A NOTE ON SEMIMULTIPLIERS IN PRIME RINGS

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Abstract. Let $R$ be a ring and $g$ be a surjective map of $R$. An additive mapping $F : R \to R$ is called a semimultiplier if
1. $F(xy) = F(x)g(y) = g(x)F(y)$
2. $F(g(x)) = g(F(x))$ for all $x, y \in R$. In this paper, we introduce the notion of semimultiplier of a ring $R$ and investigate the commutativity of prime rings admitting semimultipliers satisfying certain identities involving semimultiplier.

1. Introduction

Many considerable works have been done on left (right) multipliers in prime and semiprime rings during the last couple of decades ([9-11]). An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + yd(x)$ holds for all $x, y \in R$. Following [5], an additive mapping $F : R \to R$ is called a generalized derivation on $R$ if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ for every $x, y \in R$. Obviously, a generalized derivation with $d = 0$ covers the concept of left multipliers. Over the last few decades, several authors have investigated the relationship between the commutativity of the ring $R$ and certain specific types of derivations of $R$. The first result in this direction is due to E. C. Posner [8] who proved that if a ring $R$ admits a nonzero derivation $d$ such that $[d(x), x] \in Z(R)$ for all $x \in R$, then $R$ is commutative. This result was subsequently refined and extended by a number of authors. In [7], Bresar and Vuckman showed that a prime ring must be commutative if it admits a nonzero left derivation. Recently, many authors have obtained commutativity theorems for prime and semiprime rings admitting derivation, generalized derivation. In this paper, we introduce the notion of a semimultiplier of $R$, and investigate the commutativity of prime rings satisfying certain identities involving semimultiplier.

2. Preliminaries

Throughout $R$ will represent an associative ring with center $Z(R)$. For all $x, y \in R$, as a usual commutator, we shall write $[x, y] = xy - yx$, and $x \circ y = xy + yx$. Also,

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By hypothesis, we have

\[ R; x \in F \]

Suppose that in (1), we have

\[ x \in F \]

Proof. Since \( g \) is onto, we get

\[ x RF(r) = 0 \]

for every \( x \in I \) and \( r \in R \). Thus, we obtain

\[ xRF(r) = \{0\} \]

for every \( x \in I \) and \( r \in R \). Since \( R \) is prime and \( I \) is a nonzero right semigroup ideal of \( R \), it implies that \( F = 0 \).

\[ \square \]

Lemma 3.3. Let \( R \) be a prime ring and \( I \) be a nonzero semigroup ideal of \( R \). Suppose that \( F \) is a semimultiplier of \( R \) associated with \( g \) and \( a \in R \). If \( aF(x) = 0 \) for every \( x \in R \), then \( a = 0 \) or \( F = 0 \).

Proof. By hypothesis, we have \( aF(x) = 0 \) for any \( x \in I \) and \( a \in R \). Replacing \( x \) by \( xr \) in the last relation, we get

\[ ag(x)F(r) = 0, \forall x \in I, r \in R. \] (2)

Since \( g \) is onto, we have \( axF(r) = 0 \) for all \( x \in I \) and \( r \in R \). Now, replacing \( x \) by \( xs \) in (2), we have \( axsF(r) = 0 \) for every \( x \in I \) and \( r, s \in R \). Thus, we obtain

\[ axRF(r) = \{0\} \]

for every \( x \in I \) and \( r \in R \). Since \( R \) is prime and \( I \) is a nonzero right semigroup ideal of \( R \), it implies that \( ax = 0 \) for all \( x \in I \) or \( F(r) = 0 \) for every \( r \in R \). Hence

\[ aI = 0 \] or \( F = 0. \)
Assume that $F \neq 0$. Then we get $ax = 0$ for every $x \in I$. Replacing $x$ by $rx$ with $r \in R$ in the last equation, we have $arx = 0$ for every $x \in I, r \in R$. Thus
\[ aRx = \{0\}, \forall \ x \in I. \]
Since $R$ is prime and $I$ is a nonzero right ideal of $R$, we obtain $a = 0$.

**Lemma 3.4.** Let $R$ be a prime ring and $I$ be a nonzero semigroup ideal of $R$ and $a, b \in R$. If $ab = 0$, then $a = 0$ or $b = 0$.

