_{n=1}^{\infty} b_n z^n\) be two analytic functions defined in \(D\). Then there Hadamard product (or convolution) is the function \((f \ast g)(z)\) defined by
\[ (f \ast g)(z) = \sum_{n=1}^{\infty} a_n b_n z^n. \]
The classes of starlike and convex functions are closed under convolution with convex function. The following lemma is required for our next result.

**Lemma 6.** [10] Let \(\psi\) and \(\phi\) be convex in \(U\) and suppose \(f \prec \psi\), then
\[ f \ast \phi = \psi \ast \phi. \]

2. MAIN RESULTS

First of all, we show in which way our class is associated with the appropriate subordination.

**Theorem 1.** A function \(f(z) \in \mathcal{K}_p^{(k)}(\alpha, \beta)\) if and only if there exists \(g_k(z)\) satisfying the condition (2) such that
\[ \frac{1}{p} \frac{zf'(z)}{G(z)} < \frac{1 + \beta z}{1 - \alpha \beta z} \quad (z \in U), \]
(7)
where \(G(z)\) is given by (5).

**Proof.** Let \(f(z) \in \mathcal{K}_p^{(k)}(\alpha, \beta)\). Then, for \(\alpha \neq 1\) and \(\beta \neq 1\), squaring and expanding both sides of (4), we see that the region of \(\frac{1}{p} \frac{zf'(z)}{G(z)}\) for \(z \in U\) is contained in the disk \(C\) whose center is \(\frac{(1+\alpha \beta^2)}{(1-\alpha \beta^2)}\) and radius is \(\frac{|\beta(1+\alpha)|}{|1-\alpha \beta^2|}\). Since \(q(z) = \frac{1+\beta z}{1-\alpha \beta z}\) maps the unit disk \(U\) to the disk \(C\) and \(q(z)\) is univalent in \(U\), we obtain the relation (7). Conversely, assume that the relation (7) holds true. Then we have
\[ \frac{1}{p} \frac{zf'(z)}{G(z)} < \frac{1 + \beta z}{1 - \alpha \beta z}, \]
\[ (0 \leq \alpha \leq 1, 0 < \beta \leq 1; z \in U), \]
where \(w(z)\) is analytic in \(U\), \(w(0) = 0\) and \(|w(z)| < 1\) for \(z \in U\). Therefore from the above equation, we obtain the inequality (4), that is, \(f(z) \in \mathcal{K}_p^{(k)}(\alpha, \beta)\).

**Theorem 2.** Let \(0 \leq \alpha \leq 1, 0 < \beta \leq 1, f\) given by (1) and \(g \in S_p^* \left(\frac{(k-1)p}{k}\right)\) are such that the condition (4) holds. Then, for \(n \geq 1\), we have
\[ |ma_m - pB_m|^2 - (1+\alpha)^2 \beta^2 p^2 \leq \sum_{n=1}^{m-1} \left\{ (\alpha^2 \beta^2 - 1)n^2|a_n|^2 + (\beta^2 - 1)p^2|B_n|^2 + 2p(\alpha \beta^2 + 1)n|a_n B_n| \right\} \]
(8)
where the coefficients $B_n$ are given in (5).

**Proof.** Suppose that the condition (4) is satisfied then by lemma 4, we have

$$\frac{zf'(z)}{G(z)} = \frac{p - p\phi(z)}{1 + \alpha\phi(z)} \quad (z \in \mathbb{U}),$$

where $\phi$ is an analytic functions in $\mathbb{U}$, $\phi(z) \leq 1$ for $z \in \mathbb{U}$ and $G(z)$ is given by (5). From the above equality, we obtain that

$$[\alpha zf'(z) + pG(z)]z\phi(z) = pG(z) - zf'(z). \quad (9)$$

Now, we put

$$z\phi(z) = \sum_{n=1}^{\infty} t_n z^n \quad (z \in \mathbb{U}).$$

Thus from (9), we find that

$$\left((1 + \alpha)p + \sum_{n=1}^{\infty} \alpha(p + n)a_{p+n}z^n + p \sum_{n=1}^{\infty} B_{p+n}z^n\right)\sum_{n=1}^{\infty} t_n z^n
$$

$$= p \sum_{n=1}^{\infty} B_{p+n}z^n - \sum_{n=1}^{\infty} (p + n)a_{p+n}z^n. \quad (10)$$

