ON TRIPLE SEQUENCE SPACE OF BERNSTEIN OPERATOR
OF $\chi^3$ OF ROUGH $\lambda-$STATISTICAL CONVERGENCE IN
PROBABILITY DEFINED BY MUSIELAK-ORLICZ FUNCTION
OF $p-$METRIC

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Abstract. We introduced the triple sequence space of Bernstein polynomials $\chi^3$ of rough $\lambda-$statistical convergence in probability and discuss general properties of among these sequence spaces with respect of Musielak-Orlicz function.

1. Introduction

The idea of rough convergence was introduced by Phu [12], who also introduced the concepts of rough limit points and roughness degree. The idea of rough convergence occurs very naturally in numerical analysis and has interesting applications. Aytar [1] extended the idea of rough convergence into rough statistical convergence using the notion of natural density just as usual convergence was extended to statistical convergence. Pal et al. [11] extended the notion of rough convergence using the concept of ideals which automatically extends the earlier notions of rough convergence and rough statistical convergence.

The Bernstein operator of order $(r, s, t)$ is given by

$$B_{rst}(f, x) = \sum_{m=0}^{r} \sum_{n=0}^{s} \sum_{k=0}^{t} f\left(\frac{m}{r} \right) \left(\frac{n}{s} \right) \left(\frac{k}{t} \right) x^{m+n+k}(1-x)^{(m-r)+(n-s)+(k-t)}$$

where $f$ is a continuous (real or complex valued) function defined on $[0, 1]$.

Throughout the paper, $\mathbb{R}$ denotes the real of three dimensional space with metric $(X, d)$. Consider a triple sequence of Bernstein polynomials $(B_{mnk}(f, x))$ such that $(B_{mnk}(f, x)) \in \mathbb{R}$, $m, n, k \in \mathbb{N}$.

Let $f$ be a continuous function defined on the closed interval $[0, 1]$. A triple sequence of Bernstein polynomials $(B_{rst}(f, x))$ is said to be statistically convergent to $0 \in \mathbb{R}$, written as $st-\lim x = 0$, provided that the set

$$K_\epsilon := \{(m, n, k) \in \mathbb{N}^3 : |B_{mnk}(f, x) - f(x)| \geq \epsilon\}$$

has natural density zero for any $\epsilon > 0$. In this case, 0 is called the statistical limit of the triple sequence of Bernstein polynomials. i.e., $\delta(K_\epsilon) = 0$. That is,

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An Orlicz function (see [8]) is a function $d: \mathbb{R}^2 \to \mathbb{R}$. 

A triple sequence (real or complex) can be defined as a function $x = (x_0, x_1, x_2, \ldots)$. 

In this case, we write $\delta - \lim_{n \to \infty} B_{m,n,k}(f, x) = f(x)$ or $B_{m,n,k}(f, x) \to \lim_{n \to \infty} B_{m,n,k}(f, x)$.

A triple sequence (real or complex) can be defined as a function $x = (x_0, x_1, x_2, \ldots)$. The different types of notions of triple sequence were introduced and investigated at the initial by Sahiner et al. [13,14], Esi et al. [2-5], Datta et al. [6], Subramanian et al. [15], Deb Nath et al. [7] and many others. 

A triple sequence $x = (x_{m,n,k})$ is said to be triple analytic if

$$\sup_{m,n,k} \left| x_{m,n,k} \right|^\frac{1}{m+n+k} < \infty.$$ 

The space of all triple analytic sequences is usually denoted by $A^3$. A triple sequence $x = (x_{m,n,k})$ is called triple gai sequence if

$$((m + n + k)! |x_{m,n,k}|)^{\frac{1}{m+n+k}} \to 0 \text{ as } m, n, k \to \infty.$$ 

The space of all triple gai sequences is usually denoted by $\chi^3$.

2. Definitions and Preliminaries

2.1. Definition. An Orlicz function (see [8]) is a function $M: [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function $M$ is replaced by $M(x + y) \leq M(x) + M(y)$, then this function is called modulus function.

Lindenstrauss and Tzafriri ([9]) used the idea of Orlicz function to construct Orlicz sequence space.