**Proof.** By hypothesis, we have $ab = 0$ for any $x \in I$. Replacing $x$ by $xr$ with $r \in R$ in the last relation, we get $axrb = 0$ for all $x \in I$ and $r \in R$. Thus
\[ axrb = \{0\}, \forall x \in I. \tag{3} \]
Since $R$ is prime, we have $a = 0$ or $b = 0$. Suppose that $b \neq 0$. Then it means that $ax = 0$ for all $x \in I$. Taking $rx$ with $r \in R$ instead of $x$ in the last relation, it holds that $arx = 0$ for $x \in I, r \in R$. Hence we have
\[ aRx = \{0\}, \forall x \in I. \]
Since $R$ is prime and $I$ is a nonzero semigroup ideal of $R$, we have $a = 0$.

**Theorem 3.5.** Let $R$ be a prime ring and let $I$ be a nonzero semigroup ideal of $R$. Suppose that $R$ admits a nonzero semimultiplier $F$ associated with $g$ such that $[F(x), y] = 0$ for every $x, y \in I$. Then $R$ is commutative.

**Proof.** By hypothesis, we have $[F(x), y] = 0$ for any $x, y \in I$. Replacing $x$ by $xz$ with $z \in I$, in this relation, we have
\[ [F(x)g(z), y] = F(x)[g(z), y] + [F(x), y]g(z) = 0 \]
for every $x, y, z \in I$. Using the given hypothesis and the fact that $g$ is onto, we obtain $F(x)[z, y] = 0$ for every $x, y, z \in I$. Now, replacing $y$ by $ys$ with $s \in R$, in the last relation, we obtain
\[ F(x)y[z, s] = 0 \]
for every $x, y, z \in I$ and $s \in R$. This implies that $F(x)I[z, s] = \{0\}$ for every $x, z \in I$ and $s \in R$. Thus, by Lemma 3.4, we get $F(x) = 0$ or $[z, s] = 0$ for every $x, z \in I$ and $s \in R$. Since $F \neq 0$, we have $[z, s] = 0$ for every $z \in I$ and $s \in R$. Again, replacing $z$ by $zr$ with $r \in R$, in the last relation, we have $[zr, s] = z[r, s] + [z, s]r = z[r, s] = 0$. This implies that $xz[r, s] = 0$ for $0 \neq x \in I$, and hence $xI[r, s] = 0$. By Lemma 3.4, we have $[r, s] = 0$ for every $r, s \in R$, which implies that $R$ is commutative.

**Theorem 3.6.** Let $R$ be a prime ring and let $I$ be a nonzero semigroup ideal of $R$. Suppose that $R$ admits a nonzero semimultiplier $F$ associated with $g$ such that $F(I) \subseteq Z(R)$. Then $R$ is commutative.

**Proof.** By hypothesis, we have $F(xy) \in Z(R)$ for any $x, y \in I$, and so $F(x)g(y) \in Z(R)$ for every $x, y \in I$. This implies that $[F(x)g(y), r] = 0$ for all $x, y \in I$ and $r \in R$. This can be rewritten as following relation,
\[ F(x)[g(y), r] + [F(x), r]g(y) = 0, \forall \ x, y \in I, r \in R. \tag{4} \]
Replacing $r$ by $F(x)$ in (4), we have
\[ F(x)[g(y), F(x)] = 0, \forall \ x, y \in I. \tag{5} \]
Since $g$ is surjective, we have
\[ F(x)[y, F(x)] = 0, \ \forall \ x, y \in I. \] (6)

Again, replacing $y$ by $yz$ with $z \in I$, in (6), we get $F(x)y[z, F(x)] = 0$ for every $x, y, z \in I$. This implies that $F(x)I[z, F(x)] = \{0\}$ for every $x, z \in I$. By Lemma 3.4, we have $F(x) = 0$ or $[z, F(x)] = 0$ for every $x, z \in I$. Since $F \neq 0$, we have $[z, F(x)] = 0$ for all $x, z \in I$, which implies that $R$ is commutative by Theorem 3.5.

**Theorem 3.7.** Let $R$ be a prime ring and let $I$ be a nonzero semigroup ideal of $R$. Suppose that $R$ admits a nonzero semimultiplier $F$ associated with $g$ such that $[F(x), F(y)] = 0$, for every $x, y \in I$. Then $R$ is commutative.