Equating the coefficient of $z^m$ in (10), we have

$$pB_{p+m} - (p+n)a_{p+m} = (1 + \alpha)p t_m + (\alpha(p+1)a_{p+1} + pB_{p+1})t_{m-1} + \cdots + (\alpha(p+m-1)a_{p+m-1} + pB_{p+m-1})t_1$$

which shows that $pB_{p+n} - (p+n)a_{p+n}$ on the right hand side of (10) depends only on $a_{p+1}, B_{p+1}, a_{p+2}, B_{p+2}, \ldots, a_{p+n-1}, B_{p+n-1}$, of left-hand side. Hence, for $n \geq 1$, we can write as

$$\left(1 + \alpha\right)p + \sum_{n=1}^{m-1} (\alpha a_n + pB_n) z^n z\phi(z) = \sum_{n=1}^{m} (pB_n - na_n) z^n + \sum_{n=m+1}^{\infty} c_n z^n. \quad (11)$$

Using the fact that $|z\phi(z)| \leq \beta|z| < \beta$ for all $z \in \mathbb{U}$ in (11), this reduce to inequality

$$\left|\left(1 + \alpha\right)p + \sum_{n=1}^{m-1} (\alpha a_n + pB_n) z^n\right| < \beta \left|\sum_{n=1}^{m} (pB_n - na_n) z^n + \sum_{n=m+1}^{\infty} c_n z^n\right|.$$

Then squaring the above inequality and integrating along $|z| = r < 1$, we obtain

$$\beta^2 \int_0^{2\pi} \left|\left(1 + \alpha\right)p + \sum_{n=1}^{m-1} (\alpha a_n + pB_n) r^n e^{i\theta}\right|^2 d\theta
$$

$$> \int_0^{2\pi} \left|\sum_{n=1}^{m} (pB_n - na_n) r^n e^{i\theta} + \sum_{n=m+1}^{\infty} c_n r^n e^{i\theta}\right|^2 d\theta.$$

Using now the Paraseval’s inequality, we obtain

$$\beta^2 \left((1 + \alpha)^2 p^2 + \sum_{n=1}^{m-1} |\alpha a_n + pB_n|^2 r^{2n}\right) > \sum_{n=1}^{m} |pB_n - na_n|^2 r^{2n} + \sum_{n=m+1}^{\infty} |c_n|^2 r^{2n}.$$

Letting $r \to 1$ in this inequality, we get
\[ \sum_{n=1}^{m} |na_n - pB_n|^2 \leq \beta^2 \left( (1 + \alpha)^2 p^2 + \sum_{n=1}^{m-1} |na_n + pB_n|^2 \right). \]

Hence we deduce that
\[ |ma_m - pB_m|^2 - (1 + \alpha)^2 \beta^2 p^2 \leq \sum_{n=1}^{m-1} \left\{ (\alpha^2 \beta^2 - 1)n^2 |a_n|^2 + (\beta^2 - 1)p^2 |B_n|^2 + 2p(\alpha \beta^2 + 1)n |a_n B_n| \right\}, \]

and thus we obtain the inequality (8). Which completes the proof of Theorem 2.

**Theorem 3.** Let \( 0 \leq \alpha \leq 1, 0 < \beta \leq 1, f \) given by (1) and \( g \in S_p^*(\frac{(k-1)\beta}{k}) \) such that
\[ (1 + \alpha \beta) \sum_{n=1}^{\infty} (p + n)|a_{p+n}| + (1 + \beta)p \sum_{n=1}^{\infty} |B_{p+n}| < (1 + \alpha \beta)p, \tag{12} \]

where the coefficients \( B_{p+n} \) are given by (5), then \( f \in K_p^{(k)}(\alpha, \beta) \).

