ON SOME DOUBLY NONLINEAR SYSTEM IN INHOMOGENEOUS ORLICZ SPACES

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Abstract. Our aim in this paper is to discuss the existence of renormalized solutions of the following systems:
\[
\frac{\partial b_i(x,u_i)}{\partial t} - \text{div}(a(x,t,u_i,\nabla u_i)) - \phi_i(x,t,u_i) + f_i(x,u_1,u_2) = 0 \quad i=1,2.
\]
where the function \(b_i(x,u_i)\) verifies some regularity conditions, the term \(a(x,t,u_i,\nabla u_i)\) is a generalized Leray-Lions operator and \(\phi_i\) is a Carathéodory function assumed satisfy only a growth condition. The source term \(f_i(x,u_1,u_2)\) belongs to \(L^1(\Omega \times (0,T))\).

1. Introduction

Let \(\Omega\) be a bounded open subset of \(\mathbb{R}^N\), \((N \geq 1)\) with the segment property. Fixing a final time \(T > 0\) and let \(Q_T := (0,T) \times \Omega\). We prove the existence of a renormalized solutions for the nonlinear parabolic systems:
\[
(b_i(x,u_i))_t - \text{div}\left(a(x,t,u_i,\nabla u_i) - \Phi_i(x,t,u_i)\right) + f_i(x,u_1,u_2) = 0 \quad \text{in} \ Q, \quad \tag{1}
\]
\[
u_i = 0 \quad \text{on} \ \Gamma := (0,T) \times \partial \Omega, \quad \tag{2}
\]
\[
b_i(x,u_i)(t=0) = b_i(x,u_{i,0}) \quad \text{in} \ \Omega, \quad \tag{3}
\]
where \(i = 1,2\). Here, the vector field
\[
a : \Omega \times (0,T) \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N \quad \text{is a Carathéodory function} \quad \tag{4}
\]
where \(A(u) = -\text{div}(a(x,t,u,\nabla u))\) is a Leray-Lions operator defined on the inhomogeneous Orlicz-Sobolev space \(W^{1,x}_{0,L}(Q_T)\), \(M\) is a \(N\)-function related to the growth of \(A(u)\) (see assumptions (8)-(10)), and to the growth of the lower order Carathéodory function \(\phi(x,t,u)\) (see assumption (11)). \(b : \Omega \times R \to \mathbb{R}\) is a Carathéodory function such that for every \(x \in \Omega\), \(b(x,\cdot)\) is a strictly increasing \(C^1\)-function, the source term \(f_i\) is a Carathéodory function.

In the first time, on the Classical Sobolev space. The existence of renormalized solution has been proved by R.-Di Nardo et al. in [9] in the case \(b(x,u) = u\), by

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H. Redwane in [12] where \( b(u) = b(x,u) \), by A. Aberqi, J. Bennouna and H. Redwane, in [2], where \( |\phi(x,t,s)| < c(x,t)|s|^\gamma \) and by L. Aharouch, J. Bennouna and A. Touzani in [3] and by A. Benkirane and J. Bennouna [7] in the Orlicz spaces and degenerated spaces.

In the second time, the existence of a renormalized solution to a class of doubly nonlinear parabolic systems, in the classical Sobolev space \( b_i(u_i) = u_i \) and \( \phi_i = \phi \), \( i = 1, 2 \) has been studied by H. Redwane [12] and for the parabolic version of (1.1)-(1.3), existence and uniqueness results are already proved in [8] (see also [13]) in the case \( f_i(x,u_1,u_2) \) is replaced by \( f - \text{div}(g) \), by Azroul et al. in [6] has studied the Problem (1), where the term \( \phi \) is continuous function, who allows to eliminate it by using the Stockes formula. Recently Aberqi et al. in [2] has treated the same problem, where the right-side is \( f - \text{div}(g) \) where \( f \in L^1(Q) \) and \( g \in (L^p(Q))^N \) and the term \( \phi \) satisfy the following growth condition \( \phi(x,t,s) \leq c(x,t)|s|^\gamma \).

It is our purpose in this paper to generalize the last two results in the Orlicz-Sobolev spaces and with the condition \( \phi(x,t,s) \leq c(x,t)M^{-1}(\frac{\gamma}{2}|s|) \) and not assuming any other condition (no coercivity condition and no \( \Delta_2 \) condition on the N-function \( M \)). However the uniqueness of solution remains yet open.

To illustrate the type of problems in Orlicz-Sobolev spaces, we cite the model bellow:

\[
\left\{ \begin{array}{l}
\frac{\partial |u|^{(p(x))}}{\partial t} - \text{div}(\frac{\alpha}{1+|u|^\gamma} \cdot \log(e + u)) - \text{div}(c(x,t)|u|^{p-1}) = f & \text{in } Q_T, \\
u(x,t) = 0 & \text{on } \partial \Omega \times (0,T),
\end{array} \right.
\]

where \( b(x,u) = |u|^{q(x)-2}u \), where \( q : \Omega \rightarrow [1, +\infty[ \), with \( q(x) \leq -|x|^2 + 2 \).

\( Au = -\Delta_M u = -\text{div}(\frac{|\nabla u|^{p-2} \nabla u}{1+|u|^\gamma} \cdot \log(e + u)) \), here the N-functions \( M \) associated to the operator are \( M(t) = t^p \log^p(e + t) \), and \( P(t) = \frac{\nu}{p} \), with \( P \ll M \).

\( \phi(x,t,u) = c(x,t)|u|^{p-1} \) the term in divergentiel form which is not continuous with respect to \( x \).

This article is organized as follows: In Section 2, we give some technical lemmas. In Section 3 we give the basic assumptions and give the definition of a renormalized solution of (1.1)-(1.3) and in Section 4, we establish (Theorem 4) the existence of such a solutions.

2. Preliminaries and some technical lemmas

Let \( M : R^+ \rightarrow R^+ \) be an N-function, that is, \( M \) is continuous, convex, with \( M(t) > 0 \) for \( t > 0 \), \( M(t)/t \rightarrow 0 \) as \( t \rightarrow 0 \), and \( M(t)/t \rightarrow +\infty \) as \( t \rightarrow +\infty \). Equivalently, \( M \) admits the representation \( M(t) = \int_0^t a(s)ds \), where \( a : R^+ \rightarrow R^+ \) is nondecreasing, right continuous, with \( a(0) = 0, a(t) > 0 \) for \( t > 0 \), and \( a(t) \rightarrow +\infty \) as \( t \rightarrow +\infty \). The N-function \( \overline{M} \) conjugate to \( M \) is defined by \( \overline{M}(t) = \int_0^t \overline{a}(s)ds \), where \( \overline{a} : R^+ \rightarrow R^+ \), is given by \( \overline{a}(t) = \sup\{s : a(s) \leq t\} \).

We will extend these N-functions into even functions on all \( R \). Let \( P \) and \( Q \) be two N-functions. \( P \ll Q \) means that \( P \) grows essentially less rapidly than \( Q \), that is, for each \( \epsilon > 0 \), \( \frac{P(t)}{Q(\epsilon t)} \rightarrow 0 \) as \( t \rightarrow +\infty \). This is the case if and only if \( \lim_{t \rightarrow +\infty} \frac{Q^{-1}(t)}{P^{-1}(t)} = 0 \).
The Orlicz class $K_M(\Omega)$ (resp. the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence classes of) real valued measurable functions $u$ on $\Omega$ such that

$$\int_{\Omega} M(u(x))dx < +\infty \quad (\text{resp.} \quad \int_{\Omega} M\left(\frac{u(x)}{\lambda}\right)dx < +\infty \quad \text{for some} \quad \lambda > 0).$$

The set $L_M(\Omega)$ is Banach space under the norm

$$\|u\|_{M,\Omega} = \inf\{\lambda > 0 : \int_{\Omega} M\left(\frac{u(x)}{\lambda}\right)dx \leq 1\}$$

and $K_M(\Omega)$ is a convex subset of $L_M(\Omega)$. The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\Omega$ is denoted by $E_M(\Omega)$. The dual $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of the pairing $\int_{\Omega} uvdx$, and the dual norm of $L_{\overline{M}}(\Omega)$ is equivalent to $\|u\|_{\overline{M},\Omega}$. We now turn to the Orlicz-Sobolev space, $W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$) is the space of all functions $u$ such that $u$ and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ (resp. $E_M(\Omega)$). It is a Banach space under the norm

$$\|u\|_{1,M} = \sum_{|\alpha| \leq 1} \|D^\alpha u\|_{M,\Omega}$$

Thus, $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of product of $N + 1$ copies of $L_M(\Omega)$. Denoting this product by $\Pi L_M$, we will use the weak topologies $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ and $\sigma(\Pi L_M, \Pi L_{\overline{M}})$. The space $W^1_0E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $D(\Omega)$ in $W^1E_M(\Omega)$ and the space $W^1_0L_M(\Omega)$ as the $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ closure of $D(\Omega)$ in $W^1L_M(\Omega)$.

