A TWO-POINT BOUNDARY VALUE PROBLEM FOR A DIFFERENTIAL EQUATION WITH SELF-REFERENCE

NGUYEN T.T. LAN AND EDUARDO PASCALI

Abstract. In this paper, we study the following two-point boundary value problem
\[
\begin{align*}
  u'(t) &= a(t)u(u(t)), \quad t \in [-1, 1], \\
  a\alpha(-1) + \beta u(1) &= \gamma,
\end{align*}
\]
where \(a(t)\) is a given continuous, non-negative function on \([-1, 1]\); \(\alpha, \beta\) and \(\gamma\) are constants such that \(\alpha + \beta \neq 0\) and other appropriate conditions. The existence of solution of this problem is proved first by the Schauder fixed-point theorem and next by a iterative procedure.

1. Introduction

The existence, uniqueness, analyticity and analytic dependence of solutions to the following equation of a one-variable unknown function \(u : I \subset \mathbb{R} \rightarrow \mathbb{R}\) is well considered in [1]
\[
  u'(t) = u(u(t))
\]
This equation has attracted much attention. As a more general case than (1), Si and Cheng [3] investigated the functional-differential equation
\[
  u'(t) = u(at + bu(t)),
\]
where \(a \neq 1\) and \(b \neq 0\) are complex numbers; the unknown \(u : \mathbb{C} \rightarrow \mathbb{C}\) is a complex function. By using the power series method, analytic solutions of this equation are obtained. By generalizing (2), in [7] Cheng, Si and Wang considered the equation
\[
  \alpha t + \beta u'(t) = u(at + bu'(t)),
\]
where \(a, \alpha\) and \(b, \beta\) are complex numbers. Existence theorems are established for the analytic solutions, and systematic methods for deriving explicit solutions are also given. In [8], Staněk studied maximal solutions of the functional-differential equation
\[
  u(t)u'(t) = ku(u(t))
\]

2010 Mathematics Subject Classification. 47J35, 45G10.
Key words and phrases. Non-linear evolution equations; functional differential equations, existence.
with \(0 < |k| < 1\). Here \(u : I \subset \mathbb{R} \to \mathbb{R}\) is a real unknown. This author showed that properties of maximal solutions depend on the sign of the parameter \(k\) for two separate cases \(k \in (-1, 0)\) and \(k \in (0, 1)\). For earlier work of Staněk than (3), see [9]–[14].

The idea of the equation (1) is developed also for partial differential equations (see [2, 4, 5, 6]).

It is emphasized that in [1, 3, 7] and [9]–[14] any boundary-value problem of (1) has not been considered.

In this paper, by associating (1) with a two-point boundary condition, we study the solution existence of the following two-point boundary-value problem:

\[
\begin{aligned}
    u'(t) &= a(t)u(t)), \quad t \in [-1, 1], \\
    au(-1) + \beta u(1) &= \gamma,
\end{aligned}
\]

where \(a(t), \alpha, \beta\) and \(\gamma\) with \(\alpha + \beta \neq 0\) are given.

2. Solution existence by Schauder fixed-point theorem

In this section we start with the definition of an operator \(T\) such that fixed point for \(T\) are solution of the problem (4).

**Lemma 2.1.** Assume that \(a(t)\) is a given continuous, non-negative function on \([-1, 1] ; \alpha, \beta\) and \(\gamma\) are constants such that \(\alpha + \beta \neq 0\). Moreover assume

\[
\int_{-1}^{1} a(s)ds \leq \frac{|\alpha + \beta| - |\gamma|}{|\alpha + \beta| + |\beta|}.
\]

Then the problem (4) is equivalent to the following operator equation

\[
u = T(u),
\]

where the operator

\[
T_{u}(t) := \int_{-1}^{t} a(s)u(u(s))ds - \frac{\beta}{\alpha + \beta} \int_{-1}^{1} a(s)u(u(s))ds + \frac{\gamma}{\alpha + \beta}
\]

acts in the convex, closed, bounded subset \(K = C([-1, 1]; [-1, 1])\) of the Banach space \(X = C([-1, 1]; R)\) endowed with the norm \(||u|| = \max |u(t)|\).

