HARDY-SOBOLEV-MAZ’YA INEQUALITY ON TIME SCALE AND APPLICATION TO THE BOUNDARY VALUE PROBLEMS

F. Z. LADRANI AND A. BENAÏSSA CHERIF.

Abstract. In this paper, we will prove some new dynamic inequalities of Hardy-Sobolev-May’ze type on time scales. An application in the boundary value problems for dynamic equation.

1. Introduction

The classical Hardy inequality states that for \( f \geq 0 \) and integrable over any finite interval \((0, x)\) and \( f^p \) is integrable and convergent over \((0, \infty)\) and \( p > 1 \), then

\[
\int_0^x \left( \frac{1}{x} \int_0^x f(t) \, dt \right)^p \, dx \leq \left( \frac{p}{p-1} \right)^p \int_0^\infty f^p(t) \, dt
\]

holds and the constant \( \left( \frac{p}{p-1} \right)^p \) is the best possible. Inequality (1) which is usually referred to in the literature as the classical Hardy inequality, was proved in 1925 by Hardy [17]. More general Hardy integral inequalities have been studied in continuous. The inequalities of Hardy and Sobolev have a pivotal role in analysis and continue to be topics of intensive study. In its familiar basic form in \( L^p(\Omega) \); the Hardy inequality takes the form

\[
\int_\Omega |\nabla f(x)|^p \, dx \geq C(n, p) \int_\Omega \frac{|f(x)|^p}{|x|^p} \, dx, \quad \text{for all } f \in W^{1,p}_0(\Omega),
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \), containing the origin, \( p > 1 \) and \( C(n, p) \) is constant > 0.

Indeed, Rupert l. Frank and Michael loss [20] have obtained the following improved Hardy inequalities valid for any \( f \in W^{1,p}_0((a, b)) \)

\[
\int_a^b \left| f'(x) \right|^p \, dx \geq \frac{1}{4} \int_a^b \left( \frac{f(x)}{x} \right)^2 \, dx + K_p \| f \|^2_{L^p((a, b))}.
\]

where \( a, b \in \mathbb{R}, a \leq 0 < b, p > 1 \) and \( K_p \) is constant > 0.

Hardy type inequalities on time scales not only give a unification of continuous inequalities of Hardy type but also can be extended to different types of time scales.

2010 Mathematics Subject Classification. 26A15, 26D10, 26D15, 39A13, 34A40, 34N05.
Key words and phrases. Time scale, Hardy inequality, Sobolev inequality, Hardy-Sobolev-Maz’ya inequality, boundary value problem.
In 2005, Řeháč [8] stated that if \( a > 0 \), \( P > 1 \), and \( f \) be a nonnegative function such that the delta integral \( \int_{a}^{\infty} f^p(s) \Delta s \) exists as a finite number, then

\[
\int_{a}^{\infty} \left( \frac{1}{\sigma(t) - a} \int_{a}^{\sigma(t)} f(s) \Delta s \right)^p \leq \left( \frac{p}{p-1} \right)^p \int_{a}^{\infty} f^p(t) \Delta t
\]

(4)

unless \( f \equiv 0 \). If, in addition, \( \frac{\mu(t)}{t} \to 0 \) as \( t \to \infty \), then the constant \( \left( \frac{p}{p-1} \right)^p \) is the best possible.

The aim of this paper is to extend a Hardy-Sobolev inequality (2) and Hardy-Sobolev-Maz’ ye inequality (3) on time scales and we give an application of our extension of the Hardy inequality in the boundary value problems.

2. Preliminaries

A time scale \( \mathbb{T} \) is an arbitrary nonempty closed subset of the real numbers. For \( t \in \mathbb{T} \), we define the forward jump operator \( \sigma : \mathbb{T} \to \mathbb{T} \) by \( \sigma(t) = \inf \{ s \in \mathbb{T} : s > t \} \), and the backward jump operator \( \rho(t) = \sup \{ s \in \mathbb{T} : s < t \} \). (supplemented by \( \inf \emptyset := \sup \mathbb{T} \) and \( \sup \emptyset := \inf \mathbb{T} \) are well defined. If \( \sigma(t) > t \) we say that \( t \) is right-scattered, while if \( \rho(t) < t \) we say that \( t \) is left-scattered. Points that are simultaneously right-scattered and left-scattered are said to be isolated. If \( \sigma(t) = t \), then \( t \) is called right-dense; if \( \rho(t) = t \), then \( t \) is called left-dense. Points that are right-dense and left-dense at the same time are called dense. If \( \mathbb{T} \) has a left-scattered maximum \( M \), define \( \mathbb{T}^k := \mathbb{T} \setminus \{ M \} \); otherwise, set \( \mathbb{T}^k := \mathbb{T} \).