Let $W^{-1}L_{\overline{M}}(\Omega)$ (resp. $W^{-1}E_{\overline{M}}(\Omega)$) denote the space of distributions on $\Omega$ which can be written as sums of derivatives of order $\leq 1$ of functions in $L_{\overline{M}}(\Omega)$ (resp. $E_{\overline{M}}(\Omega)$). It is a Banach space under the usual quotient norm (for more details see [1]).

We recall the following Lemma:

**Lemma 1** (see [11] and [10]) For all $u \in W^1_0L_M(Q_T)$ with $meas(Q_T) < +\infty$ one has

$$\int_{Q_T} M\left(\frac{|u|}{\lambda}\right)dxdt \leq \int_{Q_T} M(|\nabla u|)dxdt? \quad (5)$$

where $\lambda = diamQ_T$, is the diameter of $Q_T$.

### 3. Assumptions and statement of main results

Throughout this paper, we assume that the following assumptions hold true:

Let $P$ and $M$ are two N-functions, such that $P << M$, and for all $i = 1, 2$:

$$b_i : \Omega \times R \rightarrow R \text{ is a Carathéodory function such that for every } x \in \Omega, \quad (6)$$

$b_i(x,.)$ is a strictly increasing $C^1(R)$-function and $b_i \in L^\infty(\Omega \times R)$ with $b_i(x,0) = 0$. Next for any $k > 0$, there exists a constant $\lambda_k^i > 0$ and functions $A_k^i \in L^\infty(\Omega)$ and $B_k^i \in L_M(\Omega)$ such that:

$$\lambda_k^i \leq \frac{\partial b_i(x,s)}{\partial s} \leq A_k^i(x) \quad \text{and} \quad |\nabla_x \left(\frac{\partial b_i(x,s)}{\partial s}\right)| \leq B_k^i(x) \quad \text{a.e.} \ x \in \Omega \text{ and } \forall |s| \leq k. \quad (7)$$
For almost every \((x, t) \in Q_T\), for every \(s \in R\) and every \(\xi, \eta \in R^N\)
\[
|a(x, t, s, \xi)| \leq d_k(x, t) + \beta_{k,1}M^{-1}P(\beta_{k,2}|\xi|),
\]
\[
a(x, t, s, \xi) \xi \geq \alpha M(|\xi|) \text{ with } \alpha > 0,
\]
\[
(a(x, t, s, \xi) - a(x, t, s, \eta))(\xi - \eta) > 0 \text{ with } \xi \neq \eta,
\]
where \(d_k(x, t) \in E^s_m(Q_T)\), and \(\beta_{k,1}, \beta_{k,2} > 0\) are the given real numbers.

Let \(\phi(x, t, s)\) be a Carathéodory function such that for a.e (\(x, t) \in Q_T\) for all \(s \in R\)
\[
|\phi_i(x, t, s)| \leq c_i(x, t)M^{-1}M(\frac{\alpha}{\lambda}|s|), \quad c_i(,.,.) \in L^\infty(Q_T), \text{ where } ||c_i(,.,.)||_\infty \leq \alpha,
\]
\[
f_i : \Omega \times R \times R \rightarrow R \text{ is a Carathéodory function with}
\]
\[
f_i(x, 0, s) = f_2(x, s, 0) = 0 \text{ a.e. } x \in \Omega, \forall s \in R,
\]
and for almost every \(x \in \Omega\), for every \(s_1, s_2 \in R\),
\[
\text{sign}(s_i)f_i(x, s_1, s_2) \geq 0.
\]

The growth assumptions on \(f_i\) are as follows: For each \(K > 0\), there exists \(\sigma_K > 0\) and a function \(F_K\) in \(L^1(\Omega)\) such that
\[
|f_1(x, s_1, s_2)| \leq F_K(x) + \sigma_K |b_2(x, s_2)|,
\]
a.e. in \(\Omega\), for all \(s_1\) such that \(|s_1| \leq K\), for all \(s_2 \in R\). For each \(K > 0\), there exists \(\lambda_K > 0\) and a function \(G_K\) in \(L^1(\Omega)\) such that
\[
|f_2(x, s_1, s_2)| \leq G_K(x) + \lambda_K |b_1(x, s_1)|,
\]
for almost every \(x \in \Omega\), for every \(s_2\) such that \(|s_2| \leq K\), and for every \(s_1 \in R\).

Finally, we assume the following condition on the initial data \(u_{i,0}\):
\[
u_{i,0} \text{ is a measurable function such that } b_i(., u_{i,0}) \in L^1(\Omega), \text{ for } i = 1, 2.
\]

In this paper, for \(K > 0\), we denote by \(T_K : r \mapsto \min(K, \max(r, -K))\) the truncation function at height \(K\). For any measurable subset \(E \subset Q_T\), we denote by \(\text{meas}(E)\) the Lebesgue measure of \(E\). For any measurable function \(v\) defined on \(Q\) and for any real number \(s, \chi_{\{v > s\}}\) (respectively, \(\chi_{\{v = s\}}, \chi_{\{v < s\}}\) denote the characteristic function of the set \(\{x, t) \in Q_T : v(x, t) > s\}\) (respectively, \(\{x, t) \in Q_T : v(x, t) < s\}\).

**Definition 2** A couple of functions \((u_1, u_2)\) defined on \(Q_T\) is called a renormalized solution of (6)-(16) if for \(i = 1, 2\) the function \(u_i\) satisfies
\[
T_K(u_i) \in W^{1,\infty}_0 L^m(Q_T) \quad \text{and} \quad b_i(x, u_i) \in L^\infty(0, T; L^1(\Omega)),
\]
\[
\int_{\{m \leq |u_i| \leq m+1\}} a(x, t, u_i, \nabla u_i) \nabla u_i \, dx \, dt \to 0 \text{ as } m \to +\infty,
\]
For every function \(S\) in \(W^{2,\infty}(R)\) which is piecewise \(C^1\) and such that \(S'\) has a compact support, we have
\[
\frac{\partial B_i(S, u_i)}{\partial t} - \text{div}(S'(u_i) a(x, t, u_i, \nabla u_i)) + S''(u_i) a(x, t, u_i, \nabla u_i) \nabla u_i
\]
\[
+ \text{div}(S''(u_i) \phi_i(x, t, u_i) - S''(u_i) \phi_i(x, t, u_i) \nabla u_i + f_i(x, u_i, u_2) S'(u_i) = 0}
\]
\[
B_i(S, u_i)(t = 0) = B_i(S, u_{i,0}) \text{ in } \Omega,
\]
where \(B_i(S(r) = \int_0^r B_i(x, s) S'(s) \, ds\).

**Remark 3**
Due to (17), each term in (19) has a meaning in $W^{-1,\infty}(Q_T) + L^1(Q_T)$. Indeed, if $K$ such that $\text{supp} S \subset [-K,K]$, the following identifications are made in (19)

- $B_{i,S}(x,u_i) \in L^\infty(Q_T)$, since $|B_{i,S}(x,u_i)| \leq K\|A^i_k\|_{L^\infty(\Omega)}\|S^i\|_{L^\infty(R)}$
- $S'(u_i)a(x,t,u_i,\nabla u_i)$ can be identified with $S'(u_i)a(x,t,T_K(u_i),\nabla T_K(u_i))$ a.e. in $Q_T$. Since indeed $|T_K(u_i)| \leq K$ a.e. in $Q_T$. As a consequence of (8), (17) and $S'(u_i) \in L^\infty(Q_T)$, it follows that

\[
S'(u_i)a(x,T_K(u_i),\nabla T_K(u_i)) \in (L^\infty(Q_T))^N.
\]

- $S'(u_i)a(x,t,u_i,\nabla u_i)\nabla u_i$ can be identified with $S'(u_i)a(x,t,T_K(u_i),\nabla T_K(u_i))\nabla T_K(u_i)$ a.e. in $Q_T$ with (7) and (17) it has

\[
S'(u_i)a(x,t,T_K(u_i),\nabla T_K(u_i))\nabla T_K(u_i) \in L^1(Q_T)
\]

- $S'(u_i)\Phi_i(u_i)$ and $S''(u_i)\Phi_i(u_i)$ respectively identify with $S'(u_i)\Phi_i(T_K(u_i))$ and $S''(u_i)\Phi_i(T_K(u_i))\nabla T_K(u_i)$. In view of the properties of $S$ and (11), the functions $S'$, $S''$ and $\Phi \circ T_K$ are bounded on $R$ so that (17) implies that $S'(u_i)\Phi_i(T_K(u_i)) \in (L^\infty(Q_T))^N$ and $S''(u_i)\Phi_i(T_K(u_i))\nabla T_K(u_i) \in (L^\infty(Q_T))^N$.