**Proof.** We prove at first that if \(u \in K\) then \(Tu\) is an element of \(K\). This is easy to prove from (5). In fact if \(||u(.)|| \leq 1\), then

\[
|Tu(t)| \leq \left(1 + \frac{|\beta|}{|\alpha + \beta|}\right) \int_{-1}^{1} a(s)ds + \frac{|\gamma|}{|\alpha + \beta|} \leq 1.
\]

Now, from (4)_1, we deduce that

\[
u(t) = u(-1) + \int_{-1}^{t} a(s)u(u(s))ds,
\]

hence, from (4)_2

\[
u(-1) = \frac{-\beta}{\alpha + \beta} \int_{-1}^{1} a(s)u(u(s))ds + \frac{\gamma}{\alpha + \beta}.
\]

From (7) and (8), we obtain

\[
u(t) = \int_{-1}^{t} a(s)u(u(s))ds - \frac{\beta}{\alpha + \beta} \int_{-1}^{1} a(s)u(u(s))ds + \frac{\gamma}{\alpha + \beta}.
\]
Moreover, if (9) holds for \( u \in X \) then (4) also holds. \( \square \)

We have the following theorem.

**Theorem 2.1.** Suppose \( a(t) \) is a given continuous, non-negative function on \([-1, 1]\) satisfying (5) where \( \alpha, \beta \) and \( \gamma \) are constants such that \( \alpha + \beta \neq 0 \). Then the operator (6) has a fixed point in \( K \).

**Proof.** From the definition of \( K \), it is clear that \( K \) is convex, closed and bounded in the Banach space \( X \).

For \( u \in K \), consider

\[
T(u) := \int_{-1}^{1} a(s)u(u(s))ds - \frac{\beta}{\alpha + \beta} \int_{-1}^{1} a(s)u(s)ds + \frac{\gamma}{\alpha + \beta}.
\]

Note that the identity \( a(Tu)(-1) + \beta(Tu)(1) = \gamma \) holds.

Moreover, we have that \( T(K) \subseteq K \). In fact, if \( u \in K \), from \(|(Tu)(t)| \leq (1 + |\beta| + |\gamma|) \int_{-1}^{1} |a(s)|ds + |\gamma| \frac{1}{|\alpha + \beta|} \) it follows

\[
|(Tu)(t)| \leq \frac{|\alpha + \beta + |\beta|| \int_{-1}^{1} |a(s)|ds + |\gamma|}{|\alpha + \beta|} \leq 1
\]

for all \( t \in [-1, 1] \). Therefore the claim is proved.

Furthermore, \( T \) is continuous. Let \((u_n)\) be a sequence in \( K \) convergent with respect to the norm \(|\cdot|_0\) to the function \( u \in K \). Note that for every \( n \in N \) and \( t \in [-1, 1], \)

\[
|u_n(u_n(t)) - u(u(t))| \leq |u_n(u_n(t)) - u(u_n(t))| + |u(u_n(t)) - u(u(t))|.
\]

From the uniform convergence of \((u_n)\) to \( u \), for a fixed \( \epsilon > 0 \) there exists \( \nu_1 \) such that \(|u_n(\rho) - u(\rho)| \leq \frac{\epsilon}{2} \) for every \( \rho \). And still, for the uniform continuity of \( u \), for a fixed \( \epsilon > 0 \) there exists \( \delta > 0 \) such that \(|u(\xi_2) - u(\xi_1)| \leq \delta \) we have \(|u(\xi_2) - u(\xi_1)| \leq \frac{\epsilon}{2} \).

Hence, there exists \( \nu_2 \) such that for \( n > \nu_2 \) we have \(|u_n(\rho) - u(\rho)| \leq \delta \). So for \( n > \max(\nu_1, \nu_2) \) we obtain that

\[
|u_n(u_n(t)) - u(u(t))| \leq |u_n(u_n(t)) - u(u_n(t))| + |u(u_n(t)) - u(u(t))| < \epsilon
\]

for all \( t \in [-1, 1] \). This proves the continuity of \( T \).