The graininess function for a time scale \( \mathbb{T} \) is defined by \( \mu(t) = \sigma(t) - t \), and for any function \( f : \mathbb{T} \to \mathbb{R} \) the notation \( f^\sigma(t) \) denotes \( f(\sigma(t)) \).

Let \( f : \mathbb{T} \to \mathbb{R} \) be a real valued function on a time scale \( \mathbb{T} \). Then, for \( t \in \mathbb{T}^k \), we define \( f^\Delta(t) \) to be the number, if one exists, such that for all \( \varepsilon > 0 \), there is a neighborhood \( U \) of \( t \) such that for all \( s \in U \),

\[
|f^\sigma(t) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s|.
\]

We say that \( f \) is delta differentiable on \( \mathbb{T} \) provided \( f^\Delta(t) \) exists for all \( t \in \mathbb{T}^k \). We will make use of the following product and quotient rules for the derivative of the product \( fg \) and the quotient \( \frac{f}{g} \) (where \( gg^\sigma \neq 0 \)) of two differentiable function \( f \) and \( g \)

\[
(fg)^\Delta = f^\Delta g^\sigma + f g^\Delta, \quad \text{and} \quad \left( \frac{f}{g} \right)^\Delta = \frac{f^\Delta g - f g^\Delta}{g g^\sigma}.
\]

A function \( f : \mathbb{T} \to \mathbb{R} \) will be called rd-continuous provided it is continuous at each right-dense point and has a left-sided limit at each point, we write \( f \in C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R}) \).

The set of functions that are differentiable and whose derivative is rd-continuous is denoted by \( C_{rd}^1(\mathbb{T}) = C_{rd}^1(\mathbb{T}, \mathbb{R}) \).

We will work with the \( L^p_{\Delta}([a, b]_{\mathbb{T}}) \) spaces, where \( [a, b]_{\mathbb{T}} = [a, b] \cap \mathbb{T} \), \( a, b \in \mathbb{T} \), \( a < b \), is an arbitrary closed subinterval of \( \mathbb{T} \) and \( [a, b]_{\mathbb{T}} = [a, b] \cap \mathbb{T} \); we state some of their properties whose proofs can be found in [6, 3, 10].

Lemma 2.1. The set of all right-scattered points of \( \mathbb{T} \) is at most countable, that is, there are \( I \subset \mathbb{N} \) and \( \{ t_i \}_{i \in I} \) such that

\[
\mathcal{R} := \{ t \in \mathbb{T} : \sigma(t) > t \} = \{ t_i \}_{i \in I}.
\]
**Proposition 2.2.** Let \( A \subset \mathbb{T} \). Then \( A \) is a \( \Delta \)-measurable if and only if, \( A \) is Lebesgue measurable. If \( b \notin A \), then
\[
\mu_\Delta (A) = \mu_L (A) + \sum_{i \in I_A} \mu (t_i),
\]
where \( I_E := \{ i \in I : t_i \in E \} \).

**Definition 2.3.** Assume \( \mu_\Delta \) and \( \mu_L \) are measures. The set \( \mathbb{R} \) of all \( \mu_\Delta \)-measurable functions \( f : \mathbb{R} \rightarrow \mathbb{R} \) is a Banach space together with the norm defined for \( f \in L_\Delta^p (E) \) as
\[
\| f \|_{L_\Delta^p} := \left\{ \left( \int_{\mathbb{R}} | f(t) |^p \Delta t \right)^{\frac{1}{p}} \right\},
\]
where \( p \in \mathbb{R} \), and \( \inf \{ C \in \mathbb{R} : | f | \leq C \Delta - a.e. on \mathbb{R} \} \) is a constant.

Moreover, \( L_\Delta^p (E) \) is a Hilbert space together with the inner product given for every \( f, g \in L_\Delta^p (E) \) by
\[
(f, g)_{L_\Delta^p} := \int_{\mathbb{R}} f(s) \cdot g(s) \Delta s.
\]

**Definition 2.5.** Assume \( n \in \mathbb{N}, n \geq 1, p \in \mathbb{R} \) and \( p \geq 1 \). Let \( \rho : [a, b]_\mathbb{T} \rightarrow \mathbb{R} \). Say that \( f \) belongs to \( W^{n,p}_\Delta ([a, b]_\mathbb{T}) \) if and only if \( f \in L_\Delta^p ([a, b]_\mathbb{T}) \) and \( f^{\Delta^j} \in L_\Delta^p ([a, \rho^j (b)]_\mathbb{T}) \), for all \( j \in [1, n-1]_\mathbb{Z} \).