A sequence $g = (g_{mn})$ defined by

$$g_{mn}(v) = \sup \{v | u - (f_{m,n,k})(u) : u \geq 0\}, m, n, k = 1, 2, \ldots$$

is called the complementary function of a Musielak-Orlicz function $f$. For a given Musielak-Orlicz function $f$, [see 10] the Musielak-Orlicz sequence space $t_f$ is defined as follows

$$t_f = \left\{ x \in w^3 : I_f(|x_{m,n,k}|)^{\frac{1}{m+n+k}} \to 0 \text{ as } m, n, k \to \infty \right\},$$

where $I_f$ is a convex modular defined by

$$I_f(x) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{m,n,k}(|x_{m,n,k}|)^{\frac{1}{m+n+k}}, x = (x_{m,n,k}) \in t_f.$$ 

We consider $t_f$ equipped with the Luxemburg metric

$$d(x, y) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{m,n,k} \left( \frac{|x_{m,n,k} - y_{m,n,k}|}{m,n,k} \right)^{\frac{1}{m+n+k}}$$

is an extended real number.

2.2. Definition. Let $X, Y$ be a real vector space of dimension $w$, where $u \leq m$. A real valued function $d_p(x_1, x_2, \ldots, x_n) = \|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p$ on $X$ satisfying the following four conditions:

(i) $\|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p = 0$ if and only if $d_1(x_1, 0), \ldots, d_n(x_n, 0)$ are linearly dependent,

(ii) $\|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p$ is invariant under permutation,

(iii) $\|(d_\alpha(x_1, 0), \ldots, d_n(x_n, 0))\|_p = |\alpha| \|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p, \alpha \in \mathbb{R}$

(iv) $d_p((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)) = (d_X(x_1, x_2, \ldots, x_n)p + d_Y(y_1, y_2, \ldots, y_n)p)^{1/p}$ for $1 \leq p < \infty$; (or)

(v) $d((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)) := \sup \left\{ d_X(x_1, x_2, \ldots, x_n), d_Y(y_1, y_2, \ldots, y_n) \right\}$, for $x_1, x_2, \ldots, x_n \in X, y_1, y_2, \ldots, y_n \in Y$ is called the $p$ product metric of the Cartesian product of $n$ metric spaces (see [16]).
2.3. Definition. Let \( \eta = (\lambda_{abc}) \) be a non-decreasing sequence of positive real numbers tending to infinity and \( \lambda_{111} = 1 \) and \( \lambda_{a+b+c+3} \leq \lambda_{a+b+c+3} + 1 \), for all \( a, b, c \in \mathbb{N} \). The collection of all such triple sequences \( \lambda \) is denoted by \( \mathcal{S} \).

The generalized de la Vallée-Poussin means are defined by
\[
  t_{abc}(x) = \lambda_{abc}^{-1} \sum_{m,n,k \in I_{abc}} B_{mnk}(f, x),
\]
where \( I_{abc} = [abc - \lambda_{abc} + 1, abc] \). Let \( f \) be a continuous function defined on the closed interval \([0, 1]\). A triple sequence of Bernstein polynomials \( (B_{mnk}(f, x)) \) of real numbers of random variables is said to be convergent to \( (V, \lambda) \)-summable to a number \( f(x) \) if \( t_{abc}(B_{mnk}(f, x)) \to f(x) \), as \( abc \to \infty \).

2.4. Definition. Let \( f \) be a continuous function defined on the closed interval \([0, 1]\). A triple sequence of Bernstein polynomials \( (B_{mnk}(f, x)) \) of real numbers of random variables is said to be strong \((V, \lambda)\)-convergent to \( f(x) \) if
\[
  \lim_{abc \to \infty} \frac{1}{\lambda_{abc}} \sum_{m \in I_a} \sum_{n \in I_b} \sum_{k \in I_c} |B_{mnk}(f, x), f(x)| = 0.
\]
In this case write \( B_{mnk}(f, x) \to [V, \lambda] f(x) \).