- $S'(u_1)f_1(x,u_1,u_2)$ identifies with $S'(u_1)f_1(x,T_K(u_1),u_2)$ a.e. in $Q_T$ or $S'(u_2)f_2(x,u_1,T_K(u_2))$ a.e. in $Q_T$. Indeed, since $|T_K(u_i)| \leq K$ a.e. in $Q_T$, assumptions (14) and (15) and using (17) and of $S'(u_i) \in L^\infty(Q)$, one has

\[
S'(u_1)f_1(x,T_K(u_1),u_2) \in L^1(Q_T) \quad \text{and} \quad S'(u_2)f_2(x,u_1,T_K(u_2)) \in L^1(Q_T).
\]

As consequence, (19) takes place in $D'(Q_T)$ and that

\[
\frac{\partial B_{i,S}(x,u_i)}{\partial t} \in W^{-1,\infty}(Q_T) + L^1(Q_T).
\] (21)

Due to the properties of $S$ and (7)

\[
B_{i,S}(x,u_i) \in W^{1,\infty}(Q_T).
\] (22)

Moreover (21) and (22) implies that $B_{i,S}(x,u_i) \in C^0([0,T],L^1(\Omega))$ so that the initial condition (20) makes sense.

4. Existence result

We shall prove the following existence theorem

**Theorem 4** Assume that (6)-(16) hold true. There is at least a renormalized solution $(u_1,u_2)$ of Problem (1).

**Proof.** We give the proof in 5 steps.

**Step 1: Approximate problem.**

Let us introduce the following regularization of the data: for $n > 0$ and $i = 1,2$

\[
b_{i,n}(x,s) = b_i(x,T_n(s)) + \frac{1}{n} \quad s \in \mathbb{R},
\] (23)

\[
a_{n}(x,t,s,\xi) = a(x,t,T_n(s),\xi) \quad \text{a.e. in } \Omega, \forall s \in \mathbb{R}, \forall \xi \in \mathbb{R}^N,
\] (24)

\[
\Phi_{i,n}(x,t,s) = \Phi_{i,n}(x,t,T_n(s)) \quad \text{a.e. } (x,t) \in Q_T, \forall s \in \mathbb{R}.
\] (25)
Using Young inequality
\[ B^{n}_{i,k}(x,u_{i,0n}) \leq k \int_{\Omega} |b_{i,n}(x,u_{i,0n})| dx \leq k \|b_{i}(x,u_{i,0})\|_{L^1(\Omega)}, \quad \forall k > 0. \]

In view of (13), we have $\int_{Q_{i}} f_{i,n} T_{k}(u_{i,n}) \, dx \, dt \geq 0$
Using Young inequality 11 and lemma 5, we obtain
\[ \int_{Q_{i}} \phi_{i,n}(x,t,u_{i,n}) \nabla T_{k}(u_{i,n}) \, dx \, dt \leq \|c_{i}\|_{L^\infty} (\alpha_{0}^{i} + 1) \int_{\Omega} M(\nabla T_{k}(u_{i,n})) \, dx \, dt. \]
We conclude that
\[ \frac{\lambda_k}{2} \int_{\Omega} |T_k(u_{i,n})|^2 \, dx + \alpha \int_{\Omega} M(\nabla T_k(u_{i,n})) \, dx \, dt \leq \]
\[ \|c_i\|_{L^\infty}(\alpha_0^i + 1) \int_{\Omega} M(\nabla T_k(u_{i,n})) \, dx \, dt + k(\|f\|_{L^1(Q_T)} + \|b(x, u_{i,0,n})\|_{L^1(\Omega)}). \]

Then
\[ \frac{\lambda_k}{2} \int_{\Omega} |T_k(u_{i,n})|^2 \, dx + [\alpha - \|c\|_{L^\infty}(\alpha_0^i + 1)] \int_{\Omega} M(\nabla T_k(u_{i,n})) \, dx \, dt \leq C_i.k. \]

If we choose \( \|c_i\|_{L^\infty} < \alpha \) and \( \alpha_0^i < \frac{\alpha - \|c_i\|_{L^\infty}}{\|c_i\|_{L^\infty}} \)
we get
\[ \int_{Q_T} M(\nabla T_k(u_{i,n})) \, dx \, dt \leq C_i.k, \quad (35) \]
then, we conclude that \( T_k(u_{i,n}) \) is bounded in \( W^{1,2}_M(Q_T) \) independently of \( n \)
and for any \( k \geq 0 \), so there exists a subsequence still denoted by \( u_n \) such that
\[ T_k(u_{i,n}) \rightharpoonup \psi_{i,k} \quad (36) \]
weakly in \( W^{1,2}_M(Q_T) \) for \( \sigma(\Pi L_M, \Pi E_M) \) strongly in \( E_M(Q_T) \) and a.e in \( Q_T \).
Since Lemma (5) and (41), we get also,
\[ M(\frac{k}{N}) \text{meas}\left\{ \{u_{i,n} > k\} \cap B_R \times [0,T] \right\} \leq \int_{[0,T]} \int_{\{u_{i,n} > k\} \cap B_R} M(\frac{T_k(u_{i,n})}{\lambda}) \, dx \, dt \]
\[ \leq \int_{Q_T} M(\frac{T_k(u_{i,n})}{\lambda}) \, dx \, dt \leq \int_{Q_T} M(\nabla T_k(u_{i,n})) \, dx \, dt. \]
Then
\[ \text{meas}\left\{ \{u_{i,n} > k\} \cap B_R \times [0,T] \right\} \leq \frac{C_i.k}{M(\frac{k}{N})}, \]
which implies that:
\[ \lim_{k \rightarrow +\infty} \text{meas}\left\{ \{u_{i,n} > k\} \cap B_R \times [0,T] \right\} = 0. \text{ uniformly in } n. \]
Now we turn to prove the almost every convergence of \( u_{i,n}, b_{i,n}(x, u_{i,n}) \) and convergence of \( a_{i,n}(x, t, T_k(u_{i,n}), \nabla T_k(u_{i,n})) \).

**Proposition 5** Let \( u_{i,n} \) be a solution of the approximate problem, then:
\[ u_{i,n} \rightarrow u_i \quad \text{a.e in } Q_T, \quad (37) \]
\[ b_{i,n}(x, u_{i,n}) \rightarrow b_i(x, u_i) \quad \text{a.e in } Q_T. \quad b_i(x, u_i) \in L^\infty(0, T, L^4(\Omega)), \quad (38) \]
\[ a_{i}(x, t, T_k(u_{i,n}), \nabla T_k(u_{i,n})) \rightarrow X_{i,k} \quad \text{in } (L\overline{M}(Q_T))^\mathbb{N} \text{ for } \sigma(\Pi L\overline{M}, \Pi E_M), \quad (39) \]
for some \( X_{i,k} \in (L\overline{M}(Q_T))^\mathbb{N} \)
\[ \lim_{m \rightarrow +\infty} \lim_{n \rightarrow +\infty} \int_{m \leq |u_{i,n}| \leq m + 1} a_{i}(x, t, u_{i,n}, \nabla u_{i,n}) \, dx \, dt = 0. \quad (40) \]

**Proof**
**Proof of (37) and (38):**
Now, consider a non decreasing function \( g_k \in C^2(R) \) such that \( g_k(s) = s \) for \( |s| \leq \frac{k}{2} \).
and $g_k(s) = k$ for $|s| \geq k$. Multiplying the approximate equation by $g'_k(u_{i,n})$, we get

$$
\frac{\partial B_{i,n}^k(x,u_{i,n})}{\partial t} + \text{div} \left( a_n(x,t,u_{i,n},\nabla u_{i,n})g'_k(u_{i,n}) \right) + a_n(x,t,u_{i,n},\nabla u_{i,n})g''_k(u_{i,n})\nabla u_{i,n} +\nabla \phi_{i,n}(x,t,u_{i,n}) - g''_k(u_{i,n})\phi_{i,n}(x,t,u_{i,n})\nabla u_{i,n} + f_{i,n}g'_k(u_{i,n}) = 0 \quad \text{in } D'(Q_T),
$$

where $B_{i,g}^k(x,z) = \int_0^z \frac{\partial b_{i,n}(x,s)}{\partial s} g'_k(s) ds$.