Where \( \rho^j (b) = \rho (\rho^{j-1} (b)) \) and \( f^{\Delta^j} = \left( f^{\Delta^j-1} \right)^{\Delta} \), for all \( j \in [1, n-1]_\mathbb{Z} \).

**Theorem 2.6.** Assume \( n \in \mathbb{N}, n \geq 1, p \in \mathbb{R} \) and \( p \geq 1 \). The set \( W^{1,p}_\Delta ([a, b]_\mathbb{T}) \) is a Banach space together with the norm defined for every \( f \in W^{1,p}_\Delta ([a, b]_\mathbb{T}) \) as
\[
\| f \|_{W^{1,p}_\Delta} := \sum_{j=0}^{n} \left\| f^{\Delta^j} \right\|_{L_\Delta^p},
\]
where \( f^{\Delta^0} = f \). Furthermore, the set \( H^{n}_\Delta ([a, b]_\mathbb{T}) = W_\Delta^{n,2} ([a, b]_\mathbb{T}) \) is a Hilbert space together with the inner product given for every \( f, g \in H^{n}_\Delta ([a, b]_\mathbb{T}) \) by
\[
(f, g)_{H^{n}_\Delta} := \sum_{j=0}^{n} \left( f^{\Delta^j}, g^{\Delta^j} \right)_{L_\Delta^2}.
\]

**Definition 2.7.** Assume \( n \in \mathbb{N}, n \geq 1, p \in \mathbb{R} \) and \( p \geq 1 \), define the set \( W^{n,p}_\Delta ([a, b]_\mathbb{T}) \) as the closure of the set \( C^{n}_{0,r} ([a, b]_\mathbb{T}) \) in \( W^{1,p}_\Delta ([a, b]_\mathbb{T}) \).

**Denote as** \( H^{n}_\Delta ([a, b]_\mathbb{T}) = W^{n,2}_\Delta ([a, b]_\mathbb{T}) \).

Where \( C^{n}_{0,r} ([a, b]_\mathbb{T}) = \{ f \in C^{n}_{r} ([a, b]_\mathbb{T}) : f(a) = f (\rho^j (b)) = 0, \text{ for all } j \in [1, n-1]_\mathbb{Z} \} \).
Proposition 2.8. Assume \( n \in \mathbb{N}, n \geq 1, p \in \mathbb{R} \) and \( p \geq 1 \). Let \( f \in W^{n,p}_{\Delta}([a,b]_T) \). Then, \( f \in W^{n,p}_{\rho,\Delta}([a,b]_T) \) if and only if \( f(a) = f(\rho^j(b)) = 0 \), for all \( j \in [1,n-1]_\mathbb{Z} \).

Proposition 2.9. Let \( p \in \mathbb{R} \) be such that \( p \geq 1 \). Then, there exists a constant \( L > 0 \), only dependent on \( (b-a) \), such that

\[
\|f\|_{W^{1,p}_{\Delta}} \leq L \|f^\Delta\|_{L^p_{\Delta}}, \quad \text{for all } f \in W^{1,p}_{0,\Delta}([a,b]_T).
\]

that is, in \( W^{1,p}_{0,\Delta}([a,b]_T) \), the norm defined for every \( f \in W^{1,p}_{0,\Delta}([a,b]_T) \) as \( \|f^\Delta\|_{L^p_{\Delta}} \) is equivalent to the norm \( \|f\|_{W^{1,p}_{\Delta}} \).

3. Main Results

In this paper, we suppose that \( \mathbb{T} \) is a particular time scale, \( a < b < \infty \) are points in \( \mathbb{T} \).

Now, we are ready to state and prove the main results in this paper. We generalize the Hardy-Sobolev-Maz'ya inequality (3) on time scales.

Theorem 3.1. Let \( q \geq 2 \). Then there exist constant \( C_q \) only on \( q \) such that the inequality

\[
\int_a^b |f^\Delta(t)|^2 \Delta t \geq \frac{1}{4} \int_a^b \frac{|f(t)|^2}{(b-t)^2} \Delta t + C_q \left( \int_a^b |f(t)|^q \Delta t \right)^{\frac{2}{q}}, \quad \text{(HSM)}
\]

holds for all \( f \in W^{1,q}_{0,\Delta}([a,b]_T) \).

If, in addition, \( t \to \frac{\mu(t)}{b-t} \) is a function nonincreasing.