**Proof.** By hypothesis, we have
\[ [F(x), F(y)] = 0, \ \forall \ x, y \in I. \] (7)
Replacing $y$ by $yz$ with $z \in I$, in (7), we have $[F(x), F(y)g(z)] = 0$, which implies that
\[ F(y)[F(x), g(z)] = 0. \]
Since $g$ is onto, we have
\[ F(y)[F(x), z] = 0, \ \forall \ x, y, z \in I. \] (8)
Now, replacing $z$ by $zs$ with $s \in R$, we have $F(y)z[F(x), s] = 0$ for every $x, y \in I$ and $s \in R$. This implies that $F(y)I[F(x), s] = \{0\}$ for every $y \in I$ and $s \in R$. Thus, by Lemma 3.4, we have $F(y) = 0$ for any $y \in I$ and $[F(x), s] = 0$ for $x \in I$ and $s \in R$. Since $F \neq 0$, we have $[F(x), s] = 0$, which implies that $F(x) \in Z(R)$ for any $x \in I$. That is, $F(I) \subseteq Z(R)$. Hence, by Theorem 3.6, $R$ is commutative.

**Theorem 3.8.** Let $R$ be a prime ring and let $I$ be a nonzero semigroup ideal of $R$. Suppose that $R$ admits a nonzero semimultiplier $F$ associated with $g$. If $F(x) \circ F(y) = 0$ holds for every $x, y \in I$, then $R$ is commutative.

**Proof.** By hypothesis, we have
\[ F(x) \circ F(y) = 0, \ \forall \ x, y \in I. \] (9)
Replacing $y$ by $zy$ with $z \in I$, in (9), we have $F(x) \circ F(yz) = F(x) \circ F(y)g(z) = 0$ for every $x, y, z \in I$, which implies that
\[ (F(x) \circ F(y))g(z) - F(y)[F(x), g(z)] = 0 \]
for every $x, y, z \in I$. Using the given relation, we have $F(y)[F(x), g(z)] = 0$ for every $x, y, z \in I$. Since $g$ is onto, we have $F(y)[F(x), z] = 0$ for every $x, y, z \in I$. Replacing $z$ by $zy$, where $x \in I$, in the last equation, we have $F(y)z[F(x), y] = 0$, which implies that $F(y)I[F(x), y] = \{0\}$ for every $x, y \in I$. By Lemma 3.4, we have $F(y) = 0$ or $[F(x), y] = 0$ for every $x, y \in I$. Since $F \neq 0$, we have $[F(x), y] = 0$ for every $x, y \in I$. Hence, by Theorem 3.5, $R$ is commutative.

**Theorem 3.9.** Let $R$ be a prime ring and let $I$ be a nonzero semigroup ideal of $R$. Suppose that $R$ admits a nonzero semimultiplier $F$ associated with $g$. If $F([x, y]) = 0$ holds for every $x, y \in I$, then $R$ is commutative.
Proof. By hypothesis, we have
\[ F([x, y]) = 0, \forall x, y \in I. \] (10)
Replacing \( y \) by \( zy \) with \( z \in I \), in (10), we have \( F[x, yz] = 0 \) for every \( x, y, z \in I \), which implies that
\[ F(y[x, z] + [x, y]z) = 0 \]
for every \( x, y, z \in I \). Hence we have \( F(y)g([x, z]) + F([x, y])g(z) = 0 \) for all \( x, y, z \in I \), and so
\[ F(y)g([x, z]) = 0, \forall x, z \in I. \] (11)
Since \( g \) is onto, we have \( F(y)[x, z] = 0 \) for every \( x, y, z \in I \). Replacing \( z \) by \( zr \), where \( r \in R \), in the last equation, we have \( F(y)z[x, r] = 0 \), which implies that
\[ F(y)I[x, r] = \{0\} \]
for every \( x, y \in I \) and \( r \in R \). By Lemma 3.4, we have \( F(y) = 0 \) or \([x, r] = 0\) for every \( x, y \in I \) and \( r \in R \). Since \( F \neq 0 \), we have \([x, r] = 0\) for every \( x \in I \) and \( r \in R \). Again, replacing \( x \) by \( xs \) in the last relation, we have \( x[s, r] = 0 \) for all \( x \in I \) and \( s, r \in R \). Hence \( I[s, r] = \{0\} \) for all \( s, r \in R \), which implies that \( IR[s, r] = \{0\} \) for all \( s, r \in R \). Since \( I \neq 0 \), we have \([s, r] = 0\) for all \( s, r \in R \), which implies that \( R \) is commutative.

Theorem 3.10. Let \( R \) be a prime ring and let \( I \) be a nonzero semigroup ideal of \( R \). Suppose that \( R \) admits a nonzero semimultiplier \( F \) associated with \( g \). If \( F(x \circ y) = 0 \) holds for every \( x, y \in I \), then \( R \) is commutative.