2.5. Definition. Let \( f \) be a continuous function defined on the closed interval \([0, 1]\). A triple sequence of Bernstein polynomials \( (B_{mnk}(f, x)) \) of real numbers of random variables is said to be \( \lambda \)-statistically convergent (or shortly: \( S_{\lambda} \)-convergent) to \( f(x) \) if for any \( \epsilon > 0 \),
\[
  \lim_{abc \to \infty} \frac{1}{\lambda_{abc}} \left\{ \{(m, nk) \in I_{abc} : |B_{mnk}(f, x), f(x)| \geq \epsilon \} \right\} = 0.
\]
In this case we write \( S_{\lambda} \to \lim B_{mnk}(f, x) = f(x) \) or by \( B_{mnk}(f, x) \to S_{\lambda}^\alpha f(x) \). Now we introduce the following main definition:

2.6. Definition. Let \( f \) be a continuous function defined on the closed interval \([0, 1]\). A triple sequence of Bernstein polynomials \( (B_{mnk}(f, x)) \) of real numbers and \( \alpha \) be non negative real number of random variables is said to be rough \([V, \lambda] \)-summable in probability to \( X : W \times W \times W \to \mathbb{R} \times \mathbb{R} \times \mathbb{R} \) with respect to the roughness of degree \( \alpha \) (or shortly: \( \alpha - [V, \lambda] \)-summable in probability to \( f(x) \)) if for any \( \epsilon > 0 \),
\[
  \lim_{abc \to \infty} \frac{1}{\lambda_{abc}} \sum_{m \in I_a} \sum_{n \in I_b} \sum_{k \in I_c} \mathbb{P}(|B_{mnk}(f, x), f(x)| \geq \alpha + \epsilon) = 0.
\]
In this case we write \( B_{mnk}(f, x) \to_{[V, \lambda]^{\alpha}} f(x) \). The class of all rough \([V, \lambda] \)-summable triple sequence space of Bernstein polynomials of random variables in probability will be denoted simply by \( \alpha [V, \lambda]^{\alpha} \).

2.7. Definition. Let \( f \) be a continuous function defined on the closed interval \([0, 1]\). A triple sequence of Bernstein polynomials \( (B_{mnk}(f, x)) \) of real numbers and \( \alpha \) be non negative real number of random variables is said to be rough \( \lambda \)-statistically convergent in probability to \( X : W \times W \times W \to \mathbb{R} \times \mathbb{R} \times \mathbb{R} \) with respect to the roughness of degree \( \alpha \) (or shortly: \( \alpha - \lambda \)-statistically convergent in probability to \( f(x) \)) if for any \( \epsilon, \delta > 0 \),
\[
  \lim_{abc \to \infty} \frac{1}{\lambda_{abc}} \left\{ \{(m, nk) \in I_{abc} : \mathbb{P}(|B_{mnk}(f, x), f(x)| \geq \alpha + \epsilon) \geq \delta \} \right\} = 0.
\]
In this case we write \( B_{mnk}(f, x) \to_{\alpha, \delta}^{\alpha} f(x) \). The class of all \( \alpha - \lambda \)-statistically convergent triple sequence space of Bernstein polynomials of random variables in probability will be denoted simply by \( \alpha S_{\lambda}^{\alpha} \).

2.8. Note. Let \( f \) be a continuous function defined on the closed interval \([0, 1]\). A triple sequence of Bernstein polynomials \( (B_{mnk}(f, x)) \) of real numbers and \( M \) be an Musielak-Orlicz function is defined by
\[
  \|\lambda_{M}, (d(x_1), d(x_2), \ldots, d(x_n-1))\|_p =
\]
\[
\left[ M_{mnk} \left( \|\mu_{mnk}(B_{mnk}(f, x)), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \right) \right],
\]
where \(\mu_{mnk}(X) = \left( (m + n + k)!B_{mnk}(f, x) \right)^{1/(m+n+k)} f(x)\).

3. Main Results

3.1. Theorem. Let \(f\) be a continuous function defined on the closed interval \([0, 1]\). A triple sequence of Bernstein polynomials \((B_{mnk}(f, x))\) of real numbers of random variables and \(M\) be an Musielak-Orlicz function are equivalent:

(i) \(\|X_M(B_{mnk}(f, x)), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\) is \(\alpha - [V, \lambda] -\) summable in probability to \(f(x)\).

(ii) \(\|X_M(D(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\) is \(\alpha - \lambda -\) statistically convergent in probability to \(f(x)\).

Proof: Similar to the proof of Theorem (3.1) in (see [18]).

3.2. Theorem. If \(\|X_M(B_{mnk}(f, x)), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \to_{S^p_{\alpha}} f(x)\) and \(\|X_M(B_{mnk}(f, y)), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \to_{S^p_{\beta}} f(y)\) then

\[ P \left( \left[ M_{mnk} \left( \|\mu_{mnk}(X), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \right) \right] \geq \alpha + \epsilon \right) = 0. \]

Proof: Similar to the proof of Theorem (3.1) in (see [17]).