**Proof.** For \( f \) given by (1) and \( g_k \) defined by (2), we set
\[ \Lambda = \left| zf'(z) - \frac{g_k(z)}{z(k-1)p} - \beta \alpha zf'(z) + p \frac{g_k(z)}{z(k-1)p} \right| \]
\[ = \left| \sum_{n=1}^{\infty} (p+n)a_{p+n}z^{p+n} - p \sum_{n=1}^{\infty} B_{p+n}z^{p+n} \right| - \beta \left| (1 + \alpha)pz^p + \alpha \sum_{n=1}^{\infty} (p+n)a_{p+n}z^{p+n} + p \sum_{n=1}^{\infty} B_{p+n}z^{p+n} \right| \]
\[ \Lambda \leq \sum_{n=1}^{\infty} (p + n)|a_{p+n}||z|^{p+n} + p \sum_{n=1}^{\infty} |B_{p+n}||z|^{p+n} \]
\[ - \beta \left( (1 + \alpha)p|z|^p - \alpha \sum_{n=1}^{\infty} (p + n)|a_{p+n}||z|^{p+n} - p \sum_{n=1}^{\infty} |B_{p+n}||z|^{p+n} \right) \]
\[ = -(1 + \alpha \beta)p|z|^p + (1 + \alpha \beta) \sum_{n=1}^{\infty} (p + n)|a_{p+n}||z|^{p+n} + (1 + \beta)p \sum_{n=1}^{\infty} |B_{p+n}||z|^{p+n} \]
\[ = - (1 + \alpha \beta)p|z|^p + (1 + \alpha \beta) \sum_{n=1}^{\infty} (p + n)|a_{p+n}| + (1 + \beta)p \sum_{n=1}^{\infty} |B_{p+n}| |z|^p. \]

From the inequality (12), we obtain that \( \Lambda < 0 \). Thus we have
\[ \left| zf'(z) - \frac{g_k(z)}{z(k-1)p} \right| < \beta \alpha \left| zf'(z) + p \frac{g_k(z)}{z(k-1)p} \right| \]
which is equivalent to (4). Hence \( f \in K_p^{(k)}(\alpha, \beta) \). This completes the proof of Theorem 3.

**Theorem 4.** If \( f \in K_p^{(k)}(\alpha, \beta) \), then for \( |z| = r \) \((0 \leq r < 1)\), we have

(i) \[ \frac{p(1 - \beta r)^{p-1}}{(1 + \alpha \beta r)(1 + r)^2} \leq |f'(z)| \leq \frac{p(1 + \beta r)^{p-1}}{(1 - \alpha \beta r)(1 - r)^2} \tag{13} \]

(ii) \[ \int_0^r \frac{p(1 - \beta \tau)^{p-1}}{(1 + \alpha \beta \tau)(1 + \tau)^2} d\tau \leq |f(z)| \leq \int_0^r \frac{p(1 + \beta \tau)^{p-1}}{(1 - \alpha \beta \tau)(1 - \tau)^2} d\tau \tag{14} \]
In order to prove the lower bound in (4) holds. (i) From Lemma 1 it follows that the function \(G(z)\) given by (5) is \(p\)-valently starlike function. Hence from [1, Theorem 1] we have
\[
\frac{r^p}{(1 + r)^2} \leq |G(z)| \leq \frac{r^p}{(1 - r)^2} \quad (|z| = r \ (0 \leq r < 1)).
\] (15)

Let us define \(\Psi(z)\) by
\[
\Psi(z) = \frac{zf'(z)}{G(z)} \quad (z \in U),
\]
then by (7), we have
\[
\frac{(p - p\beta r)}{(1 + \alpha\beta r)} \leq |\Psi(z)| \leq \frac{(p + p\beta r)}{(1 - \alpha\beta r)} \quad (z \in U).
\] (16)

Thus from (15) and (16), we get the inequalities (13).

(ii) Let \(z = re^{\theta} (0 < r < 1)\). If \(l\) denotes the closed line-segment in the complex \(\zeta\)-plane from \(\zeta = 0\) and \(\zeta = z\), i.e. \(l = [0, re^{\theta}]\), then we have
\[
f(z) = \int_l f'(\zeta)d\zeta = \int_0^r f'(\tau e^{\theta})e^{\theta}d\tau \quad (|z| = r \ (0 \leq r < 1)).
\]
Thus, by using the upper estimate in (13), we have
\[
|f(z)| = \left| \int_l f'(\zeta)d\zeta \right| \leq \int_0^r |f'(\tau e^{\theta})|d\tau \leq \int_0^r \frac{p(1 + \beta\tau)(1 - \beta\tau)}{(1 - \alpha\beta\tau)(1 - \alpha\beta\tau)}d\tau \quad (|z| = r \ (0 \leq r < 1)),
\]
which yields the right hand of the inequality in (14).