Proof. By hypothesis, we have
\[ F(x \circ y) = 0, \forall x, y \in I. \] (12)
Replacing \( y \) by \( yz \) with \( z \in I \), in (12), we have \( F(x \circ yz) = 0 \) for every \( x, y, z \in I \), which implies that
\[ F((x \circ y)z - y[x, z]) = 0 \]
for every \( x, y, z \in I \). Hence we have \( F(x \circ y)g(z) - F(y)g([x, z]) = 0 \) for all \( x, y, z \in I \), and so
\[ F(y)g([x, z]) = 0, \forall x, z \in I. \] (13)
Since \( g \) is onto, we have \( F(y)[x, z] = 0 \) for every \( x, y, z \in I \). Replacing \( z \) by \( zr \), where \( r \in R \), in the last equation, we have \( F(y)z[x, r] = 0 \), which implies that
\[ F(y)I[x, r] = \{0\} \]
for every \( x, y \in I \) and \( r \in R \). By Lemma 3.4, we have \( F(y) = 0 \) or \([x, r] = 0\) for every \( x, y \in I \) and \( r \in R \). Since \( F \neq 0 \), we have \([x, r] = 0\) for every \( x \in I \) and \( r \in R \). Again, replacing \( x \) by \( xs \) in the last relation, we have \( x[s, r] = 0 \) for all \( x \in I \) and \( s, r \in R \). Hence \( I[s, r] = \{0\} \) for all \( s, r \in R \), which implies that \( IR[s, r] = \{0\} \) for all \( s, r \in R \). Since \( I \neq 0 \), we have \([s, r] = 0\) for all \( s, r \in R \), which implies that \( R \) is commutative.

Theorem 3.11. Let \( R \) be a prime ring and let \( I \) be a nonzero semigroup ideal of \( R \). Suppose that \( R \) admits a semimultiplier \( F \) associated with \( g \) and \( F(x) \neq x \) for all \( x \in I \). If \( F(xy) = F(x)F(y) \) holds for every \( x, y \in I \), then \( F = 0 \).

Proof. By hypothesis, we have
\[ F(xy) = F(x)g(y) = F(x)F(y), \forall x \in I. \] (14)
Replacing \( x \) by \( xw \) in (14), we have \( F(xw)g(y) = F(xw)F(y) \), that is, \( F(x)g(w)g(y) = F(x)g(w)F(y) \) for all \( x, y, w \in I \). This implies that \( F(x)g(w)(g(y) - F(y)) = 0 \) for
all \(x, y, w \in I\). Since \(g\) is onto, we have \(F(x)R(y - F(y)) = \{0\}\) for all \(x, y \in R\). Since \(R\) is prime, we have \(F(x) = 0\) or \(y - F(y) = 0\) for all \(x, y \in R\). But \(F(x) \neq x\) for all \(x \in I\), and so \(F(x) = 0\) for all \(x \in I\), which implies that \(F = 0\) by Lemma 3.2.

\[\square\]

**Theorem 3.12.** Let \(R\) be a prime ring and let \(I\) be a nonzero semigroup ideal of \(R\). Suppose that \(R\) admits a nonzero semimultiplier \(F\) associated with \(g\) and \(g(x) \neq x\) for all \(x \in I\). If \(F(xy) = [x, y]\) holds for every \(x, y \in I\), then \(R\) is commutative.

**Proof.** By hypothesis, we have

\[F(xy) = [x, y], \; \forall \; x \in I.\] (15)

Replacing \(x\) by \(xy\) in (15), we have \(F(xy)g(y) = [xy, y]\), that is, \([x, y]g(y) = [x, y]y\) for all \(x, y \in I\). This implies that \([x, y]g(y) = [x, y]y\) for all \(x, y \in I\). Also, replacing \(x\) by \(sx\) in the last relation, we have \([s, y]x(g(y) - y) = 0\) for all \(x \in I\) and \(s \in R\). This implies that \([s, y]I(g(y) - y) = \{0\}\) for all \(x, y \in I\) and \(s \in R\). Since \(R\) is prime, we have \([s, y] = 0\) for all \(y \in I\) and \(s \in R\) or \(g(y) - y = 0\) for all \(y \in I\). But \(g(x) \neq x\) for all \(x \in I\), and so \([s, y] = 0\) for all \(x \in I\) and \(s \in R\). Again, replacing \(y\) by \(ry\) with \(r \in R\) in this relation, we have \([s, r]y = 0\), which implies that \([s, r]I = \{0\}\) for all \(r, s \in R\). Hence \([s, r]RI = \{0\}\). Since \(I \neq 0\), we have \([s, r] = 0\), which means that \(R\) is commutative.