Using (41), we can deduce that $g_k(u_{i,n})$ is bounded in $W^{1,\infty}_0 L_M(Q_T)$ and $\frac{\partial B_{i,n}^k(x,u_{i,n})}{\partial t}$ is bounded in $L^1(Q_T) + W^{-1,\infty} L^\infty(Q_T)$ independently of $n$, thanks to (11) and properties of $g_k$, it follows that

$$
\left| \int_{Q_T} \phi_{i,n}(x,t,u_{i,n})g'_k(u_{i,n}) dx dt \right| \leq \|g'_k\|_\infty \left( \int_{Q_T} \alpha_0 M(\nabla T_k(u_{i,n})) dx dt + \int_{Q_T} M(\|c_i(x,t)\|_\infty) dx dt \right) \leq C_{i,k}^1,
$$

and

$$
\left| \int_{Q_T} g''_k(u_{i,n})\phi_{i,n}(x,t,u_{i,n})\nabla u_{i,n} dx dt \right| \leq \|g''_k\|_\infty \left( \|c_i\|_{L^\infty}(\alpha_0^1 + 1) \int_{\Omega} M(\nabla T_k(u_{i,n})) dx dt \right) \leq C_{i,k}^2.
$$

where $C_{i,k}^1$ and $C_{i,k}^2$ constants independently of $n$.

We conclude that $\frac{\partial g_k(u_{i,n})}{\partial t}$ is bounded in $L^1(Q_T) + W^{-1,\infty} L^\infty(Q_T)$ for $k < n$, which implies that $g_k(u_{i,n})$ is compact in $L^1(Q_T)$. Due to the choice of $g_k$, we conclude that for each $k$, the sequence $T_k(u_{i,n})$ converges almost everywhere in $Q_T$, which implies that the sequence $u_{i,n}$ converge almost everywhere to some measurable function $u_i$ in $Q_T$.

Then by the same argument in [5], we have

$$
u_{i,n} \to u_i \text{ a.e. } Q_T,
$$

where $u_i$ is a measurable function defined on $Q_T$. and

$$b_{i,n}(x,u_{i,n}) \to b_i(x,u_i) \text{ a.e. in } Q_T,
$$

by (36) and (42) we have

$$T_k(u_{i,n}) \to T_k(u_i)
$$

weakly in $W^{1,\infty}_0 L_M(Q_T)$ for $\sigma(\Pi L_M, \Pi E_{Q_T})$ strongly in $E_{M}(Q_T)$ and a.e in $Q_T$.

We now show that $b_i(x,u_i) \in L^\infty(0,T;L^1(\Omega))$. Indeed using $\frac{1}{\varepsilon} T_\varepsilon(u_{i,n})$ as a test function in (29),

$$
\frac{1}{\varepsilon} \int_\Omega b_{i,n}^\varepsilon(x,u_{i,n})(t) dx + \frac{1}{\varepsilon} \int_{Q_T} a_n(x,u_{i,n},\nabla u_{i,n})\nabla T_\varepsilon(u_{i,n}) dx dt
\quad - \frac{1}{\varepsilon} \int_{Q_T} \Phi_{i,n}(x,u_{i,n})\nabla T_\varepsilon(u_{i,n}) dx dt + \frac{1}{\varepsilon} \int_{Q_T} f_{i,n}(x,u_{i,n},u_{2,n})T_\varepsilon(u_{i,n}) dx dt
\quad = \frac{1}{\varepsilon} \int_\Omega b_{i,n}^\varepsilon(x,u_{i,0,n}) dx,
$$

for almost any $t$ in $(0,T)$, where, $b_{i,n}^\varepsilon(r) = \int_0^t b_{i,n,\varepsilon}^\varepsilon(s)T_\varepsilon(s) ds$.
Since $a_n$ satisfies (9) and $f_{i,n}$ satisfies (13), we get
\[
\int_\Omega b_{i,n}(x,u_{i,n})(t)\,dx \leq \int_{Q_T} \Phi_{i,n}(x,t,u_{i,n})\nabla T_\varepsilon(u_{i,n})\,dx\,dt + \int_\Omega b_{i,n}(x,u_{i,0n})\,dx,
\]  
By Young inequality and (11), we get
\[
\int_{Q_T} \Phi_{i,n}(x,t,u_{i,n})\nabla T_\varepsilon(u_{i,n})\,dx\,dt \leq \int_{|u_{i,n}| \leq \varepsilon} M(\Phi_{i,n}(x,t,u_{i,n}))\,dx\,dt + \int_{|u_{i,n}| \geq \varepsilon} M(\nabla T_\varepsilon(u_{i,n}))\,dx\,dt
\]  
\[
\leq \|c_i\|_{L^\infty}(a_0^i + 1) \int_{|u_{i,n}| \leq \varepsilon} M(\nabla T_\varepsilon(u_{i,n}))\,dx\,dt.
\]  
Using the Lebesgue’s Theorem and $M(\nabla T_\varepsilon(u_{i,n})) \in W^{1,\infty}_0 L_M(Q_T)$ in second term of the left hand side of (46) and letting $\varepsilon \to 0$ in (45) we obtain
\[
\int_\Omega |b_{i,n}(x,u_{i,n})(t)|\,dx \leq \|b_{i,n}(x,u_{i,0n})\|_{L^1(\Omega)}
\]  
(47)
for almost $t \in (0,T)$, thanks to (28), (37), and passing to the limit-inf in (47), we obtain $b_i(x,u_t) \in L^\infty(0,T;L^1(\Omega))$.

**Proof of (39):**
Following the same way in (4), we deduce that $a_n(x,t,T_k(u_{i,n}),\nabla T_k(u_{i,n}))$ is a bounded sequence in $(L^\infty_0(Q_T))^N$, and we obtain (39).

**Proof of (40):**
Multiplying the approximating equation (29) by the test function $\theta_m(u_{i,n}) = T_{m+1}(u_{i,n}) - T_m(u_{i,n})$
\[
\int_\Omega B_{i,m}(x,u_{i,n}(T))\,dx + \int_{Q_T} a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla \theta_m(u_{i,n})\,dx\,dt + \int_{Q_T} \phi_{i,n}(x,t,u_{i,n})\nabla \theta_m(u_{i,n})\,dx\,dt
\]  
\[
+ \int_{Q_T} f_{i,n}\theta_m(u_{i,n})\,dx\,dt \leq \int_\Omega B_{i,m}(x,u_{i,0n})\,dx,
\]  
where $B_{i,m}(x,r) = \int_0^r \theta_m(s)\frac{\partial b_{i,n}(x,s)}{\partial s}\,ds$.

By (11), we have
\[
\int_{Q_T} \phi_{i,n}(x,t,u_{i,n})\nabla \theta_m(u_{i,n})\,dx\,dt \leq \|c_i\|_{L^\infty}(a_0^i + 1) \int_\Omega M(\nabla \theta_m(u_{i,n}))\,dx\,dt
\]  
Also $\int_{Q_T} f_{i,n}\theta_m(u_{i,n})\,dx\,dt \geq 0$ in view of (13). Then, the same argument in step 2, we obtain,
\[
\int_{Q_T} M(\nabla \theta_m(u_{i,n}))\,dx\,dt \leq C_i \int_\Omega B_{i,m}(x,u_{i,0n})\,dx
\]  
passing to limit as $n \to +\infty$, since the pointwise convergence of $u_{i,n}$ and strongly convergence in $L^1(Q_T)$ of $B_{i,m}(x,u_{i,0n})$ we get
\[
\lim_{n \to +\infty} \int_{Q_T} M(\nabla \theta_m(u_{i,n}))\,dx\,dt \leq C_i \int_\Omega B_{i,m}(x,u_{i,0})\,dx
\]
By using Lebesgue’s theorem and passing to limit as $m \to +\infty$, in the all term of the right-hand side, we get

\[
\lim_{m \to +\infty} \lim_{n \to +\infty} \int_{m \leq |u_i| \leq m+1} M(\nabla \theta_m(u_{i,n}))dxdt = 0, \tag{49}
\]

and on the other hand, we have

\[
\lim_{m \to +\infty} \lim_{n \to +\infty} \int_{Q_T} \phi_{i,n}(x,t,u_{i,n})\nabla \theta_m(u_{i,n})dxdt \leq \lim_{m \to +\infty} \lim_{n \to +\infty} \int_{m \leq |u_i| \leq m+1} M(\nabla \theta_m(u_{i,n}))dxdt
\]

\[+ \lim_{m \to +\infty} \lim_{n \to +\infty} \int_{m \leq |u_i| \leq m+1} \mathcal{M}(\phi_{i,n}(x,t,u_{i,n}))dxdt
\]