In order to prove the lower bound in (14), let \(z_0 \in U\) with \(|z_0| = r \ (0 < r < 1)\), such that
\[
|f(z_0)| = \min\{|f(z)| : |z| = r\}
\]
It is sufficient to prove that the left-hand side inequality holds for this point \(z_0\).
Moreover, we have
\[
|f(z)| \geq |f(z_0)| \quad (|z| = r \ (0 \leq r < 1)).
\]
The image of the closed line-segment \(l_0 = [0,f(z_0)]\) by \(f^{-1}\) is a piece of arc \(\Gamma\) included in the closed disk \(U_r\) given by
\[
U_r = \{z : z \in \mathbb{C} \text{ and } |z| \leq r \ (0 \leq r < 1)\},
\]
that is, \(\Gamma = f^{-1}(l_0) \subset U_r\). Hence, in accordance with (13), we obtain
\[
|f(z_0)| = \int_{l_0} |dw| = \int_\Gamma |f'(\zeta)||d\zeta| \geq \int_0^r \frac{p(1 - \beta\tau)(1 - \beta\tau)}{(1 + \alpha\beta\tau)(1 + \beta\tau)^2}d\tau.
\]
This finishes the proof of the inequality (14).

**Theorem 5.** Let \(-1 \leq -\alpha_2\beta_2 \leq -\alpha_1\beta_1 < \beta_1 \leq \beta_2 \leq 1\) Then,
\[
\mathcal{K}_{p}^{(k)}(\alpha_1, \beta_1) \subset \mathcal{K}_{p}^{(k)}(\alpha_2, \beta_2)
\]

**Proof.** Suppose that \(f \in \mathcal{K}_{p}^{(k)}(\alpha_1, \beta_1)\) Then
\[
\frac{1}{p} \frac{zf'(z)}{G(z)} \leq \frac{1 + \beta_1 z}{1 - \alpha_1 \beta_1 z}
\]
since \(-1 \leq -\alpha_2\beta_2 \leq -\alpha_1\beta_1 < \beta_1 \leq \beta_2 \leq 1\). By Lemma 5, we have
\[
\frac{1}{p} \frac{zf'(z)}{G(z)} \leq \frac{1 + \beta_1 z}{1 - \alpha_1 \beta_1 z} \leq \frac{1 + \beta_2 z}{1 - \alpha_2 \beta_2 z}
\]
it follows that \( f(z) \in \mathcal{K}_p^{(k)}(\alpha_2, \beta_2) \), which implies the inclusion result.

**Theorem 6.** For a function \( f(z) \) given by (1) is in the class \( \mathcal{K}_p^{(k)}(\alpha, \beta) \) and \( \mu \in \mathbb{C} \), the following estimates holds.

\[
|a_{p+2} - \mu a_{p+1}^2| \leq 2(1 + \alpha)\beta p \left| \frac{p}{p + 2} - \frac{2 \mu p^2}{(p + 1)^2} \right| + \frac{p^2}{p + 2} \mu_1 + \frac{(1 + \alpha)(1 + \alpha \beta)\beta p}{p + 2} \mu_2
\]

(17)

where 
\[
\mu_1 = \max\left\{ 1, \left| 1 + 2p - \frac{2 \mu p(p + 2)}{(p + 1)^2} \right| \right\}
\]

and 
\[
\mu_2 = \max\left\{ 1, \left| \frac{(1 + \alpha)(p + 2)\beta \mu - (1 + \alpha \beta)(p + 1)^2}{(1 + \alpha \beta)(p + 1)} \right| \right\}
\]

(18)

and 
\[
|a_{p+2} - \mu a_{p+1}^2| \leq 2(1 + \alpha)\beta p \left| \frac{p}{p + 2} - \frac{2 \mu p^2}{(p + 1)^2} \right| + \frac{p^2}{p + 2} \mu_1 + \frac{(1 + \alpha)(1 + \alpha \beta)\beta p}{p + 2} \mu_2
\]

(19)

**Proof.** Let \( f \in \mathcal{K}_p^{(k)}(\alpha, \beta) \), then

\[
\frac{1}{p} \frac{z f(z)}{G(z)} = \frac{1 + \beta w(z)}{1 - \alpha \beta w(z)} \quad (z \in \mathbb{U}),
\]

(20)

where \( G(z) \) is given by (5) and \( w(z) \) is schwarz function given by (6) which is analytic in \( \mathbb{U} \) with \( w(0) = 0 \) and \( |w(z)| < 1 \).