\[\square\]

**Theorem 3.13.** Let \(R\) be a prime ring and let \(I\) be a nonzero semigroup ideal of \(R\). Suppose that \(R\) admits a nonzero semimultiplier \(F\) associated with \(g\) and \(g(x) \neq x\) for all \(x \in I\). If \(F(xy) = x \circ y\) holds for every \(x, y \in I\), then \(R\) is commutative.

**Proof.** By hypothesis, we have

\[F(xy) = x \circ y, \; \forall \; x \in I.\] (16)

Replacing \(x\) by \(xy\) in (16), we have \(F(xy)g(y) = (x \circ y)y\), that is, \((x \circ y)g(y) = (x \circ y)y\) for all \(x, y \in I\). This implies that \((x \circ y)g(y) = (x \circ y)y\) for all \(x, y \in I\). Also, replacing \(x\) by \(xy\) in the last relation, we have \((x \circ y)y(g(y) - y) = 0\) for all \(x, y \in I\). This implies that \((x \circ y)I(g(y) - y) = \{0\}\) for all \(x, y \in I\). By Lemma 3.4, we have \(x \circ y = 0\) for all \(x, y \in I\) or \(g(y) - y = 0\) for all \(y \in I\). But \(g(x) \neq x\) for all \(x \in I\), and so \(x \circ y = 0\) for all \(x, y \in I\). Again, replacing \(y\) by \(ys\) with \(s \in R\) in this relation, we have \(y[x, s] = 0\) for all \(x \in I\) and \(s \in R\). Taking \(x\) instead of \(x\) with \(r \in R\) in the last relation, we have \(y[x, s] = 0\), that is, \(yI[r, s] = 0\) for all \(r, s \in R\). Since \(I \neq 0\), we have \([r, s] = 0\) for all \(r, s \in R\). Hence \(R\) is commutative.

\[\square\]

**Theorem 3.14.** Let \(R\) be a prime ring and let \(I\) be a nonzero semigroup ideal of \(R\). Suppose that \(R\) admits a nonzero semimultiplier \(F\) associated with \(g\) and \(g(x) \neq x\) for all \(x \in I\). If \(F([x, y]) = x \circ y\) holds for every \(x, y \in I\), then \(R\) is commutative.

**Proof.** By hypothesis, we have

\[F([x, y]) = x \circ y, \; \forall \; x, y \in I.\] (17)

Replacing \(y\) by \(yz\) in (17), we have \(F(y[x, z] + [x, y]z) = x \circ yz\), that is, \(F(yg([x, z]) + F([x, y])g(z)) = (x \circ y)z - y[x, z]\) for all \(x, y, z \in I\). Taking \(z\) instead of \(x\) in this
Let, by hypothesis, we have

\[(a) \quad (z \circ y)g(z) = (z \circ y)z \quad \text{for all } y, z \in I. \]

By hypothesis, we obtain

\[(z \circ y)(g(z) - z) = 0 \quad \text{for all } y, z \in I.\]

Using the similar arguments of the last part proof Theorem 3.13, we get the required result.

\[\square\]

**Theorem 3.15.** Let \( R \) be a prime ring and let \( I \) be a nonzero semigroup ideal of \( R \). Suppose that \( R \) admits a nonzero semimultiplier \( F \) associated with \( g \) and \( g(x) \neq x \) for all \( x \in I \). If \( F(x \circ y) = [x, y] \) holds for every \( x, y \in I \), then \( R \) is commutative.

**Proof.** By hypothesis, we have

\[F(x \circ y) = [x, y], \quad \forall \ x, y \in I. \quad (18)\]

Replacing \( y \) by \( yz \) in (18), we have \( F((x \circ y)z - y[x, z]) = [x, yz] \), that is, \( F(x \circ y)g(z) - F(y)g([x, z]) = y[x, z] + [x, y]z \) for all \( x, y, z \in I \). Taking \( z \) instead of \( z \) in this relation, we have

\[F(x \circ y)g(x) = [x, y]x\]

for all \( x, y \in I \). By the hypothesis, we get \( [x, y]g(x) = [x, y]x \), and so \( [x, y](g(x) - x) = 0 \) for all \( x, y \in I \).