Using the pointwise convergence of $u_{i,n}$ and by Lebesgue’s theorem, in the second term of the right side, we get

\[
\lim_{n \to +\infty} \int_{m \leq |u_i| \leq m+1} \mathcal{M}(\phi_{i,n}(x,t,u_{i,n}))dxdt = \int_{m \leq |u_i| \leq m+1} \mathcal{M}(\phi_{i}(x,t,u_{i}))dxdt
\]

and also, by Lebesgue’s theorem

\[
\lim_{m \to +\infty} \int_{m \leq |u_i| \leq m+1} \mathcal{M}(\phi_{i}(x,t,u_{i}))dxdt = 0 \tag{50}
\]

we obtain with (49) and (50),

\[
\lim_{m \to +\infty} \lim_{n \to +\infty} \int_{Q_T} \phi_{i,n}(x,t,u_{i,n})\nabla \theta_m(u_{i,n})dxdt = 0
\]

then passing to the limit in (48), we get (40).

**Step 3:**

Let $v_{i,j} \in \mathcal{D}(Q_T)$ be a sequence such that $v_{i,j} \to u_i$ in $W^{1,\infty}_0 L_M(Q_T)$ for the modular convergence.

This specific time regularization of $T_k(v_{i,j})$ (for fixed $k \geq 0$) is defined as follows. Let $(\alpha_{i,0}^\mu)_{\mu}$ be a sequence of functions defined on $\Omega$ such that

\[
\alpha_{i,0}^\mu \in L^\infty(\Omega) \cap W^{1,\infty}_0 L_M(\Omega) \quad \text{for all} \quad \mu > 0 \tag{51}
\]

\[\|\alpha_{i,0}^\mu\|_{L^\infty(\Omega)} \leq k \quad \text{for all} \quad \mu > 0,
\]

and $\alpha_{i,0}^\mu$ converges to $T_k(u_{i,0})$ a.e. in $\Omega$ and $\frac{1}{\mu}\|\alpha_{i,0}^\mu\|_{M,\Omega}$ converges to 0 $\mu \to +\infty$.

For $k \geq 0$ and $\mu > 0$, let us consider the unique solution $(T_k(v_{i,j}))_{\mu} \in L^\infty(Q) \cap W^{1,\infty}_0 L_M(Q_T)$ of the monotone problem:

\[
\frac{\partial(T_k(v_{i,j}))_{\mu}}{\partial t} + \mu((T_k(v_{i,j}))_{\mu} - T_k(v_{i,j})) = 0 \quad \text{in} \quad D'(\Omega), \tag{52}
\]

\[(T_k(v_{i,j}))_{\mu}(t = 0) = \alpha_{i,0}^\mu \quad \text{in} \quad \Omega. \tag{53}
\]

Remark that due to

\[
\frac{\partial(T_k(v_{i,j}))_{\mu}}{\partial t} \in W^{1,\infty}_0 L_M(Q_T) \tag{54}
\]

We just recall that,

\[(T_k(v_{i,j}))_{\mu} \to T_k(u_i) \quad \text{a.e. in} \quad Q_T, \quad \text{weakly * in} \quad L^\infty(Q_T) \quad \text{and} \quad (55)\]
\[(T_\mu^k(u_{i,j}))_\mu \rightarrow (T_\mu^k(u_i))_\mu \quad \text{in} \quad W_0^{1,x} L_M(Q_T) \] (56)

for the modular convergence as \(j \rightarrow +\infty\).

\[(T_\mu^k(u_i))_\mu \rightarrow T_\mu^k(u_i) \quad \text{in} \quad W_0^{1,x} L_M(Q_T) \] (57)

for the modular convergence as \(\mu \rightarrow +\infty\).

\[
\| (T_\mu^k(u_{i,j}))_\mu \|_{L^\infty(Q_T)} \leq \max(\| (T_\mu^k(u_i))_\mu \|_{L^\infty(Q_T)}, \| a_0^\mu \|_{L^\infty(\Omega)}) \leq k, \quad \forall \mu > 0 , \forall k > 0. \] (58)

Now, we introduce a sequence of increasing \(C^\infty(R)\)-functions \(S_m\) such that, for any \(m \geq 1\).

\[ S_m(r) = r \text{ for } |r| \leq m, \quad \text{supp}(S'_m) \subset \left[-(m+1), (m+1)\right], \quad \|S''_m\|_{L^\infty(R)} \leq 1. \] (59)

Through setting, for fixed \(K \geq 0\),

\[ W_{i,j,\mu}^n = T_K^i(u_{i,n}) - T_K^i(u_{i,j})_\mu \quad \text{and} \quad W_{i,\mu}^n = T_K^i(u_{i,n}) - T_K^i(u_i)_\mu \] (60)

we obtain upon integration,

\[
\int_{Q_T} \left( \frac{\partial b_{i,m}(u_{i,n})}{\partial t}, W_{i,j,\mu}^n \right) dx \, dt \\
+ \int_{Q_T} S'_m(u_{i,n}) a_n(x, u_{i,n}, \nabla u_{i,n}) \nabla W_{i,j,\mu}^n dx \, dt + \int_{Q_T} S''_m(u_{i,n}) W_{i,j,\mu}^n a_n(x, u_{i,n}, \nabla u_{i,n}) \nabla u_{i,n} dx \, dt \\
+ \int_{Q_T} \Phi_{i,n}(x, t, u_{i,n}) S'_m(u_{i,n}) \nabla W_{i,j,\mu}^n dx \, dt + \int_{Q_T} S''_m(u_{i,n}) W_{i,j,\mu}^n \Phi_{i,n}(x, t, u_{i,n}) \nabla u_{i,n} dx \, dt \\
+ \int_{Q_T} f_{i,n}(x, u_{i,n}, u_{2,n}) S'_m(u_{i,n}) W_{i,j,\mu}^n dx \, dt = 0. \] (61)

We pass to limit, as \(n \rightarrow +\infty, j \rightarrow +\infty, \mu \rightarrow +\infty\) and then \(m\) tends to \(+\infty\), the real number \(K \geq 0\) being kept fixed. In order to perform this task we prove below
the following results for fixed $K \geq 1$:
\[
\liminf_{\mu \to +\infty} \lim_{j \to +\infty} \lim_{n \to +\infty} \int_{Q_T} \left\langle \frac{\partial b_{i,n}(x,u_{i,n})}{\partial t}, S'_m(u_{i,n})W^n_{i,j,\mu} \right\rangle \, dx \, dt \geq 0 \quad \text{for any } m \geq K, \tag{62}
\]
\[
\lim_{\mu \to +\infty} \lim_{j \to +\infty} \lim_{n \to +\infty} \int_{Q_T} S'_m(u_{i,n})\Phi_{i,n}(x,t,u_{i,n})W^n_{i,j,\mu} \, dx \, dt = 0 \quad \text{for any } m \geq 1, \tag{63}
\]
\[
\lim_{\mu \to +\infty} \lim_{j \to +\infty} \lim_{n \to +\infty} \int_{Q_T} S''_m(u_{i,n})W^n_{i,j,\mu} \, dx \, dt = 0 \quad \text{for any } m, \tag{64}
\]
\[
\lim_{\mu \to +\infty} \lim_{j \to +\infty} \lim_{n \to +\infty} \left| \int_{Q_T} S''_m(u_{i,n})W^n_{i,j,\mu} a_n(x, t, u_{i,n}, \nabla u_{i,n}) \nabla u_{i,n} \, dx \, dt \right| = 0, \tag{65}
\]
\[
\lim_{\mu \to +\infty} \lim_{j \to +\infty} \lim_{n \to +\infty} \int_{Q_T} f_{i,n}(x, u_{i,n}, u_{2,n})S'_m(u_{i,n})W^n_{i,j,\mu} \, dx \, dt = 0 \quad \text{for any } m \geq 1. \tag{66}
\]
\[
\limsup_{n \to +\infty} \int_{Q_T} a(x, t, u_{i,n}, \nabla T_K(u_{i,n}))\nabla T_K(u_{i,n}) \, dx \, dt \leq \int_{Q_T} X_{i,K} \nabla T_K(u_i) \, dx \, dt. \tag{67}
\]
\[
\int_{Q_T} [a(x, t, T_k(u_{i,n}), \nabla T_k(u_{i,n})) - a(x, t, T_k(u_{i,n}), \nabla T_k(u_i))] [\nabla T_k(u_{i,n}) - \nabla T_k(u_i)] \, dx \, dt \to 0. \tag{68}
\]
**Proof of (62):**
**Lemma 6**
\[
\int_{Q_T} \left\langle \frac{\partial b_{i,n}(x,u_{i,n})}{\partial t}, S'_m(u_{i,n})W^n_{i,j,\mu} \right\rangle \, dx \, dt \geq \epsilon(n, j, \mu, m) \tag{69}
\]
**Proof:** This follows from the proof in [13].
**Proof of (63):**
If we take $n > m + 1$, we get
\[
\phi_{i,n}(x, t, u_{i,n})S'_m(u_{i,n}) = \phi_i(x, t, T_{m+1}(u_{i,n}))S'_m(u_{i,n})
\]
Using (11), we have:
\[
M(\phi_{i,n}(x, t, T_{m+1}(u_{i,n}))S'_m(u_{i,n})) \leq (m + 1)M(\phi_i(x, t, T_{m+1}(u_{i,n})))
\]
\[
\leq (m + 1)M(\|\epsilon_i(x, t)\|_{L^\infty(Q_T)} M^{-1}(\alpha_0 / \lambda (m + 1)))
\]
Then $\phi_{i,n}(x, t, u_{i,n})S_m(u_{i,n})$ is bounded in $L_M(Q_T)$, thus, by using the pointwise convergence of $u_{i,n}$ and Lebesgue’s theorem we obtain $\phi_{i,n}(x, t, u_{i,n})S_m(u_{i,n}) \to \phi_i(x, t, u_i)S_m(u_i)$ with the modular convergence as $n \to +\infty$, then $\phi_{i,n}(x, t, u_{i,n})S_m(u_{i,n}) \to \phi_i(x, t, u_i)S_m(u_i)$ for $\sigma(\prod L_{M_i}, \prod L_M)$.