Since \( a \) is continuous on \([-1, 1]\), there exists \( M \in \mathbb{R} \) such that \( a(s) \leq M \).

We are proving that \( T(K) \) is relatively compact with respect to the norm \(|\cdot|_0\). Let \((Tu_n)\) be a sequence with \( u_n \in K \) for all \( n \in N \). It is obvious that \((Tu_n)\) is bounded, recalled Lemma 3.2. From the continuity of \( u_n \) and \( a \), we have that \( Tu_n \in C^1 \) and \((Tu_n)(t) = a(t)u_n(u_n(t))\). Therefore, \(|(Tu_n)(t)| \leq M \) for all \( t \). Then \((Tu_n)\) is an equi-bounded, equi-Lipschitz sequence. By the Ascoli-Arzelà theorem, there exists a convergent subsequence of \((Tu_n)\).

In conclusion, we have that \( T : K \rightarrow K \) is a continuous operator and \( T(K) \) is relatively compact. By Schauder fixed point theorem, \( T \) has a fixed point in \( K \). \( \square \)

3. An iterative scheme for existence of solutions

The Schauder theorem applied in the previous section say that a solution of the problem (4) exists; now we consider a sequence of functions, defined by iteration, for which the uniform limit exists and is a solution of the problem (4). We need an other condition on function \( a = a(t) \) and \( \alpha, \beta, \gamma \).
Consider the following sequence of functions \( \{u_n\}_n \)

\[
\begin{align*}
    u_{n+1}(t) &= \int_{-1}^{t} a(s)u_n(s)ds - \frac{\beta}{\alpha + \beta} \int_{-1}^{1} a(s)u_n(s)ds + \frac{\gamma}{\alpha + \beta}, \\
    u_0(t) &= \frac{\alpha}{\alpha + \beta},
\end{align*}
\]  

(10)

for all \( t \in [-1, 1] \).

Assume also the following condition

\[
    \left| \frac{\gamma}{\alpha + \beta} \right| \leq 1,
\]  

(11)

from the definition of the operator \( T \), as in the previous section, it is easy to prove the following lemma.

**Lemma 3.2.** The sequence defined by (10) is equibounded and every \( u_n \) is a \( C^1 \) function provided that \( a(t) \) is a given continuous, non-negative function on \([-1, 1]\) such that (5) and (11) hold.

More precisely we have that, \( \forall n \in \mathbb{N} \),

\[
    |u_n(t)| \leq 1; \quad |u_n'(t)| \leq M
\]

and so

\[
    |u_n(t_2) - u_n(t_1)| \leq M|t_2 - t_1| \quad \forall t_2, t_1 \in [-1, 1]
\]

where \( M = \max_{t} a(t) \).

Hence we are able to prove the following theorem.

**Theorem 3.2.** Suppose \( a(t) \) is a given continuous, non-negative function on \([-1, 1]\) satisfying (5) and (11) where \( \alpha, \beta \) and \( \gamma \) are constants such that \( \alpha + \beta \neq 0 \).

Assume, if \( 1 \leq M \),

\[
    \max_{t} \left[ \frac{|\alpha|}{|\alpha + \beta|} \int_{-1}^{t} a(s)ds + \frac{|\beta|}{|\alpha + \beta|} \int_{t}^{1} a(s)ds \right] < \frac{1}{2M}
\]  

(12)

or, if \( \alpha, \beta \) are non-negative,

\[
    \int_{-1}^{1} a(s)ds < \frac{1}{2M}
\]  

(13)

or, if \( M \leq 1 \), the same previous conditions with \( M \) replaced with 1. Then the sequence (10) is uniformly convergent to a solution of the problem (4).