Using the similar arguments of the last part proof Theorem 3.12, we get the required result.

\[\square\]

**Theorem 3.16.** Let \( R \) be a prime ring and let \( I \) be a nonzero semigroup ideal of \( R \). Suppose that \( R \) admits a nonzero semimultiplier \( F \) associated with \( g \) such that \( F \) satisfies any one of the following conditions:

(a) \( [F(x), F(y)] = xy \) for every \( x, y \in I \),

(b) \( [F(x), F(y)] = yx \) for every \( x, y \in I \).

Then \( R \) is commutative.

**Proof.** (a) By hypothesis, we have

\[F(x), F(y)] = xy, \quad \forall \ x, y \in I. \quad (19)\]

Replacing \( y \) by \( yz \) in (19), we get \( F(x), F(yz)] = [F(x), F(y)]g(z) = xyz \) for every \( x, y \in I \). Using (19) and the fact that \( g \) is onto, we obtain

\[F(y)[F(x), z] = 0, \quad \forall \ x, y \in I. \quad (20)\]

Again, replacing \( z \) by \( zs \) with \( s \in R \), in (20), we have \( F(y)z[F(x), s] = 0 \), which implies that \( F(y)I[F(x), s] = \{0\} \) for any \( x, y \in I \) and \( s \in R \). Thus, by Lemma 3.4, we have \( F(y) = 0 \) or \( [F(x), s] = 0 \) for every \( x, y \in I \) and \( s \in R \). Since \( F \neq 0 \), we have \( [F(x), s] = 0 \), which implies that \( F(I) \subseteq Z(R) \). Hence, by Theorem 3.6, \( R \) is commutative.

(b) By hypothesis, we have

\[F(x), F(y)] = yx, \quad \forall \ x, y \in I. \quad (21)\]

Replacing \( x \) by \( xz \) with \( z \in I \), in (21), we get \( [F(xz), F(y)] = [F(x), g(z), F(y)] = 0 \) for every \( x, y \in I \). Using (21) and the fact that \( g \) is onto, we obtain

\[F(x)[z, F(y)] = 0, \quad \forall \ x, y, z \in I. \quad (22)\]

Using the same methods as we used in the last part proof of (a), we get the required result.

\[\square\]
Theorem 3.17. Let $R$ be a prime ring and let $I$ be a nonzero semigroup ideal of $R$. Suppose that $R$ admits a nonzero semimultiplier $F$ associated with $g$ such that $F$ satisfies any one of the following conditions:

(a) $F(x)F(y) = [x, y]$ for every $x, y \in I$,
(b) $F(y)F(x) = [x, y]$ for every $x, y \in I$,
(c) $F(x)F(y) = x \circ y$ for every $x, y \in I$.

Then $R$ is commutative.

Proof. (a) By hypothesis, we have

$$F(x)F(y) = [x, y], \ \forall \ x, y \in I. \quad (23)$$

Replacing $y$ by $yz$ in (23), we get $F(x)F(yz) = [x, yz]$ for every $x, y \in I$. This implies that $F(x)F(y)g(z) = y[x, z] + [x, y]z$ for every $x, y, z \in I$. Using (23), we obtain

$$[x, y]g(z) = y[x, z] + [x, y]z, \ \forall \ x, y, z \in I. \quad (24)$$

Taking $y$ in place of $x$, we have $y[y, z] = 0$ for all $y, z \in I$. Replacing $z$ by $zs$ with $z \in I$, in (24), we have $yz[y, s] = 0$, which implies that $y[I[x, s] = \{0\}$ for every $x, y \in I$ and $s \in R$. Thus, by Lemma 3.4, we have $y = 0$ for $[y, s] = 0$ for all $x, y \in I$ and $s \in R$. If $y = 0$, then $I = \{0\}$, a contradiction, and so $[x, s] = 0$ for every $x \in I$ and $s \in R$. Now, replacing $x$ by $xr$ in the last relation, we have $x[r, s] = 0$, which means that $I[r, s] = \{0\}$ for all $r, s \in R$. Hence we get $x[I[r, s] = \{0\}$ for $0 \neq x \in I$ and $r, s \in R$. By Lemma 3.4, we have $[r, s] = 0$, which implies that $R$ is commutative.

In cases of (b) and (c), using the same methods as we used in the last part proof of (a), we get the required result.

\[\square\]

References