Proof. Let \( g \) is function define by:

\[
f(t) = \eta(t) g(t), \quad t \in [a,b]_T.
\]

Where \( \eta(t) = \sqrt{b-t} \), for all \( t \in [a,b]_T \). Then \( \eta \in C^1_{rd}([a,b]_T) \) and

\[
\eta^\Delta(t) = \frac{-1}{\eta(t) + \eta^\sigma(t)}. \quad (6)
\]

Using propertie (6), we obtain that

\[
\eta^\sigma(t) g^\Delta(t) = f^\Delta(t) + \frac{f(t)}{\eta^2(t) + \eta(t) \eta^\sigma(t)}. \quad (7)
\]

By (6), we have \( \eta^\sigma(t) \leq \eta(t) \), and

\[
|\eta^\sigma(t) g^\Delta(t)|^2 = |f^\Delta(t)|^2 + \frac{f^2(t)}{(\eta^2(t) + \eta(t) \eta^\sigma(t))^2} + \frac{2f(t) f^\Delta(t)}{\eta^2(t) + \eta(t) \eta^\sigma(t)} \leq |f^\Delta(t)|^2 + \frac{\left\{ f(t) \left( \frac{f(t)}{(\eta^2(t) + \eta(t) \eta^\sigma(t))} + f^\Delta(t) \right) \right\}}{4(b-t)^2} \leq |f^\Delta(t)|^2 + \xi(t) g^\Delta(t) g(t) - \frac{1}{4(b-t)^2}. \quad (8)
\]

Where \( \xi(t) := -2\eta^\Delta(t) \eta^\sigma(t) \), for all \( t \in [a, \rho(b)]_T \). Then \( \xi \) is \( \Delta \)-differentiable for all the points right-scattered. Let \( t \in [a, b]_T \) such that \( t \) is point right-dense, then \( t \) is point accumulation, we have two cases.
(a) First case, there exists \( c, d \in [a, b]_T \) such that \( t \in [c, d] \subset [a, b]_T \), then \( \xi \) is \( \Delta \)-differentiable in \( t \) and \( \xi^\Delta (t) = 0 \).

(b) Second case, there exists a sequence \( \{ t_k \}_{k \in \mathbb{N}} \in \mathcal{R} \cap [a, b]_T \), such that, for all \( k \in \mathbb{N} \) one has \( t_k \) is point isolat and \( t_k \to t \) as \( k \to \infty \). In this case, \( \xi^\Delta (t) \) do not exist.

By the proposition 2.2, we get

\[
\mu_\Delta \left( \left\{ t \in [a, b]_T : \sigma (t) = t \text{ and } t = \lim_{k \to \infty} t_k, (t_k)_{k \in \mathbb{N}} \subset \mathcal{R} \right\} \right) = 0.
\]

Consequently, we obtain that \( \xi^\Delta \) is \( \Delta \)-differentiable a.e on \( [a, b]_T \).

Let \( t, s \in [a, b]_T \) whish that \( t > s \), we have

\[
\xi (t) - \xi (s) = \frac{1}{2} \xi (t) \xi (s) \left\{ \frac{\eta (s)}{\eta^\sigma (s)} - \frac{\eta (t)}{\eta^\sigma (t)} \right\}
\]

\[
= \frac{1}{2} \xi (t) \xi (s) \left\{ \sqrt{1 + \frac{\mu (s)}{b - \sigma (s)}} - \sqrt{1 + \frac{\mu (t)}{b - \sigma (t)}} \right\}.
\]

Then \( \xi \) is function increasing.

Therefore,

\[
\int_a^b \xi (t) g^\Delta (t) g (t) \Delta t = - \int_a^b [\xi, g] g^\Delta (t) g^\sigma (t) \Delta t
\]

\[
= - \int_a^b \xi (t) |g^\sigma (t)|^2 \Delta t - \int_a^b \xi (t) g^\Delta (t) g^\sigma (t) \Delta t
\]

\[
\leq - \int_a^b \xi (t) g^\Delta (t) g (t) \Delta t - \int_a^b \xi (t) \mu (t) |g^\Delta (t)|^2 \Delta t
\]

\[
\leq - \int_a^b \xi (t) g^\Delta (t) g (t) \Delta t.
\]

Using the above inequality we have

\[
\int_a^b |\eta^\sigma (t) g^\Delta (t)|^2 \Delta t \leq \int_a^b \left( |f^\Delta (t)|^2 - \frac{|f (t)|^2}{4 (b - t)^2} \right) \Delta t \tag{9}
\]

Bötsche rule [1], we see that

\[
|g (t)|^{\frac{q + 2}{2}} \leq \frac{q + 2}{2} \left| g^\Delta (t) \right| \int_0^1 |h g (t) + (1 - h) g^\sigma (t)|^2 \, dh
\]

\[
\leq \frac{q + 2}{2} \left| g^\Delta (t) \right| |g_1 (t)|^2.
\]