Using the series expansions in (20), we have

\[
1 + \left( \frac{p + 1}{p} a_{p+1} - B_{p+1} \right) z + \left( \frac{p + 1}{p} a_{p+2} - \frac{p + 1}{p} a_{p+1} B_{p+1} + (B_{p+1}^2 - B_{p+2}) \right) z^2 + \ldots
\]

\[
= 1 + (1 + \alpha)\beta w_1 z + (1 + \alpha)(1 + \alpha \beta)\beta (w_1^2 + w_2) z^2 + \ldots
\]

(21)

Equating of coefficients in (21) gives us

\[
a_{p+1} = \frac{p}{p + 1} ((1 + \alpha)\beta w_1 + B_{p+1}),
\]

\[
a_{p+2} = \frac{p}{p + 2} ((1 + \alpha)\beta w_1 B_{p+1} + (1 + \alpha)(1 + \alpha \beta)\beta (w_1^2 + w_2) + B_{p+2}).
\]

Therefore, we have

\[
|a_{p+2} - \mu a_{p+1}^2| \leq \frac{p}{p + 2} \left| B_{p+2} - \mu \frac{(p^2 + 2p)}{(p + 1)^2} B_{p+1}^2 \right| + \frac{p^2}{p + 2} \left| w_1 \right| \left| B_{p+1} \right|
\]

\[
+ \frac{p}{p + 2} (1 + \alpha)(1 + \alpha \beta)\beta \left| w_2 - \left( \frac{(1 + \alpha)(\beta (p^2 + 2p) - (1 + \alpha \beta)(p + 1)^2)}{(1 + \alpha \beta)(p + 1)} \right) w_1^2 \right|.
\]

(22)

Now, the desired result follows upon using lemma 2 and lemma 3 in (22).

**Theorem 7.** If \( f(z) \in \mathcal{K}_p^{(k)}(\alpha, \beta) \), then there exists

\[
q(z) < \frac{1 + \beta z}{1 - \alpha \beta z}
\]

such that for all \( s \) and \( t \) with \( |s| \leq 1 \) and \( |t| \leq 1 \),

\[
\frac{p^r - 1}{s^{p-1}} f'(s z) q(t z) < \left( \frac{1 - tz}{1 - sz} \right)^{2p}.
\]

(23)

**Proof.** Let \( f(z) \in \mathcal{K}_p^{(k)}(\alpha, \beta) \), then there exist \( g(z) \in \mathcal{S}_p^{*} \left( \frac{(k - 1)z}{k} \right) \).

Suppose

\[
q(z) = \frac{1}{p} \frac{z f'(z)}{G(z)}.
\]

(24)
where
\[ G(z) = \frac{g_k(z)}{z^{(k-1)p}}. \]

Then by (7), we have
\[ q(z) \prec \frac{1 + \beta z}{1 - \alpha \beta z} \]

logarithmic derivative of (24), implies
\[
\frac{zf''(z)}{f'(z)} - \frac{zq'(z)}{q(z)} + 1 - p = \frac{zG'(z)}{G(z)} - p. \tag{25}
\]

Since \( G(z) \in S_p^* \),
\[
\frac{1}{p} \frac{zG'(z)}{G(z)} < \frac{1 + z}{1 - z},
\]
so
\[
\frac{zG'(z)}{G(z)} - p \prec 2pz \tag{26}
\]

From (25) and (26), we have
\[
\frac{zf''(z)}{f'(z)} - \frac{zq'(z)}{q(z)} + 1 - p \prec 2pz \tag{27}
\]

For \( s \) and \( t \) such that \(|s| \leq 1\) and \(|t| \leq 1\), the function
\[
h(z) = \int_0^z \left( \frac{s}{1 - su} - \frac{t}{1 - tu} \right) du \tag{28}
\]
is convex in \( U \).

Applying Lemma 6, we have
\[
\left( \frac{zf''(z)}{f'(z)} - \frac{zq'(z)}{q(z)} + 1 - p \right) * h(z) \prec 2pz \frac{1}{1 - z} * h(z).
\]

Given any function \( k(z) \) analytic in \( U \), with \( k(0) = 0 \), we have
\[
(k * h)(z) = \int_{it}^{sz} k(u) \frac{du}{u} \quad (z \in U),
\]
which implies that
\[
\log \left( \frac{(sz)^{1-p}f'(sz)q(sz)}{(t z)^{1-p}f'(t z)q(t z)} \right) \prec \log \left[ \frac{1 - tz}{1 - sz} \right]^{2p}
\]
which is equivalent to (23). This completes the proof of Theorem 7.

REFERENCES