**Proof.** We remark that

\[
    |u_1(t) - u_0(t)| = \frac{\gamma}{(\alpha + \beta)^2} \left[ \alpha \int_{-1}^{t} a(s)ds - \beta \int_{t}^{1} a(s)ds \right],
\]

hence

\[
    |u_1(t) - u_0(t)| \leq \frac{\gamma}{(\alpha + \beta)^2} \left[ |\alpha| \int_{-1}^{t} a(s)ds + |\beta| \int_{t}^{1} a(s)ds \right] = \frac{|\gamma|}{|\alpha + \beta|^2} g_1(t)
\]

where \( g_1(t) = \left[ |\alpha| \int_{-1}^{t} a(s)ds + |\beta| \int_{t}^{1} a(s)ds \right] \).

Assume that \( 1 \leq M \).

From

\[
    |u_2(t) - u_1(t)| = \frac{1}{\alpha + \beta} \left[ \alpha \int_{-1}^{t} a(s)[u_1(u_1(s)) - u_0(u_0(s))]ds + \right.
\]

\[-\beta \int_t^1 a(s)[u_1(u_1(s)) - u_0(u_0(s))]ds]\]

we obtain, for the previous step, condition on \( M \) and the Lipschitz property of all \( u_n \) of the previous lemma,

\[|u_2(t) - u_1(t)| \leq \frac{M|\gamma|}{|\alpha + \beta|^3} g_2(t)\]

where

\[g_2(t) = \left[ |\alpha| \int_{-1}^t a(s)g_1(s)ds + |\beta| \int_t^1 a(s)g_1(s)ds \right] + \left[ |\alpha| \int_{-1}^t a(s)g_1(u_0(s))ds + |\beta| \int_t^1 a(s)g_1(u_0(s))ds \right].\]

It is easy to prove, by induction, that for all \( n \in N \) and \( t \in [-1, 1] \)

\[|u_{n+1}(t) - u_n(t)| \leq \frac{M^n|\gamma|}{|\alpha + \beta|^{n+2}} g_{n+1}(t)\]

where

\[g_{n+1}(t) = \left[ |\alpha| \int_{-1}^t a(s)g_n(s)ds + |\beta| \int_t^1 a(s)g_n(s)ds \right] + \left[ |\alpha| \int_{-1}^t a(s)g_n(u_{n-1}(s))ds + |\beta| \int_t^1 a(s)g_n(u_{n-1}(s))ds \right].\]

Now we consider \( H = \max_t g_1(t) \) and remark that

\[0 \leq g_2(t) \leq 2H^2, \quad 0 \leq g_3(t) \leq 2^2H^3.\]

Hence, by induction, it is easy to prove that for all \( n \in N, \quad t \in [-1, 1] \)

\[0 \leq g_n(t) \leq 2^{n-1}H^n.\]

Then the following inequalities hold

\[|u(t)n + 1 - u_n(t)| \leq \frac{M^n|\gamma|}{|\alpha + \beta|^{n+2}} H^{n+1} = |\gamma|H^{2} \frac{2^2}{M^2} \left[ \frac{2MH}{|\alpha + \beta|^3} \right]^{n+2}.\]

From condition (12) (or (13)) follows, for all \( n \in N, \quad \left[ \frac{2MH}{|\alpha + \beta|^3} \right] < 1.\)

If we assume \( M \leq 1, \) the proof is analogous.

Hence the sequence \( (u_n)_n \) is uniformly convergent to a function \( u_\infty \in K \) and this limit is obviously a solution for the problem (4).

\[\square\]

**References**

[10] S. Stanék, Global properties of decreasing solutions for the equation $u'(t) = u(u(t)) - bu(t)$, $b \in (0, 1)$, Soochow J. Math. 26, 123–134, 2000.

Nguyen T.T. Lan
Faculty of Mathematics and Applications, Saigon University, Ho Chi Minh City, Vietnam.
E-mail address: nguyenttlan@sgu.edu.vn; nguyenttlan@gmail.com

Eduardo Pascali
Department of Mathematics "Ennio De Giorgi", University of Salento, Italy.
E-mail address: eduardo.pascali@unisalento.it