On the other hand $\nabla W^n_{i,j,\mu} = \nabla T_k(u_{i,n}) - \nabla (T_k(u_{i,n}))_\mu$ for converge to $\nabla T_k(u_i) - \nabla (T_k(u_i))_\mu$ weakly in $(L_M(Q_T))^N$, then
\[
\int_{Q_T} \phi_{i,n}(x, t, u_{i,n})S_m(u_{i,n})\nabla W^n_{i,j,\mu} \, dx \, dt \to \int_{Q_T} \phi_i(x, t, u_i)S_m(u_i)\nabla W_{i,j,\mu} \, dx \, dt
\]
as \( n \to +\infty \).

By using the modular convergence of \( W_{i,j}\mu \) as \( j \to +\infty \) and letting \( \mu \) tends to infinity, we get (63).

**Proof of (64):**

For \( n > m + 1 > k \), we have \( \nabla u_{i,n}S''_m(u_{i,n}) = \nabla T_{m+1}(u_{i,n}) \) a.e. in \( Q_T \). By the almost everywhere convergence of \( u_{i,n} \) we have \( W_{i,j}\mu \to W_{i,j}\mu \) in \( L^\infty(Q_T) \) weak-* and since the sequence \( \{\phi_{i,n}(x,t, T_{m+1}(u_{i,n}))\}_n \) converges strongly in \( E_{TV}(Q_T) \), then

\[
\phi_{i,n}(x,t, T_{m+1}(u_{i,n})) W_{i,j}\mu \to \phi_{i}(x,t, T_{m+1}(u_{i})) \quad \text{as} \quad n \to +\infty.
\]

Converge strongly in \( E_{TV}(Q_T) \) as \( n \to +\infty \). By virtue of \( \nabla T_{m+1}(u_n) \to \nabla T_{m+1}(u_{i}) \) weakly in \( (L^1(Q_T))^N \) as \( n \to +\infty \) we have

\[
\int_{m \leq |u_{i,n}| \leq m + 1} \phi_{i,n}(x,t, T_{m+1}(u_{i,n})) \nabla u_{i,n}S''_m(u_{i,n}) W_{i,j}\mu^n dx dt \to \int_{m \leq |u_{i}| \leq m + 1} \phi(x,t,u_{i}) \nabla u_{i} W_{i,j}\mu dx dt
\]

as \( n \to +\infty \).

With the modular convergence of \( W_{i,j}\mu \) as \( j \to +\infty \) and letting \( \mu \to +\infty \) we get (64).

**Proof of (65):**

For any \( m \geq 1 \) fixed, we have

\[
\left| \int_{Q_T} S''_m(u_{i,n}) a_n(x,t,u_{i,n},\nabla u_{i,n}) \nabla u_{i,n} W_{i,j}\mu^n dx dt \right|
\]

\[
\leq \|S''_m\|_{L^\infty(R)} \|W_{i,j}\mu^n\|_{L^\infty(Q_T)} \int_{m \leq |u_{i,n}| \leq m + 1} a_n(x,t,u_{i,n},\nabla u_{i,n}) \nabla u_{i,n} dx dt,
\]

for any \( m \geq 1 \), and any \( \mu > 0 \). In view (58) and (59), we can obtain

\[
\limsup_{n \to +\infty} \int_{Q_T} S''_m(u_{i,n}) a_n(x,t,u_{i,n},\nabla u_{i,n}) \nabla u_{i,n} W_{i,j}\mu^n dx dt
\]

\[
\leq 2K \limsup_{n \to +\infty} \int_{m \leq |u_{i,n}| \leq m + 1} a_n(x,t,u_{i,n},\nabla u_{i,n}) \nabla u_{i,n} dx dt,
\]

for any \( m \geq 1 \). Using (40) we pass to the limit as \( m \to +\infty \) in (70) and we obtain (65).

**Proof of (66):**

For fixed \( n \geq 1 \) and \( n > m + 1 \), we have

\[
f_{1,n}(x,u_{1,n},u_{2,n}) S'_m(u_{1,n}) = f_1(x,T_{m+1}(u_{1,n}),T_n(u_{2,n})) S'_m(u_{1,n}),
\]

\[
f_{2,n}(x,u_{1,n},u_{2,n}) S'_m(u_{2,n}) = f_2(x,T_n(u_{1,n}),T_{m+1}(u_{2,n})) S'_m(u_{2,n}),
\]

In view (14),(15),(43) and Lebesgue's theorem allow us to get, for

\[
\lim_{n \to +\infty} \int_{Q_T} f_{1,n}(x,u_{1,n},u_{2,n}) S'_m(u_{i,n}) W_{i,j}\mu^n dx dt = \int_{Q_T} f_{1}(x,u_{1},u_{2}) S'_m(u_{i}) W_{i,j}\mu dx dt
\]

Using (56), we follow a similar way we get as \( j \to +\infty \)

\[
\lim_{j \to +\infty} \int_{Q_T} f_{1}(x,u_{1},u_{2}) S'_m(u_{i}) W_{i,j}\mu dx dt = \int_{Q_T} f_{1}(x,u_{1},u_{2}) S'_m(u_{i})(T_{K}(u_{i}) - T_{K}(u_{i})) dx dt
\]
we fixed \( m > 1 \), and using (57), we have
\[
\lim_{\mu \to +\infty} \int_{Q_T} f_i(x, u_1, u_2) S'_m(u_i)(T_K(u_i) - T_K(u_i)_\mu) \, dx \, dt = 0
\]

Then we conclude the proof of (66).

**Proof of (67):**

If we pass to the lim-sup when \( n, j \) and \( \mu \) tends to \(+\infty\) and then to the limit as \( m \) tends to \(+\infty\) in (61). We obtain using (62)-(66), for any \( K \geq 0 \),

\[
\lim_{m \to +\infty} \limsup_{j \to +\infty} \limsup_{n \to +\infty} \int_{Q_T} S'_m(u_{i,n}) a_n(x, t, u_{i,n}, \nabla u_{i,n}) (\nabla T_K(u_{i,n}) - \nabla T_K(u_{i,j}, \mu)) \, dx \, dt \leq 0.
\]

Since \( S'_m(u_{i,n}) a_n(x, t, u_{i,n}, \nabla u_{i,n}) \nabla T_K(u_{i,n}) = a_n(x, t, u_{i,n}, \nabla u_{i,n}) \nabla T_K(u_{i,n}) \)
for \( n > K \) and \( K \leq m \).