3.3. Theorem. Let \(f\) be a continuous function defined on the closed interval \([0, 1]\). A triple sequence of Bernstein polynomials \((B_{mnk}(f, x))\) of real numbers of random variables and \(M\) be an Musielak-Orlicz function. If \(\lambda \in \mathbb{N}\) such that \(\frac{\lambda_{abc}}{(abc)} = 1\) then \(\alpha S^p_{\lambda} \subset \alpha S^p\).

Proof: Let \(0 < \eta < 1\) be given. Since \(\lim_{abc \to \infty} \frac{\lambda_{abc}}{(abc)} = 1\), we can choose \((u, v, w) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}\) such that \(\left| \frac{\lambda_{abc}}{(abc)} - 1 \right| < \frac{\eta}{2} \) for all \((a, b, c) > (u, v, w)\). Now, for \(\epsilon, \delta > 0\)

\[
\frac{1}{abc} \left\{ \left( \text{mmk} \leq (abc) : P \left( \left[ f_{mnk} \left( \|\mu_{mnk}(X), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \right) \right] \geq \alpha + \epsilon \right) \geq \delta \right) \right\} = \frac{1}{abc} \left\{ \left( \text{mmk} \leq (abc) - \lambda_{abc} : P \left( \left[ f_{mnk} \left( \|\mu_{mnk}(X), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \right) \right] \geq \alpha + \epsilon \right) \geq \delta \right) \right\}
\]

\[
\leq \frac{1}{abc} \left\{ \left( \text{mmk} \in I_{abc} : P \left( \left[ f_{mnk} \left( \|\mu_{mnk}(X), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \right) \right] \geq \alpha + \epsilon \right) \geq \delta \right) \right\} + \frac{1}{abc} \left\{ \left( \text{mmk} \in I_{abc} : P \left( \left[ f_{mnk} \left( \|\mu_{mnk}(X), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \right) \right] \geq \alpha + \epsilon \right) \geq \delta \right) \right\}
\]

\[
\leq 1 - (1 - \frac{\eta}{2}) + \frac{\lambda_{abc}}{\lambda_{abc}} \left\{ \left( \text{mmk} \in I_{abc} : P \left( \left[ f_{mnk} \left( \|\mu_{mnk}(X), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \right) \right] \geq \alpha + \epsilon \right) \geq \delta \right) \right\}
\]

\[
< \frac{\eta}{2} + \frac{\lambda_{abc}}{\lambda_{abc}} \left\{ \left( \text{mmk} \in I_{abc} : P \left( \left[ f_{mnk} \left( \|\mu_{mnk}(X), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \right) \right] \geq \alpha + \epsilon \right) \geq \delta \right) \right\}
\]

holds for all \((a, b, c) \geq (u, v, w)\).
3.4. Theorem. Let \( f \) be a continuous function defined on the closed interval \([0, 1]\). A triple sequence of Bernstein polynomials \( B_{mnk} (f, x) \) of real numbers of random variables and \( M \) be an Musielak-Orlicz function and \( \alpha S^p \subset \alpha S^p \lambda \) if and only if \( \lim_{abc} \frac{\lambda_{abc}}{\alpha_{abc}} > 0 \).