Using the fact that \( \eta \) is decreasing and we find that

\[
|f (t)|^{\frac{q + 2}{2}} = |\eta (t)|^{\frac{q + 2}{2}} \int_a^t \left( |g (s)|^{\frac{q + 2}{2}} \right)^\Delta s
\]

\[
\leq \frac{q + 2}{2} \int_a^t \left| \eta (t) \right|^{\frac{q + 2}{2}} |g^\Delta (s)| |G (s)|^\frac{q}{2} \Delta s
\]

\[
\leq \frac{q + 2}{2} \int_a^b \left| g^\Delta (s) \right| |G (s)|^\frac{q}{2} |\eta (s)|^{\frac{q + 2}{2}} \Delta s.
\]
Where $G := \max \{|g|, |g^\sigma|\}.$

Using the Hölder inequality we find

$$|f(t)|^{q+2} \leq m_q \left( \int_a^b |g^\Delta(t)|^{2} \eta^2(t) \Delta t \right) \left( \int_a^b |G(t)|^q \eta^q(t) \Delta t \right) \leq m_q \int_a^b \left( |f^\Delta(t)|^{2} - \frac{|f(t)|^{2}}{4(b-t)^2} \right) \Delta t \left( \int_a^b |F(t)|^q \Delta t \right).$$

Where $m_q = \frac{1}{q} (q + 2)^2$ and $F := \max \{|f|, |f^\sigma|\}.$

Then

$$\int_a^b |f_i(t)|^q \Delta t \leq (m_q)^{\frac{q}{q+2}} \left( \int_a^b \left( |f^\Delta(t)|^{2} - \frac{|f(t)|^{2}}{4(b-t)^2} \right) \Delta t \right)^{\frac{2}{q+2}} \left( \int_a^b |F(t)|^q \Delta t \right)^{\frac{q}{q+2}}.$$

Thus

$$\int_a^b |f^\Delta(t)|^2 \geq \frac{1}{4} \int_a^b \frac{|f(t)|^{2}}{(b-t)^2} \Delta t + \frac{1}{m_q} \left( \int_a^b |F(t)|^q \Delta t \right)^{\frac{q}{q+2}}.$$

The intended inequality (HSM) is proved.

4. APPLICATION

We are concerned with the existence of positive solutions of the $p$-Laplacian dynamic equation on a time scale

$$\begin{cases}
[r \phi_p(u^\Delta)]^\Delta + \frac{\xi}{(\sigma(t) - a)^p} \phi_p(u^\sigma) = -f & \text{in } [a, \rho^2(b)]_T, \\
u(a) = u(b) = 0,
\end{cases} \quad (10)$$

where $\phi_p(s)$ is $p$-Laplacian operator, i.e., $\phi_p(s) = |s|^{p-1} s, p > 1, f \in L^q_\Delta ([a, b]_T), \frac{1}{p} + \frac{1}{q} = 1, r \in C_{rd}([a, b]_T)$ and $\alpha \xi \geq C_p$ (Define in the Theorem 3.1).

Consider again the functional

$$E_p(u) := \frac{1}{p} \int_a^b r |u^\Delta|^p \Delta t - \frac{1}{p} \int_a^b h |u^\sigma|^p \Delta t - \int_a^b f u^\sigma \Delta t,$$

is then well defined on the Sobolev space $W^{1,p}_\Delta ([a, b]_T).$ The (weak) solutions of the problem (10) are then the critical points of the functional $(E_p).$

The classical results in the Calculus of Variations characterize the weak. Then, the problem (10) has weak solution in $W^{1,p}_{0,\Delta} ([a, b]_T) \cap W^{2,p}_{0,\Delta} ([a, b]_T).$

REFERENCES


G.H. Hardy, Notes on some points in the integral calculus, An inequality between integrals, Messenger. Math. 54(1925), 150–156.


Fatima Zohra Ladrani
Department of Mathematics, Higher normal school of Oran, BP 1523, 31000 Oran, Algeria.
Laboratory of Biomathematics, Sidi Bel Abbes University, 22000 Sidi Bel Abbes, Algeria
E-mail address: f.z.ladrani@gmail.com

Amine Benaissa Cherif
Department of Mathematics, University of Ain Temouchent, BP 284, 46000 Ain Temouchent, Algeria
Laboratory of Biomathematics, Sidi Bel Abbes University, 22000 Sidi Bel Abbes, Algeria
E-mail address: amine.banche@gmail.com