Then, for \( K \leq m \),
\[
\limsup_{j \to +\infty} \limsup_{n \to +\infty} \int_{Q_T} a_n(x, t, u_{i,n}, \nabla u_{i,n}) \nabla T_K(u_{i,n}) \, dx \, dt
\leq \lim_{m \to +\infty} \limsup_{\mu \to +\infty} \limsup_{j \to +\infty} \limsup_{n \to +\infty} \int_{Q_T} S'_m(u_{i,n}) a_n(x, t, u_{i,n}, \nabla u_{i,n}) \nabla T_K(u_{i,j}, \mu) \, dx \, dt
\]

(71)

Thanks to (59), we have in the right hand side of (71) for \( n > m + 1 \) that,
\[
S'_m(u_{i,n}) a_n(x, t, u_{i,n}, \nabla u_{i,n}) = S'_m(u_{i,n}) a(x, t, T_{m+1}(u_{i,n}), \nabla T_{m+1}(u_{i,n})) \text{ a.e. in } Q_T.
\]

Using (39), and fixing \( m \geq 1 \), we get
\[
S'_m(u_{i,n}) a_n(u_{i,n}, \nabla u_{i,n}) \rightharpoonup S'_m(u_i) X_{i,m+1} \text{ weakly in } (L^{m'}(Q_T))^N.
\]

when \( n \to +\infty \).

We can pass to limit as \( j \to +\infty \) and \( \mu \to +\infty \), and using (56)-(57)

\[
\limsup_{j \to +\infty} \limsup_{n \to +\infty} \int_{Q_T} S'_m(u_{i,n}) a(x, t, u_{i,n}, \nabla u_{i,n}) \nabla T_K(u_{i,j}, \mu) \, dx \, dt
\]

\[
= \int_{Q_T} S'_m(u_i) X_{i,m+1} \nabla T_K(u_i) \, dx \, dt
\]

(72)

where \( K \leq m \), since \( S'_m(r) = 1 \) for \(|r| \leq m \).

On the other hand, for \( K \leq m \), we have
\[
a(x, t, T_{m+1}(u_{i,n}), \nabla T_{m+1}(u_{i,n})) \chi_{\{|u_{i,n}| < K\}} = a(x, t, T_K(u_{i,n}), \nabla T_K(u_{i,n})) \chi_{\{|u_{i,n}| < K\}},
\]
a.e. in \( Q_T \). Passing to the limit as \( n \to +\infty \), we obtain
\[
X_{i,m+1} \chi_{\{|u_i| < K\}} = X_{i,k} \chi_{\{|u_i| < K\}} \text{ a.e. in } Q_T - \{|u_i| = K\} \text{ for } K \leq n.
\]

(73)

Then
\[
X_{m+1} \nabla T_K(u_i) = X_K \nabla T_K(u_i) \text{ a.e. in } Q_T.
\]

(74)

Then we obtain (67).
Proof of (68):
Let $K \geq 0$ be fixed. Using (10) we have
\[
\int_{Q_T} \left[ a(x,t,T_K(u_{i,n}),\nabla T_K(u_{i,n})) - a(x,t,T_K(u_{i,n}),\nabla T_K(u_i)) \right] \left[ \nabla T_K(u_{i,n}) - \nabla T_K(u_i) \right] \, dx \, dt \geq 0,
\]
(75)
In view of (4) and (43), we get
\[
a(x,t,T_K(u_{i,n}),\nabla T_K(u_i)) \to a(x,t,T_K(u_i),\nabla T_K(u_i)) \quad \text{a.e. in } Q_T,
\]
as $n \to +\infty$, and by (8) and Lebesgue’s theorem, we obtain
\[
a(x,t,T_K(u_{i,n}),\nabla T_K(u_i)) \to a(x,t,T_K(u_i),\nabla T_K(u_i)) \quad \text{strongly in } (L^\infty(Q_T))^N.
\]
(76)
Using (67), (43), (39) and (76), we can pass to the lim-sup as $n \to +\infty$ in (75) to obtain (68).
To finish this step, we prove this lemma:
**Lemma 7** For $i = 1,2$ and fixed $K \geq 0$, we have
\[
X_{i,K} = a(x,t,T_K(u_i),\nabla T_K(u_i)) \quad \text{a.e. in } Q.
\]
(77)
Also, as $n \to +\infty$,
\[
a(x,t,T_K(u_{i,n}),\nabla T_K(u_i)) \nabla T_K(u_{i,n}) \to a(x,t,T_K(u_i),D_T(u_i)) \nabla T_K(u_i),
\]
weakly in $L^1(Q_T)$.

**Proof**

Proof of (77):
It’s easy to see that
\[
a_n(x,t,T_K(u_{i,n}),\xi) = a(x,t,T_K(u_{i,n}),\xi) = a(x,t,T_K(u_{i,n}),\xi) \quad \text{a.e. in } Q_T
\]
for any $K \geq 0$, any $n > K$ and any $\xi \in \mathbb{R}^N$.
In view of (39), (68) and (76) we obtain
\[
\lim_{n \to +\infty} \int_{Q_T} a_K \left( x,t,T_K(u_{i,n}),\nabla T_K(u_{i,n}) \right) \nabla T_K(u_{i,n}) \, dx \, dt = \int_{Q_T} X_{i,K} \nabla T_K(u_i) \, dx \, dt.
\]
(79)
Since (4), (8) and (43), imply that the function $a_K(x,s,\xi)$ is continuous and bounded with respect to $s$. Then we conclude that (77).

Proof of (78):
Using (10) and (68), for any $K \geq 0$ and any $T' < T$, we have
\[
[a(x,t,T_K(u_{i,n}),\nabla T_K(u_{i,n}))-a(x,t,T_K(u_{i,n}),\nabla T_K(u_i))] \left[ \nabla T_K(u_{i,n})-\nabla T_K(u_i) \right] \to 0
\]
strongly in $L^1(Q_{T'})$ as $n \to +\infty$.
On the other hand with (43), (39), (76) and (77), we get
\[
a(x,t,T_K(u_{i,n}),\nabla T_K(u_i)) \nabla T_K(u_i) \to a(x,t,T_K(u_i),\nabla T_K(u_i)) \nabla T_K(u_i)
\]
weakly in $L^1(Q_T)$,
\[
a(x,t,T_K(u_{i,n}),\nabla T_K(u_i)) \nabla T_K(u_{i,n}) \to a(x,t,T_K(u_i),\nabla T_K(u_i)) \nabla T_K(u_i)
\]
weakly in $L^1(Q_T)$,
\[
a(x, t, T_K(u_{i,n}), \nabla T_K(u_i))\nabla T_K(u_i) \to a(x, t, T_K(u_i), \nabla T_K(u_i))\nabla T_K(u_i),
\]
strongly in $L^1(Q)$, as $n \to +\infty$.
It’s results from (80), for any $K \geq 0$ and any $T' < T$,
\[
a(x, t, T_K(u_{i,n}), \nabla T_K(u_{i,n}))\nabla T_K(u_{i,n}) \to a(x, t, T_K(u_i), \nabla T_K(u_i))\nabla T_K(u_i)
\]
weakly in $L^1(Q_{T'})$ as $n \to +\infty$ then for $T' = T$, we have (78).

Finally we should prove that $u_i$ satisfies (18).
**Step 4: Pass to the limit.**

We first show that $u$ satisfies (18)
\[
\int_{m \leq |u_{i,n}| \leq m+1} a(x, t, u_{i,n}, \nabla u_{i,n})\nabla u_{i,n} \, dx \, dt
\]
\[
= \int_{Q_T} a_n(x, t, u_{i,n}, \nabla u_{i,n}) [\nabla T_{m+1}(u_{i,n}) - \nabla T_m(u_{i,n})] \, dx \, dt
\]
\[
= \int_{Q_T} a_n(x, t, T_{m+1}(u_{i,n}), \nabla T_{m+1}(u_{i,n}))\nabla T_{m+1}(u_{i,n}) \, dx \, dt
\]
\[
- \int_{Q_T} a_n(x, t, T_{m}(u_{i,n}), \nabla T_{m}(u_{i,n}))\nabla T_{m}(u_{i,n}) \, dx \, dt
\]
for $n > m + 1$. According to (78), one can pass to the limit as $n \to +\infty$; for fixed $m \geq 0$ to obtain
\[
\lim_{n \to +\infty} \int_{m \leq |u_{i,n}| \leq m+1} a(x, t, u_{i,n}, \nabla u_{i,n})\nabla u_{i,n} \, dx \, dt
\]
\[
= \int_{Q} a(x, t, T_{m+1}(u_i), \nabla T_{m+1}(u_i))\nabla T_{m+1}(u_i) \, dx \, dt
\]
\[
- \int_{Q} a(x, t, T_{m}(u_i), \nabla T_{m}(u_i))\nabla T_{m}(u_i) \, dx \, dt
\]
\[
= \int_{m \leq |u_{i}| \leq m+1} a(x, t, u_i, \nabla u_i)\nabla u_i \, dx \, dt
\]
Pass to limit as $m$ tends to $+\infty$ in (82) and using (40) show that $u_i$ satisfies (18).