Proof: Let \( \lim_{abc} \frac{\lambda_{abc}}{\alpha_{abc}} > 0 \). Then for \( \epsilon, \delta > 0 \), we have
\[
\frac{1}{\lambda_{abc}} \left\{ (mnk) \in \left\{ \begin{array}{l}
\left( f_{mnk} \left( \left\| \mu_{mnk} (X) , (d (x_1) , d (x_2) , \cdots , d (x_{n-1})) \|_p \right\| \right) \right. \\
\left. \left\| \mu_{mnk} (X) , (d (x_1) , d (x_2) , \cdots , d (x_{n-1})) \|_p \right\| \right) \right. \\
\left. \right\} \left( \begin{array}{l}
\epsilon \right) \geq \delta \right\}
\]
\[
\geq \frac{1}{\lambda_{abc}} \left\{ (mnk) \in I_{abc} : P \left( \left\{ \begin{array}{l}
\left( f_{mnk} \left( \left\| \mu_{mnk} (X) , (d (x_1) , d (x_2) , \cdots , d (x_{n-1})) \|_p \right\| \right) \right. \\
\left. \left\| \mu_{mnk} (X) , (d (x_1) , d (x_2) , \cdots , d (x_{n-1})) \|_p \right\| \right) \right. \\
\left. \right\} \left( \begin{array}{l}
\epsilon \right) \geq \delta \right\}
\]
\[
= \frac{1}{\lambda_{abc}} \left\{ (mnk) \in I_{abc} : P \left( \left\{ \begin{array}{l}
\left( f_{mnk} \left( \left\| \mu_{mnk} (X) , (d (x_1) , d (x_2) , \cdots , d (x_{n-1})) \|_p \right\| \right) \right. \\
\left. \left\| \mu_{mnk} (X) , (d (x_1) , d (x_2) , \cdots , d (x_{n-1})) \|_p \right\| \right) \right. \\
\left. \right\} \left( \begin{array}{l}
\epsilon \right) \geq \delta \right\}
\]
\[
\text{Taking limit } abc \to \infty \text{ we get } \| \chi^3_M (B_{mnk} (f, x)) , (d (x_1) , d (x_2) , \cdots , d (x_{n-1})) \|_p \to_{\alpha}^{S^p} f (x).
\]
Conversely, let \( \lim_{abc} \frac{\lambda_{abc}}{\alpha_{abc}} = 0 \) then we can choose a subsequence \( (a_u,b_v,c_w) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} \) such that \( \frac{\lambda_{a_u,b_v,c_w}}{\alpha_{a_u,b_v,c_w}} < \frac{1}{uvw} \) for all \( u, v, w \in \mathbb{N} \). Define a triple sequence space of Bernstein polynomials of \( B_{mnk} (f, x) \) of random numbers of random variables whose probability density function is
\[
\mu_{abc} (X) = \begin{cases} 
1, & \text{if } 0 < X < 1 \\
0, & \text{otherwise , where (abc) } \in I_{abc} \text{ for some (uvw) } \in \mathbb{N} 
\end{cases}
\]
Let \( 0 < \epsilon, \delta < 1 \). Then
\[
P \left( \left\{ \begin{array}{l}
\left( f_{mnk} \left( \left\| \mu_{mnk} (X) , (d (x_1) , d (x_2) , \cdots , d (x_{n-1})) \|_p \right\| \right) \right. \\
\left. \left\| \mu_{mnk} (X) , (d (x_1) , d (x_2) , \cdots , d (x_{n-1})) \|_p \right\| \right) \right. \\
\left. \right\} \left( \begin{array}{l}
\epsilon \right) \geq \delta \right\}
\]
\[
= \begin{cases} 
1, & \text{if } (abc) \in I_{abc} \text{ for some (uvw) } \in \mathbb{N} \\
\left( 1 - \frac{\epsilon}{2} \right)^n, & \text{otherwise .}
\end{cases}
\]
We have
\[
\frac{1}{\lambda_{abc}} \left\{ (mnk) \in I_{abc} : P \left( \left\{ \begin{array}{l}
\left( f_{mnk} \left( \left\| \mu_{mnk} (X) , (d (x_1) , d (x_2) , \cdots , d (x_{n-1})) \|_p \right\| \right) \right. \\
\left. \left\| \mu_{mnk} (X) , (d (x_1) , d (x_2) , \cdots , d (x_{n-1})) \|_p \right\| \right) \right. \\
\left. \right\} \left( \begin{array}{l}
\epsilon \right) \geq \delta \right\}
\]
\[
= \begin{cases} 
1, & \text{if } (abc) \in I_{abc} \text{ for some (uvw) } \in \mathbb{N} \\
0, & \text{otherwise .}
\end{cases}
\]
Hence \( \| \chi^3_M (B_{mnk} (f, x)) , (d (x_1) , d (x_2) , \cdots , d (x_{n-1})) \|_p \notin_{\alpha}^{S^p} \).

4. Conclusions and Future Work

We introduced triple sequence space of Bernstein polynomials of random numbers of random variables of rough \( \lambda \)- statistical convergence in probability with respect sequence of Musielak-Orlicz function. For the reference sections, consider the following introduction described the main results are motivating the research.

5. Competing Interests

The authors declare that there is not any conflict of interests regarding the publication of this manuscript.

References


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