Now we shown that $u_i$ to satisfy (19) and (20).
Let $S$ be a function in $W^{2,\infty}(R)$ such that $S'$ has a compact support. Let $K$ be a positive real number such that supp$S' \subset [-K,K]$. The pointwise multiplication of the approximate equation (1) by $S'(u_{i,n})$ leads to
\[
\frac{\partial B_{1}^{n}(u_{i,n})}{\partial t} - \text{div} \left(S'(u_{i,n})a_n(x, u_{i,n}, \nabla u_{i,n})\right) + S''(u_{i,n})a_n(x, u_{i,n}, \nabla u_{i,n})\nabla u_{i,n}
\]
\[
- \text{div} \left(S'(u_{i,n})\Phi_{i,n}(x, t, u_{i,n})\right) + S''(u_{i,n})\Phi_{i,n}(x, t, u_{i,n})\nabla u_{i,n} = f_{i,n}(x, u_{1,n}, u_{1,n})S'(u_{i,n})
\]
in $D'(Q_T)$, for $i = 1, 2$.
Now we pass to the limit in each term of (83).
Limit of $\frac{\partial B^n_{i,S}(u_{i,n})}{\partial t}$: Since $B^n_{i,S}(u_{i,n})$ converges to $B_{i,S}(u_i)$ a.e. in $Q_T$ and in $L^\infty(Q_T)$ weak $\star$ and $S$ is bounded and continuous. Then $\frac{\partial B^n_{i,S}(u_{i,n})}{\partial t}$ converges to $\frac{\partial B_{i,S}(u_i)}{\partial t}$ in $D'(Q_T)$ as $n$ tends to $+\infty$.

Limit of $\text{div}\left( S'(u_{i,n})a_n(x,t,u_{i,n},\nabla u_{i,n}) \right)$: Since supp $S' \subset [-K,K]$, for $n > K$, we have

$$S'(u_{i,n})a_n(x,t,u_{i,n},\nabla u_{i,n}) = S'(u_{i,n})a_n(x,t,T_K(u_{i,n}),\nabla T_K(u_{i,n}))$$

a.e. in $Q_T$.

Using the pointwise convergence of $u_{i,n}$, (59), (39) and (77), imply that

$$S'(u_{i,n})a_n\left( x,t,T_K(u_{i,n}),\nabla T_K(u_{i,n}) \right) \rightharpoonup S'(u_i) a\left( x,t,T_K(u_i),\nabla T_K(u_i) \right)$$

weakly in $(L^\infty(Q_T))^N$, for $\sigma(\Pi L_{\Pi T}, \Pi E_M)$ as $n \to +\infty$, since $S'(u_i) = 0$ for $|u_i| \geq K$ a.e. in $Q_T$. And

$$S'(u_i) a\left( x,t,T_K(u_i),\nabla T_K(u_i) \right) = S'(u_i) a(x,t,u_i,\nabla u_i)$$

a.e. in $Q_T$.

Limit of $S''(u_{i,n})a_n(x,t,u_{i,n},\nabla u_{i,n}) \nabla u_{i,n}$: Since supp $S'' \subset [-K,K]$, for $n > K$, we have

$$S''(u_{i,n})a_n(x,t,u_{i,n},\nabla u_{i,n}) \nabla u_{i,n} = S''(u_{i,n})a_n\left( x,t,T_K(u_{i,n}),\nabla T_K(u_{i,n}) \right) \nabla T_K(u_{i,n})$$

a.e. in $Q_T$.

The pointwise convergence of $S''(u_{i,n})$ to $S''(u_i)$ as $n \to +\infty$, (59) and (78) we have

$$S''(u_{i,n})a_n(x,t,u_{i,n},\nabla u_{i,n}) \nabla u_{i,n} \to S''(u_i) a\left( x,t,T_K(u_i),\nabla T_K(u_i) \right) \nabla T_K(u_i)$$

weakly in $L^1(Q_T)$, as $n \to +\infty$. And

$$S''(u_i) a\left( x,t,T_K(u_i),\nabla T_K(u_i) \right) \nabla T_K(u_i) = S''(u_i) a(x,t,u_i,\nabla u_i) \nabla u_i$$

a.e. in $Q_T$.

Limit of $S'(u_{i,n})\Phi_{i,n}(x,t,u_{i,n})$: We have $S'(u_{i,n})\Phi_{i,n}(x,t,u_{i,n}) = S'(u_{i,n})\Phi_{i,n}(x,t,T_K(u_{i,n}))$ a.e. in $Q_T$. Since supp $S' \subset [-K,K]$. Using (11), (45) and (37), it’s easy to see that $S'(u_{i,n})\Phi_{i,n}(x,t,u_{i,n}) \rightharpoonup S'(u_i)\Phi_i(x,t,T_K(u_i))$ weakly for $\sigma(\Pi L_{\Pi T}, \Pi L_M)$ as $n \to +\infty$. And $S'(u_i)\Phi_i(x,t,T_K(u_i)) = S''(u_i)\Phi(x,t,u_i)$ a.e. in $Q_T$.

Limit of $S''(u_{i,n})\Phi_{i,n}(x,t,u_{i,n})\nabla u_{i,n}$: Since $S'' \in W^{1,\infty}(R)$ with supp $S'' \subset [-K,K]$, we have $S''(u_{i,n})\Phi_{i,n}(x,t,u_{i,n}) \nabla u_{i,n} = \Phi_{i,n}(x,t,T_K(u_{i,n})) \nabla S'(T_K(u_{i,n}))$ a.e. in $Q_T$. The weakly convergence of truncation allows us to prove that

$$S''(u_{i,n})\Phi_{i,n}(x,t,u_{i,n}) \nabla u_{i,n} \rightharpoonup \Phi(x,t,u_i) \nabla S'(u_i)$$

strongly in $L^1(Q_T)$.

Limit of $f_{i,n}(x,u_{i,n},u_{2,n})S'(u_{i,n})$: Using (14), (15), (26) and (27), we have $f_{i,n}(x,u_{i,n},u_{2,n})S'(u_{i,n}) \rightharpoonup f_i(x,u_{1,2})S'(u_i)$ strongly in $L^1(Q_T)$, as $n \to +\infty$. It remains to show that $B_S(x,u_i)$ satisfies the initial condition (20) for $i=1,2$.

To this end, firstly remark that, in view of the definition of $S_M$, we have $B_{M}(x,u_{i,n})$ is bounded in $L^\infty(Q_T)$.

Secondly, by (62) we show that $\frac{\partial B_M(x,u_{i,n})}{\partial t}$ is bounded in $L^1(Q_T) + W^{-1,\Sigma} L_{\Pi T}(Q_T)$.

As a consequence, an Aubin’s type Lemma (see e.g., [14], Corollary 4) implies that $B_M(x,u_{i,n})$ lies in a compact set of $C^0([0,T]; L^1(\Omega))$.

It follows that, on one hand $B_M(x,u_{i,n})(t=0)$ converges to $B_M(x,u_i)(t=0)$ strongly in $L^1(\Omega)$. On the other hand, the smoothness of $B_M$ imply that $B_M(x,u_{i,n})(t=0)$ converges to $B_M(x,u_i)(t=0)$ strongly in $L^1(\Omega)$, we conclude that $B_M(x,u_{i,n})(t=0)$ converges to $B_M(x,u_i)(t=0)$ strongly in $L^1(\Omega)$.
0) converges to \( B_M(x, u_i(t = 0)) \) strongly in \( L^1(\Omega) \), we obtain \( B_M(x, u_i(t = 0)) = B_M(x, u_i,0) \) a.e. in \( \Omega \) and for all \( M > 0 \), now letting \( M \) to +\( \infty \), we conclude that \( b(x, u_i)(t = 0) = b(x, u_i,0) \) a.e. in \( \Omega \).

As a conclusion, the proof of Theorem (4) is complete.

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